Infinite graph theory II: exercises on 07/04/2022

1. Assume that $D = (V, E)$ is a (possibly infinite) digraph, $a \neq b \in V$ and $\mathcal{P}$ is a system of edge-disjoint ab-paths. Prove that either one can choose exactly one edge from each $P \in \mathcal{P}$ in such a way that the resulting edge set is an ab-cut (i.e. meets every ab-path) or there exists another system $\mathcal{Q}$ of edge-disjoint ab-paths such that $\delta^+_{\mathcal{Q}}(a) \supset \delta^+_{\mathcal{P}}(a)$ and $\delta^-_{\mathcal{Q}}(b) \supset \delta^-_{\mathcal{P}}(b)$ with $\left|\delta^+_{\mathcal{Q}}(a) \setminus \delta^+_{\mathcal{P}}(a)\right| = \left|\delta^-_{\mathcal{Q}}(b) \setminus \delta^-_{\mathcal{P}}(b)\right| = 1$. (Here $\delta^+_{\mathcal{Q}}(a)$ stands for the set of the outgoing edges of a in the digraph consisting of the paths in $\mathcal{Q}$ and $\delta^-_{\mathcal{Q}}(b)$ is defined similarly but with ingoing edges.)

   Hint: Consider the digraph D' that we obtain from D by reversing the edges in $E(\mathcal{P})$. Show that if D' admits an ab-path, then the second possibility occurs and otherwise the first one.

2. Formulate and prove a version of the statement from the previous exercise in which there are vertex-disjoint AB-paths for some $A, B \subseteq V$.

   Hint: Reduce it to the edge version by splitting vertices into edges.

3. State and prove the undirected versions of the statements above.