Infinite graph theory II: exercises on 16/06/2022

1. Let $G = (V, E)$ be a graph and suppose that V admits a well-order in which for each $v \in V$ less than κ neighbours of v are smaller than v. Prove that $\chi(G) \leq \kappa$.

2. Let $G = (V, E)$ be a graph for which there is a partition $V = \bigcup_{i<\kappa} V_i$ such that for every $i < \kappa$: $\chi(G[V_i]) \leq \aleph_0$ and each $v \in V_i$ has only finitely many neighbours in $\bigcup_{j<i} V_j$. Show that $\chi(G) \leq \aleph_0$. (10p)

3. Prove that for every 2-connected graph $G = (V, E)$ and $u, v \in V$ there is a finite 2-connected subgraph of G containing u and v. Is the analogous statement true for arbitrary $k \in \mathbb{N}$? What if we weaken 'finite' to 'countable'? (10p)

4. Let κ be an infinite cardinal and let κ^+ be the smallest cardinal that is larger than κ. Show that κ^+ cannot be obtained as the sum of less than κ^+ many cardinals each of which smaller than κ^+ (i.e. successor infinite cardinals are regular). (10p)