Game Theory, exercise sheet 9

1. (4 points) Compute the man-optimal and the woman-optimal stable matchings with the Gale-Shapley algorithm.

$$\begin{array}{ll} m_1 : w_2, w_3, w_1 & w_1 : m_2, m_3, m_1 \\ m_2 : w_2, w_3, w_1 & w_2 : m_3, m_2, m_1 \\ m_3 : w_3, w_2, w_1 & w_3 : m_2, m_3, m_1 \end{array}$$

2. (2 +4 points)
A, B, C, D are men and a, b, c, d are women.

$$\begin{array}{ll} A : a, b, c, d & a : D, C, B, A \\ B : b, a, d, c & b : C, D, A, B \\ C : c, d, a, b & c : B, A, D, C \\ D : d, c, b, a & d : A, B, C, D \end{array}$$

a) Given two stable matchings $M = \{Ac, Ba, Cd, Db\}$ and $M' = \{Ab, Bd, Ca, Dc\}$, if each man chooses his best mate from $M(m)$ and $M'(m)$, what matching do we get?
b) What are all the stable matchings in this example?

3. (5 points) Given k stable matchings, $S_1 ... S_k$, every man selects from his k possible partners the l^{th} best one (counted with multiplicity, if some of $S_1(m) \dots S_k(m)$ are equal). $1 \leq l \leq k$.
Show that this gives a stable matching.

4. (4 points) Show that in the one-to-one model, the lattice of stable matchings is distributive, i.e.

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

5. (2+2 points) There are n boys and n girls, and their preference lists contains all the members of the other sex. In the algorithm, "all boys make a proposal" is counted as one step.
a) Suppose that the boys all have different favorite girls. How many steps does it take for the Gale-Shapley algorithm to terminate?
b) Suppose that the boys have identical preferences. How many steps does it take for the Gale-Shapley algorithm to terminate?

6. (4 points) Determine a list of four men and four women (each ranking all the four people of the opposite sex) where no one obtains his or her first choice in the Gale-Shapley algorithm, regardless of which sex proposes.

7. (4 points) Prove that if the men propose then at most one of the men gets his last choice (assuming that everyone has ranked everyone of the opposite sex).

8. (4 points) Roommate problem: There are four people, Pat, Chris, Bob, and Leslie. They must pair off (each pair will share a two-bed suite). Each has preferences over which of the others they would prefer to have as a roommate. The preferences are:
Leslie: Pat > Chris > Bob
Chris: Leslie > Pat > Bob
Pat: Chris > Leslie > Bob
Bob: Chris > Leslie > Pat
Show that no stable matching exists. (That is, no matter who you put together, they will always be two potential roommates who are not matched, but prefer each other to their current roommate.)