Game Theory, exercise sheet 8

Reminder (we showed this on the lecture)
Theorem 8.1.8.
Let G be a network where one unit of traffic is routed from a source s to a destination t. Suppose that the latency function on each edge e is affine; that is, $\ell_e(x) = a_e x + b_e$, for constants $a_e, b_e \geq 0$. Let $\mathbf{f}$ be an equilibrium flow in this network and let $\mathbf{f}^*$ be an optimal flow; that is,

$$L(\mathbf{f}^*) = \min\{L(\tilde{\mathbf{f}}) : \tilde{\mathbf{f}} \in \Delta(\mathcal{P}_{st})\}$$

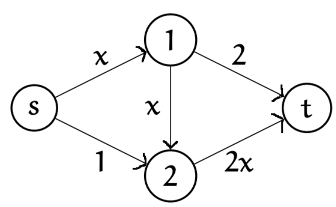
Then the price of anarchy is at most 4/3; i.e.,

$$L(\mathbf{f}) \leq \frac{4}{3} L(\mathbf{f}^*)$$

1. (4 points) Show that Theorem 8.1.8 holds in the presence of multiple traffic flows. Specifically, let G be a network where $r_i > 0$ units of traffic are routed from source s_i to destination t_i, for each $i = 1, \ldots, k$. Suppose that the latency function on each edge e is affine; that is, $\ell_e(x) = a_e x + b_e$, for constants $a_e, b_e \geq 0$. Show that the price of anarchy is at most 4/3; that is, the total latency in equilibrium is at most 4/3 that of the optimal flow.

2. (4 points) Let G be a network where one unit of traffic is routed from a source s to a destination t. Suppose that the latency function on each edge e is linear; that is, $\ell_e(x) = a_e x$ for constants $a_e \geq 0$. Show that the price of anarchy in such a network is 1.
Hint: use the inequality $xy \leq (x^2 + y^2)/2$

3. (5 points) We have a network with demand d = 1. Compute the equilibrium flow, the socially optimal flow, as well as the social cost of both flows and the Price of Anarchy.



4. (4 points) Suppose that all players in a k-player game have the same set of pure strategies S. Denote by $u_j(s; \mathbf{x})$ the utility of player j when he plays pure strategy $s \in S$ and all other players play the mixed strategy $\mathbf{x}$. We say the game is symmetric if $u_i(s; \mathbf{x}) = u_j(s; \mathbf{x})$ for every pair of players i, j, pure strategy s, and mixed strategy $\mathbf{x}$.
Prove the following:
In a symmetric game, there is a symmetric Nash equilibrium (where all players use the same strategy).

5. (5 points) Consider two-player general-sum games, and for the following categories, either give an example or show it is impossible:

- A symmetric game, where only symmetric Nash eq. exist.
- A symmetric game, where symmetric Nash eq. and asymmetric eq. both exist.
- A not symmetric game, where only symmetric Nash eq. exist.
- A not symmetric game, where only asymmetric Nash eq. exist.
- A not symmetric game, where symmetric Nash eq. and asymmetric eq. both exist.

6. (3 points) We have a two-player symmetric game. One outcome is a pure Nash equilibrium. Is it always true that is it evolutionary stable?
On the exercise class, we discussed this and said a matrix with $(1, 1)$ everywhere is counterexample. Can you find another counterexample for which the matrix is not the same everywhere?