Infinite graph theory II: exercises on 23/06/2022

1. Show that every infinite partial order has either an infinite chain or an infinite set of pairwise incomparable elements.

2. Prove that any infinite linear order has either an infinite increasing or an infinite decreasing sequence.

3. Show that K_{n,ω_1} is a subgraph of every uncountably chromatic graph.

4. Let κ be an infinite cardinal and let $G = ([(2^\kappa)^+]^2, E)$ be the graph where

   $$E = \{\{\{\alpha, \beta\}, \{\beta, \gamma\}\} : \ \alpha < \beta < \gamma < (2^\kappa)^+\}.$$

   Prove that $\chi(G) > \kappa$. (10p)

   Hint: Use $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$.

5. Show that there is no strictly increasing function $f : \omega_1 \to \mathbb{R}$. More generally, prove that if κ is an infinite cardinal then the lexicographical order of ${}^\kappa 2$ does not contain any strictly increasing or decreasing κ^+-sequence.

6. Let κ be an uncountable regular cardinal and let $U \subseteq \kappa$ be unbounded. Show that

   $$\mathsf{acc}(U) := \{\alpha < \kappa : \ \sup(\alpha \cap U) = \alpha\}$$

   is a club.

7. Let κ be an uncountable regular cardinal and let $f : \kappa \to \kappa$ be a bijection. Show that there is a club $C \subseteq \kappa$ such that $f \restriction C$ is increasing. (10p)

   Hint: Build greedily a set on which f is increasing and use Fodor's lemma to show that its complement cannot be stationary.