# Infinite graph theory II: exercises on 30/06/2022

1. Let $M$ be an elementary submodel. Show that for every $x \in \mathbb{R}^2 \setminus M$ there is at most one $y \in \mathbb{R}^2 \cap M$ such that $d(x, y) \in \mathbb{Q}$ where $d(x, y)$ denotes the distance between $x$ and $y$.

2. Let $G = (V, E)$ be the *rational distance graph* on the plane, i.e. $V = \mathbb{R}^2$ and $E = \{\{u, v\} \in [V]^2 : \ d(u, v) \in \mathbb{Q}\}$. Prove $\chi(G) \leq \aleph_0$ by showing that $K_{2,\omega_1}$ is not a subgraph of $G$ (we have seen that uncountably chromatic graphs must contain $K_{n,\omega_1}$ as a subgraph for every $n < \omega$).

3. Show that for every $f : \omega_1 \to [\omega_1]^\omega$ one can find an $\alpha < \omega_1$ such that $f(\beta) \subseteq \alpha$ for every $\beta < \alpha$.[1](10p)

   Hint: try to get $\alpha$ as a limit of an increasing sequence $(\alpha_n)_{n<\omega}$ where $\alpha_{n+1}$ "fixes the failure" of $\alpha_n$.

4. Prove that if $c$ is a colouring of the edges of $K_{\omega_1}$ with the colours $\omega_1$ in such a way that every uncountable induced subgraph contains all the colours, then there is a countably infinite monochromatic induced subgraph in each colour. (10p)

[1]Fo a set $X$ and cardinal $\kappa$ we denote the set of the $\kappa$-sized subsets of $X$ by $[X]^\kappa$.