Discrete Mathematics Exercises
2019 Summer semester

1 Exercise sheet 1

1. Find relations R, S on some set X such that $R \circ S \neq S \circ R$.

2. Let us imagine we intend to buy a refrigerator. We simplify the complicated real situation by a mathematical abstraction, and we suppose that we only look at three numerical parameters of refrigerators: their cost, electricity consumption, and the volume of the inner space. If we consider two types of refrigerators, and if the first type is more expensive, consumes more power, and a smaller amount of food fits into it, then the second type can be considered a better one?a large majority of buyers of refrigerators would agree with that.
The relation "to be clearly worse" (denote it by $\preceq$) in this sense is the following: on the set of triples (c, p, v) of real numbers (c stands for cost, p for power consumption, and v for volume), defined as follows: $(c_1, p_1, v_1) \preceq (c_2, p_2, v_2)$ if and only if $c_1 \geq c_2$, $p_1 \geq p_2$, and $v_1 \leq v_2$.
Show that this $\preceq$ is a partial ordering.

3. Are the following relations R over set X equivalence relations?

1. $X = \mathbb{R}^2$, $((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$.
2. $X = \mathbb{R}^2$, $((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow x_1 \cdot y_2 = x_2 \cdot y_1$.
3. $X = \mathbb{R}^2 \setminus \{(0, 0)\}$, $((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow x_1 \cdot y_2 = x_2 \cdot y_1$.

4. Let R and S be arbitrary equivalences on a set X. Decide which of the following relations are necessarily also equivalences (if yes, prove; if not, give a counterexample).

1. $R \cap S$
2. $R \cup S$
3. $R \setminus S$
4. $R \circ S$

5. The following (false) proof tries to prove that every symmetric and transitive relation is also reflexive: Let R be symmetric and transitive relation on set X, then for every $x, y \in X$ with $(x, y) \in R$ since the symmetry $(y, x) \in R$ and using transitivity, $(x, y) \in R$ and $(y, x) \in R$ therefore $(x, x) \in R$. Thus the relation R is an equivalence relation.

Give a counterexample for this statement, and show where is the mistake.

2 Exercise sheet 2

6. (1 point each)

1. Show that a largest element is always maximal.
2. Find an example of a poset with a maximal element but no largest element.

3. Find a poset having no smallest element and no minimal element either, but possessing a largest element.

7. (2 points)
Consider the set $\{1, 2, \dots, n\}$ ordered by the divisibility relation. What is the maximum possible number of elements of a set $X \subseteq \{1, 2, \dots, n\}$ that is ordered linearly by the relation? (such a set X is called a chain)

8. (3 points)
Prove that a relation R on a set X satisfies $R \cap R^{-1} = \Delta_X$ if and only if R is reflexive and antisymmetric.

9. (3 points)
Let X, Y, Z be finite sets, let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be relations, and let A_R and A_S be their adjacency matrices, respectively. (A_R has $|X|$ rows and $|Y|$ coloumns and $a_{Rij} = 1$ if $x_i R y_j$ otherwise 0.) Their matrix product is $A_R A_S$. Discover and describe the connection of the composed relation $R \circ S$ to the matrix product $A_R A_S$.

10. (for handing in, 8 points)
Let R be a relation on a set X such that there is no finite sequence of elements $x_1, x_2, \dots x_k$ of X satisfying $x_1 R x_2$, $x_2 R x_3, \dots,$ $x_{k-1} R x_k, x_k R x_1$ (we say that such an R is acyclic). Prove that there exists an ordering $\preceq$ on X such that $R \subseteq \preceq$. You may assume that X is finite if this helps.

3 Exercise sheet 3

For problems 1-4, each subproblem is worth 1 point.

11. **a)** There is a building with n floors (counting the ground floor as well).
How many ways can we paint the levels to red, yellow or blue?
b) What happens if two consecutive levels cannot have the same color?

Solution: a, 3^n (independent choice for every floor)
b, $3 \cdot 2^{n-1}$ starting from the ground floor, going up, we can choose from 3 colors for the ground floor, and from 2 colors for every floor above it, since the color of the lower neighbor is forbidden.

12. **a)** How many ways can a lion, a pengiun, a tiger and a polar bear stand in a row?
b) What if we have one more lion?
c) What if we have yet one more lion?
d) We have 4 lions, 2 tigers and 3 polar bears. (We do not distinguish between animals of the same species.)

Solution: **a)** $4!$; **b)** $5!/2$: differentiating between the lions, it would be $5!$, in the end we divide by two; **c)** $6!/3!$ similarly, we counted each ordering $3!$ times; **d)** $\frac{9!}{4! \cdot 2! \cdot 3!}$ permutation with repetition.

13. On a 8×8 chessboard, how many ways can we place
a) one black and one white stones; **b)** two white stones;
c) one black, one white, and one green stone; **d)** three white stones;
e) three black and four white stones?

Solution: **a)** $64 \cdot 63$ (independent decisions); **b)** $64 \cdot 63/2$ (divide the solution of **a** by 2); **c)** $64 \cdot 63 \cdot 62$; **d)** $\frac{64 \cdot 63 \cdot 62}{3!}$; **e)** $\frac{64 \cdot \dots \cdot 58}{3! \cdot 4!}$.

14. How many 8-digit numbers are there? How many 8-digit number are there with the following property:
a) the consecutive digits are different.
b) it does not contain the digit 5.
c) it contains the digit 5.
d) there are two digits that are the same (there may be more).
e) there are two *consecutive* digits that are the same (there may be more).
f) there are exactly two digits that are the same.

g) there are exactly two *consecutive* digits that are the same.

Solution: There are $9 \cdot 10^7$ eight-digit numbers, the first digit cannot be zero.
a) 9^8; **b)** $8 \cdot 9^7$; **c)** all minus bad ones: $9 \cdot 10^7 - 8 \cdot 9^7$;
d) $9 \cdot 10^7 - 9 \cdot 9 \cdot 8 \cdot 7 \cdot \dots \cdot 3$;
e) $9 \cdot 10^7 - 9^8$.
f) We can also solve it by case analysis. A tricky proof: Select the place of the two digits that are the same. Then from left to right write the digits, when we reach the first selected place write the same digit to the second place as well. The solution is $\binom{8}{2} \cdot 9 \cdot 9 \cdot 8 \cdot 7 \cdot \ldots \cdot 4$.
g) First version: the number has all different digits, except for two, which are consecutive. Take all possible 7-digit numbers with all differetn digits, and double one of the digits: $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 7$
Second version: Non-consecutive digits are allowed to be the same. Take a 7-digit numbers where the consecutive digits are different (similar to subproblem **a)**) then double one digit. This gives $7 \cdot 9^7$ as a solution.

15. (2 points) Alice goes the florist, and would like to buy 7 flowers. The shop has roses, tulips and carnations. How many ways can she buy 7 flowers? (We do not distinguish between flowers of the same type.)

Solution: This is the same as the chocolate problem, or the equation problem. Place the 7 flowers and 2 separators (anywhere, we don't have to buy ≥ 1 from each) this gives us $\binom{9}{2} = \frac{9 \cdot 8}{2} = 36$ variations. With $n = 3, k = 7$ this is $\binom{n+k-1}{k} = \binom{9}{7} = \binom{9}{2} = 36$.

16. (2 points) George is in Manhattan, and he wants to walk from the corner of 8^{th} Avenue and 42^{nd} Street to the corner of 11^{th} Avenue and 57^{th} Street. He wants to walk one of the shortest possible paths. How many ways can he do it? (The streets and avenues form a grid.)

Solution: (Looking at the map, with approximate directions) This walk involves 3 corners (Avenues) going west, and 15 corners (Streets) going north. In total he takes 18 "steps" and 3 of them is going west. This gives $\binom{18}{3} = \frac{18 \cdot 17 \cdot 16}{3!} = 816$ possibilities.

17. (3 points) How many ways can n boys and n girls stand in a line, if two boys cannot stand next to each other, and two girls cannot stand next to each other?

Solution: $2 \cdot (n!)^2$, they must stand in a boy-girl-boy-girl alternating order. First we decide if a boy or a girl starts the row. Among the boys there are $n!$ possibilities, among the girls, also $n!$.

18. (5 points) 13 green, 15 gray, and 17 red chameleons live in Madagascar. They always meet in pairs. They are easily frightened, if two chameleons of different colors meet, they get so frightened that they both switch to the third color. Is it possible, that after some time, all of them acquire the same color?

Solution: Look at the number of chameleons of each color modulo 3. They represent 3 different residue classes. When two chamelaeons meet, two numbers decrease by one, one increases by 2, so modulo 3 they all change the same way. In other words, the difference between two color classes modulo 3 does not change. Therefore, we cannot reach 0, 0, 45 at any time.

19. For handing in.
a) (4 points) How many ways can we place 8 rooks on a 8×8 chessboard, such that no pair of them can capture each other? (A rook can capture an another rook if they are in the same row or in the same coloumn.)
b) (6 points) What if they are not allowed to capture each other and the arrangement of the pieces should be centrally symmetric to the center of the chessboard?

Solution: a, Every row and every coloumn has exactly one rook. Putting down the rooks row by row we have 8! possibilities.
b, By symmetry, the place of the rook in the first row determines the place of the rook on the last row. Then we can place the second rook in 6 places. The total number of possibilities is $8 \cdot 6 \cdot 4 \cdot 2$.

4 Exercise sheet 4

For problems 1-2, each subproblem is worth 1 point, except for 1/e.

20. In a German lottery, players are required to choose six main numbers between 1 and 49 plus an additional number, known as the Superzahl, between 0 and 9. To win the jackpot, a player must match all seven numbers, but prizes are available for matching as few as two main numbers plus the Superzahl.
a) What is the probability of getting all seven numbers right?
b) What is the probability of getting 6 numbers right, but not the Superzahl?
c) What is the probability of getting the 6 numbers right? (We don't care about the Superzahl)
d) What is the probability getting exactly 5 numbers of the main 6 right? (We don't care about the Superzahl)
e) (2 p) What is the probability getting at least 3 numbers of the main 6 right, and getting the Superzahl wrong?
Solution:

$$a, \qquad \frac{1}{\binom{49}{6}} \cdot \frac{1}{10} \quad \text{and} \quad \binom{49}{6} = 13983816, \text{ so the probability is } \frac{1}{139838160}$$

$$b, \ \frac{1}{\binom{49}{6}} \cdot \frac{9}{10} \qquad c, \ \frac{1}{\binom{49}{6}}$$

$$d, \qquad \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}}$$

$$e, \qquad \left(\frac{1}{\binom{49}{6}} + \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}} + \frac{\binom{6}{4}\binom{43}{2}}{\binom{49}{6}} + \frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}} \right) \cdot \frac{9}{10}$$

21. There are 10 red, 20 yellow and 40 green balls in a box. With closed eyes, at least how many balls should we pick, to surely have
a) one yellow ball?
b) three balls with different colors?
c) three balls of the same color?
d) 15 balls of the same color?
e) two green balls that were drawn right after each other?
Solution: **a)** $10 + 40 + 1 = 51$; **b)** $40 + 20 + 1 = 61$; **c)** $2 + 2 + 2 + 1 = 7$; **d)** $10 + 14 + 14 + 1 = 39$; **e)** $2 \cdot 10 + 2 \cdot 20 + 2 = 62$ (10 green-red, 20 green-yellow and two more).

22. (1 point) Show that

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$$

Solution:

$$\frac{n-k}{k+1}\binom{n}{k} = \frac{n-k}{k+1} \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(k+1)!(n-k-1)!} = \binom{n}{k+1}$$

23. (4 points) Prove the following equality

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

Solution: Let us choose n elements out of $\{1, 2, 3, \ldots 2n\}$. If we chose k elements from the fist half, then we have to choose $n-k$ elements of the second half.

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

24. (2 points) How many ways can we cover a $2 \times n$ "chessboard" with 1×2 dominoes?

Solution: If $n = 1$: 1 way, if $n = 2$: two ways, if $n = 3$: three ways. We will show that the number of possibilities is F_{n+1}. Use induction. Take a $2 \times n$ "chessboard". If the last domino is vertical, we should fill the remaining $2 \times n - 1$ place with dominoes. We know from the induction hypothesis that this can be done F_n ways. If the last domino is horizontal, actually there has to be two horizontal dominoes. The remaining $2 \times n - 2$ part can be filled F_{n-1} ways. In total, we can fill it $F_n + F_{n-1} = F_{n+1}$ ways.

25. (3 points) Show that the product of n consecutive positive integers is always divisible by $n!$.

Solution: $\frac{m(m-1)\cdots(m-n+1)}{n!} = \binom{m}{n}$, which is an integer.

26. (2 points) How many ways can we choose three different numbers from the set $\{1, 2, 3, \dots 100\}$ in a way that the sum of these three numbers is divisible by 3?

Solution: Either the three numbers have all different residues modulo 3, or all the same. In $\{1, 2, 3, \dots 100\}$ there are 33 numbers having residue 0, 34 numbers have residue 1, and 33 numbers have residue 2. The total number of possibilities is $\binom{33}{3} + \binom{34}{3} + \binom{33}{3} + 33 \cdot 34 \cdot 33$.

27. **For handing in.** (7 points)
Prove that for the Fibonacci numbers $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ for every $n \geq 0$.

Solution: Use induction. For $n = 0$, the statement is true, since $0 = F_0 = F_2 - 1 = 1 - 1$
Induction hypothesis: $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$
Then, for $n + 1$, $F_0 + F_1 + F_2 + \cdots + F_n + F_{n+1} = F_{n+1} + F_{n+2} - 1 = F_{n+3} - 1$. Done.

5 Exercise sheet 5

28. (3 points) The inclusion-exclusion principle states the following: For finite sets $A_1, A_2 \dots A_n$:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \dots (-1)^{n-1} |A_1 \cap A_1 \cap \dots A_n|$$

Prove this statement with induction. (So, suppose we already know that the statement is true for $n-1$ sets, and using this, prove for n sets.)

29. (3 points) Express the following sum in a closed form.

$$\binom{n}{0} + \binom{n}{1} 2 + \binom{n}{2} 4 + \cdots + \binom{n}{n} 2^n$$

Solution: Use the binomial theorem: $\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k \cdot 1^{n-k} = (2+1)^n = 3^n$

30. (2 points) Prove that F_n and F_{n-1} are relative primes. (F_n is the n^{th} Fibonacci number.)

Solution: Suppose F_n and F_{n-1} are not relative primes. This means there is a $p > 1$, $p | F_n$ and $p | F_{n-1}$. Since $F_{n-2} = F_n - F_{n-1}$, $p | F_{n-2}$. Therefore F_{n-1} and F_{n-2} are not relative primes either. After some steps, we reach that $F_1 = 1$ and $F_2 = 1$ are not relative primes. But they are. Contradiction.

31. (2 points) There are 350 farmers in a large region. 260 of them farm beetroot, 100 farm potatoes, 70 farm radish, 40 farm beetroot and radish, 40 farm potatoes and radish, and 30 farm beetroot and potatoes. All of them farm something out of these three vegetables.
Determine the number of farmers that farm beetroot, potatoes, and radish.

Solution: Let x be the number of people who farm all three. Use the inclusion exclusion procíple.

$$\left|\bigcup_{i=1}^{3} A_i\right| = \sum_{i=1}^{3} |A_i| - \sum_{1 \leq i < j \leq 3} |A_i \cap A_j| + |A_1 \cap A_2 \cap A_2|$$

$$350 = 260 + 100 + 70 - (40 + 40 + 30) + x$$
$$350 = 430 - 110 + x$$
$$350 = 320 + x$$
$$x = 30$$

Therefore, 30 farmers farm beetroot, potatoes, and radish.

32. (4 points) There is a necklace with n beads ($n \geq 2$) and one big assymetric jewel. (The jewel is needed so that every bead is identifiable, we can say "this is the third bead to the left from the jewel"). The beads are colored with k possible colors. Neighboring beads must have different colors. Every bead has 2 neighbors. (The big jewel does not count as a neighbor and it is not colored.)
How many different ways can we color the necklace?

Solution: The number of good colorings is $(k-1)^n + (-1)^n(k-1)$.
We will use the inclusion-exclusion principle. The total number of coloring (without any restriction) is k^n. Let the beads be $b_1, b_2, \ldots b_n$. Let A_i be the set of all coloring where b_i and b_{i-1} have the same color. $|A_i| = k^{n-1}$. (k possible colors for the pair, k^{n-2} for everything else.)
$|A_i \cap A_j| = k^{n-2}$. The colors of 2 beads are determined by their left neighbors, and we are free to choose the colors of the other beads. With this reasoning we can see that $|\bigcap_{i \in I} A_i| = k^{|I|}$ for $1 \leq |I| \leq n-1$.
$|\bigcap_{i=1}^{n} A_i| = k$ (and not 1). The colors of all beads are the same, so we have k possibilities.
The number of good colorings is $k^n - n \cdot k^{n-1} + \binom{n}{2} \cdot k^{n-2} - \binom{n}{2} \cdot k^{n-3} + \cdots + (-1)^{n-1}\binom{n}{n-1} \cdot k + (-1)^{n-1}\binom{n}{n} \cdot k$
This equals $k^n - n \cdot k^{n-1} + \binom{n}{2} \cdot k^{n-2} - \binom{n}{2} \cdot k^{n-3} + \cdots + (-1)^{n-1}\binom{n}{n-1} \cdot k + (-1)^{n-1}\binom{n}{n} \cdot 1 + (-1)^{n-1}\binom{n}{n} \cdot (k-1)$
Using the binomial theorem, the solution is $(k-1)^n + (-1)^n(k-1)$.

(We can also check our result for small examples. If $k = 1, n \geq 2$, then there is no good coloring, the answer is 0. If $k = 2$, and n is even, we have 2 good colorings, if n is odd, then 0.)

Second solution: We use induction. Let $P(n,k)$ denote the number of good colorings of a necklace with n beads and k possible colors. For $n = 2$, $P(2,k) = k(k-1)$, which satisfies the formula $(k-1)^n + (-1)^n(k-1) = (k-1)^2 + (-1)^2(k-1) = (k-1)^2 + (k-1) = k(k-1)$.
Take a chain with n beads. The neighboring beads must have different colors, but the two endpoints only have one neighbor, so the number of good colorings is $k(k-1)^{n-1}$. We can choose from k colors for the first bead, and from $k-1$ for all the others, the color of the previous bead is forbidden. If the two endpoint have different colors, this is a good coloring for the necklace as well. If the two endpoint have same color, merge them into one, and we get a good coloring for a necklace with $n-1$ beads.
Therefore, $k(k-1)^{n-1} = P(n,k) + P(n-1,k)$. Using the induction hypothesis for $n-1$,
$P(n,k) = k(k-1)^{n-1} - P(n-1,k) = k(k-1)^{n-1} - ((k-1)^{n-1} + (-1)^{n-1}(k-1)) =$
$k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1) = k(k-1)^{n-1} - (k-1)^{n-1} + (-1)^n(k-1) = (k-1)^n + (-1)^n(k-1)$.

33. (5 points) A convex polygon with n sides is cut into triangles by connecting vertices with non-crossing line segments (polygon triangulation). The number of triangles formed is $n-2$.
How many different ways can this be achieved? (Solutions that can be transformed to each other via rotation of reflection still count as different solutions.)

Solution: The number of different ways that this can be achieved is the Catalan number C_{n-2}. For example: triangle: 1 way, quadrilateral: 2 ways, pentagon: 5 ways.
We know that $C_0 = 1$ and $C_{n+1} = \sum_{i=0}^{n} C_i\, C_{n-i}$ for $n \geq 0$. ($C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14\ldots$).
Let T_n denote the number of triangulations of an n-gon, We can see that $T_n = C_{n-2}$ is correct for $n = 3, 4, 5$.
We will use induction. Induction step: Take a convex polygon with $n+2$ sides. The vertices are $v_0, v_1 \ldots v_{n+1}$.
The side $v_0 v_{n+1}$ should be included in a triangle, the third node of the traingle can be any one of $v_1 \ldots v_n$.
If it is v_i, removing this triangle, we need to find a triangulation of $v_0 v_1 \ldots v_i$ (a polygon with $i+1$ sides) and $v_i v_{i+1} \ldots v_{n+1}$ (a polygon with $n-i+2$ sides). In the degenerate cases when $i = 1$ or $i = n$, we should "triangulate a segment", here the number of solutions equals the T_{n+1}, so we can define T_2 as 1, which satisfies

$T_2 = C_0$. Using this recursion and the induction hypothesis,

$$T_{n+2} = \sum_{i=1}^{n} T_{i+1}\,T_{n-i+2} = \sum_{i=1}^{n} C_{i-1}\,C_{n-i} = \sum_{j=0}^{n-1} C_j\,C_{n-j-1} = C_n$$

34. **For handing in.** (8 points)
Let $x_1, x_2 \ldots x_{100}$ be integers. Prove that there exist integers i and j such that $1 \le i \le j \le 100$ and

$$\sum_{k=i}^{j} x_k \text{ is divisible by } 100.$$

Solution:
Let s_i be $s_i = \sum_{k=1}^{i} x_k$ for every $1 \le i \le 100$. If there is an i such that $100|s_i$, we are done, $100|\sum_{k=1}^{i} x_k$.
If none of $s_1 \ldots s_{100}$ is divisible by 100, only 99 residue classes are possible, so by the pigeonhole principle there is an i and j such that $s_i \equiv s_j \pmod{100}$.
$\sum_{k=1}^{i} x_k \equiv \sum_{k=1}^{j} x_k \pmod{100}$.
We can suppose without loss of generality that $j > i$.
$100|\sum_{k=1}^{j} x_k - \sum_{k=1}^{i} x_k$
$100|\sum_{k=i+1}^{j} x_k$. So we found a good sum.

6 Exercise sheet 6

35. (2 points) In a group of 8 people, some of them shake hands. Is it possible that everyone shaked hands with a different number of people?

Solution: Everyone had at least 0, at most 7 handshakes. It is not possible that someone shaked hands with everyone and someone else with no one. So, by the pigeonhole principle there has to be two people with the same number of handshakes.

36. (2 points) In a simple, connected graph on 6 vertices, the degrees of 5 vertices are 1, 2, 3, 4, 5 respectively. What may be the degree of the 6^{th} vertex?

Solution: Let us call the 5 vertices with known degree $v_1, v_2, \ldots v_5$, where $d(v_i) = i$. The degree of v_6 is unknown. Node v_5 is connected by an edge to every node, so the only neighbor of v_1 is v_5. Node v_4 is connected to very node except v_1. Therefore the two neighbors of v_2 are v_5 and v_4.
For v_6, we know that it is connected by an edge to v_5 and v_4 and not connected to v_1 and v_2. The same is true for v_3. Since the degree of v_3 is 3, v_3 is connected by an edge to v_6, therefore the degree of v_6 is 3.

Second solution: Since the sum of the degrees is even, the missing number has to be odd: 1, 3, or 5. Use the first half of the first solution. For v_6, we know that it is connected by an edge to v_5 and v_4 and not connected to v_1 and v_2, this rules out the degree being 5 or 1.

37. (2 points) Draw all simple graphs on 4, 5, or 6 vertixes that are isomorphic to their complement.
(The *complement* of a graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are connected by an edge if and only if they are not connected by an edge in G.
Two graphs are *isomorphic* if there exists a one-to-one correpondence between the nodes of the first graph and the nodes of the second graph so that two nodes in the first graph that are connected by an edge correspond to nodes in the second graph that are connected by an edge, and vice versa.)
Solution:
$n = 4$ A path with 4 nodes is a good solution.
$n = 5$ A cycle of length 5 is good, and there is another solution: a triangle plus 2 edges: $v_1v_2, v_2v_3, v_3v_1, v_1v_3, v_2v_5$
$n = 6$ It is not possible. K_6 has $\binom{6}{2} = 15$ edges, half of the edges should be in G, half of them in the complement. This cannot be done with an odd umber of edges.

38. (1 point each) Is there a simple graph where the degrees of the vertices are
a) 3, 3, 3, 2, 2, 2, 1, 1, 1; **b)** 6, 6, 5, 4, 4, 3, 2, 2, 1;
c) 7, 7, 7, 6, 6, 6, 5, 5, 5; **d)** 1, 3, 3, 4, 5, 6, 6?

Solution: **a** is possible.
b is impossible beacause the sum of the degrees is odd.
c is possible, if we have a solution for **a**, the complement of that graph is a solution for **c**.
d is impossible. The graph is simple and has 7 nodes. Therefore two nodes are connected to every other node, there cannot be a node with degree 1.

39. (2 points) In a simple graph, vertex v has an odd degree. Prove there is a path from v to another vertex with odd degree.

Solution: Let v be an odd degree node, and let G' be the connected component containing v. The sum of the degrees in G' is even, so there has to be another node w with odd degree. Since they are in the same connected component, there is a path from v to w.

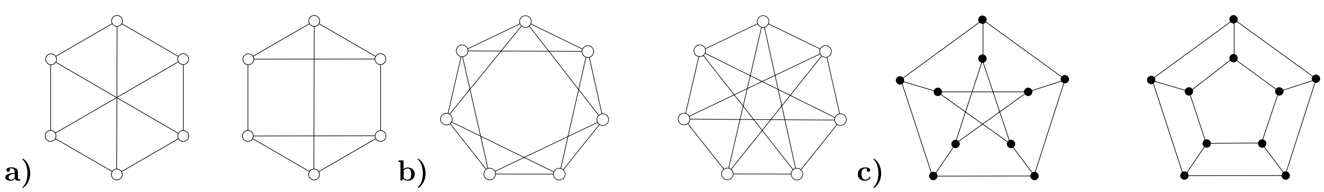
40. (3 points) Characterize the graphs with the following property: any two edges have a common endpoint.

Note: In graph theory, a star S_k is the complete bipartite graph $K_{1,k}$: a tree with one internal node and k leaves.

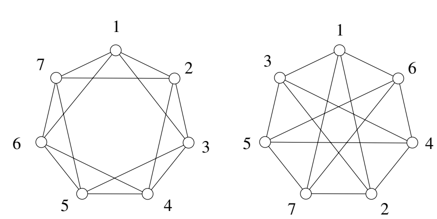
Solution: First, consider the case where the graph is simple. If the graph has at most 2 edges, the answer is a path of length 1 or 2. If the graph has at least 3 edges: Let $e = \{u, v\}$ and $f = \{u, z\}$ be two edges of the graph (their common node is u). A third edge can either be edge $\{v, z\}$, or contain u and another vertex. If the three edges form a triangle we cannot add any more edges, if they form a star, we can add more edges, all of the new edges should contain u, so the result is still a star.
Therefore, the answer is, this graph is either a star or a cycle of length 3.

If the graph is not simple, we can add parallel edges to any edge and if the graph was a star, we can add loops to its internal node.

41. (3 points) Which pairs of graphs are isomorphic?



Solution: **a)** Not isomorphic, the right-hand side graph contains a triangle, but the left-hand side graph does not.
b) Yes, they are isomorphic, see the picture.
c) Not isomorphic. The left is the Petersen graph, and it does not contain a cycle of length 4, while the right graph contains a cycle of length 4.



42. (2 points) Draw the tree with the Prüfer code 4 3 0 1 1 3.

Solution: We recreate the two-line "long" Prüfer code

2	4	5	6	7	1	3
4	3	0	1	1	3	0

The columns are the edges of the tree.

43. (5 points) At most how many intersections do the diagonals of a convex n-sided polygon have?

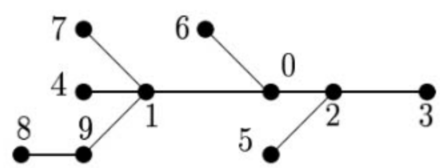
Solution: Any two intersecting diagonals cover 4 nodes, and 4 nodes gives exacly one pair of intersenting diagonals. (Other intersection points fall outside of the convex polygon). Therefore there is a one-to-one correspondence between sets of 4 nodes and intersection points. The number of intersections is $\binom{n}{4}$.

44. For handing in. How many trees are there on n labeled vertices such that
a) (3 points) the degree of each node is at most 2.
b) (5 points) the node with label 1 has degree 1.

Solution: **a)** A tree where each degree is at most 2 is a path. There are $n!$ possibilities to list n vertices in a row, but this way we counted each path twice (from left to right, from right to left) thus the number of labeled paths is $n!/2$.
b) removing the node with label 1, we get a labeled tree on $n-1$ nodes. Using the Cayley formula, there are $(n-1)^{n-3}$ such trees. We can reconnect node with label 1 to any of the other nodes, so in total there are $(n-1)(n-1)^{n-3} = (n-1)^{n-2}$ possibilities.

7 Exercise sheet 7

45. (2 points) Give the Prüfer code of the following tree:



Solution: The two-line Prüfer code is

3	4	5	2	6	7	8	9	1
2	1	2	0	0	1	9	1	0

Therefore the "short" Prüfer code is $2, 1, 2, 0, 0, 1, 9, 1$.

46. (2 points) Select the value of x such that $1, 1, 5, x, 6, 6$ is a Prüfer code of a tree, in which every degree is odd. Give the tree as well.

Solution: Only $x = 5$ can be good, otherwise 5 appears in the code only once, thus the degree of node with label 5 is 2. Choosing $x = 5$ is actually good, we get a tree with three wides with degree 3 and all the other nodes have degree 1. (I omit the picture now.)

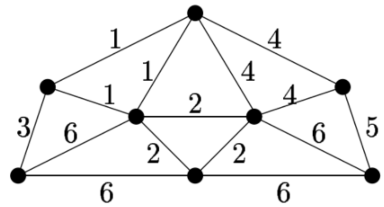
47. (2 points) Show that if a tree has a k-degree node, then it has at least k leaves. Is the reverse statement true?

Solution: From every edge of the node with degree k, start a path. All of this paths must end in a leaf, and since the graph does not contain a cycle, the path cannot merge, all these endpoints are different.

48. (2 points) How many trees are there on n labelled nodes, that have at least 3 leaves?

Solution: A tree has at least 3 leaves if and only if it is not a path. Using Cayley's formula and an earlier problem, the answer is $n^{n-2} - n!/2$, if $n \geq 2$. (If $n = 1$, there are no such trees.)

49. (2 points) Find a minimum cost spanning tree of this graph. How many minimum cost spanning trees are there?



Solution: The costs of the edges in the spanning tree are 1, 1, 2, 2, 3, 4, 5, the total cost is 18. From the three edges with cost 1, we can choose any two. Same for the edges with cost 2. The edge with cost 3 is unique. From the edges with cost 4, to avoid cycles, we can choose one of two possibilities. The edge with cost 5 is unique again, and we do not choose any of the cost 6 edges. Therefore the number of possibilies is $\binom{3}{2}\binom{3}{2}\binom{2}{1} = 18$.

50. (2 points) G is a simple graph, its vertices are labelled with $1, 2, \ldots, 100$. Nodes i and j are connected by an edge in G if and only if $|i - j| \le 2$. Does G contain an Eulerian circuit or an Eulerian walk?

Solution: G is connected. Nodes 2 and 99 have degree 3. Every other node has even degree (two or four). Therefore there is an Eulerian walk in the graph. (It starts in 2 and ends is 99 or vice versa.) There is no Eulerian circuit in the graph.

51. (2 points) Is there a graph on 10 nodes that contains an Eulerian circuit and the sum of the degrees is 34?

Solution: Yes, there is. For example, if the degree sequence is 2 2 2 4 4 4 4 4 4 4.

52. (1+1 point) **a)** Find a graph, where every degree is even, and it does not contain an Eulerian circuit. **b)** Find a graph that is not conneted, and contains an Eulerian circuit.

Solution: This problem is about the importance of connectivity and isolated nodes. **a)** Two disjoint cycles. **b)** Take a connected graph with an Eulerian circuit and add some isolated nodes. (Example: A cycle and one isolated node.)

53. (4 points) In a group everyone knows 4 other people. (We assume that acquaintance is mutual.) Show that they can sit down around some round tables in a way that everyone knows his/her two neighbors.

Solution: The problem is the following: in a 4-regular graph G, find a spanning subgraph H, sucht that every component of H is a cycle. Suppose that G is connected. Then it has an Eulerian circuit. $2|E| = \sum_{v \in V} d(v) = 4|V|$, thus $|E| = 2|V|$. Following the Eulerian circuit, color the edges of the graph red and blue in an alternating way. Since the graph has an even number of edges, the alternating coloring is kept even for the starting node. Let H be the subgraph formed by the red edges. Every degree in H is two, so it is an union of disjoint cycles.
If G is not connected, we do the previous method for each connected component.

54. (5 points) A government wants to connect cities with roads, (i. e. they want to build a spanning tree). Optimists and pessimists win in unpredictable order. This means that sometimes they build the cheapest line that does not create a cycle with those lines already constructed; sometimes they mark the most expensive lines "impossible" until they get to a line that cannot be marked impossible without disconnecting the network, and then they build it. Prove that they still end up with an optimal cost spanning tree.

Solution: See in the Lecture notes.

55. For handing in. (6 points) Tree T has 17 nodes and the degree of each node is either 1 or 4. After Alice added some edges to this graph, it has an Eulerian circuit. At least how many edges did she add?

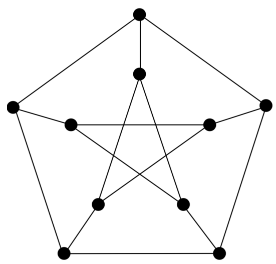
Solution: 6 edges. Let k be the number of nodes with degree 4. The tree has 16 edges, so the sum of the degrees is $\sum_{v \in V} d(v) = 4k + (17 - k) = 32$. We get that $k = 5$. The tree has 12 nodes with odd degree. By adding 6 edges, Alice can achieve that every degree of the graph is even, thus it contains an Eulerian circuit.

8 Exercise sheet 8

56. (2 points) Is the complement of the cycle of length 6 (C_6) a planar graph?

Solution: Yes, it is planar.

57. (3 points) Show that the Petersen graph is not planar.



Solution: Suppose that it has a planar drawing. The Petersen graph does not contain a cycle of lenght 3 or 4, so every country has at least 5 sides. From this, $e \geq \frac{5f}{2}$. We know that $v = 10$ and $e + 15$. From the Euler formula, $f + v = e + 2$, $f + 10 = 15 + 2$, so $f + 7$. This contradicts $e \geq \frac{5f}{2}$, therefore the graph cannot be planar.
Second solution: We can find a subdivision of $K_{3,3}$ as a subgraph of the Petersen graph.

58. (2 points each) **a)** Show that the edges of the Petersen graph cannot be colored with 3 colors.
b) Show that the Petersen graph does not have a Hamiltonian cycle, but deleting any vertex, the remaining graph has a Hamiltonian cycle.

Solution: **a)** Color the edges of the outer cycle with 3 colors: red, blue, red, blue, green. This defines the colors of the 5 edges that connect the outer cycle and the inner cycle, 3 of them is colored green. We can see that we cannot use the color green in the inner cycle, so we should color a cycle of lenght 5 with 2 colors. That is not possible.
b) Suppose the graph contains a Hamiltonian cycle, this is a cycle of lenght 10. Color the edges of the cycle red and blue in an alternating way. Since the Petersen graph is 3-regular, if we remove the Hamiltonian cycle from the graph, the remaining part is a perfect matching, color the edges of this matching green. This way we got a 3-coloring of the edges. From part **a)** we know that this is not possible.

59. (4 points) A group of musicians are traveling. Everyone has 3 enemies in the group. Show that they can be divided to sit on two buses in a way that everyone has at most one enemy who is traveling on the same bus as him.

Solution: As a graph theory problem: we have a finite, 3-regular graph. We want to find a partition of the vertices, $V = A \cup B, A \cap B = \emptyset$, such that every $v \in A$ has at most one neighbor in A, and every $v \in B$ has at most one neighbor in B. There are finitely many partitions, choose one where the number of edges between A and B is maximal.
Suppose for contradiction that (w.l.o.g.) there is an $x \in A$ such that x has at least two neighbors in A, move this node to B. $A := A \setminus \{x\}$, $B := B \cup \{x\}$. With this step, the number of edges between A and B increases. But it was already maximal. Contradiction.

60. (4 points) A *regular polyhedron* is a (3 dimensional) polyhedron whose faces are identical regular polygons. All side lengths are equal, and all angles are equal. In every vertex the same number of faces meet. Using Euler's Formula, show that only five convex regular polyhedra exist.

Solution: For a regular polytope, the resulting topological planar graph has the same degree, d, of each vertex (where $d \geq 3$), and each face has the same number, $k \geq 3$, of vertices on its boundary.
Let us denote the number of vertices of the considered graph $G = (V, E)$ by n, the number of edges by e, and the number of faces by f. First we use the equation $\sum_{v \in V} d(v) = 2|E|$ which in our case specializes to $dn = 2e$. Similarly, $kf = 2e$.

Using Euler's formula

$$e+2=n+f=\frac{2e}{d}+\frac{2e}{k}$$

Divide by $2e$.

$$\frac{1}{2}+\frac{1}{e}=\frac{1}{d}+\frac{1}{k}$$

Hence if both d and k are known, the other parameters n, e, and f are already determined uniquely. Min $(d,k)=3$, for otherwise $\frac{1}{d}+\frac{1}{k}\leq\frac{1}{2}$.
For $d=3$, if $k\geq 6$, then $\frac{1}{d}+\frac{1}{k}\leq\frac{1}{2}$. Therefore we get $k\in\{3,4,5\}$. Hence one of the following possibilities must occur:

d	k	n	e	f
3	3	4	6	4
3	4	8	12	6
3	5	20	30	12
4	3	6	12	8
5	3	12	30	20

61. (2 points) Is there a bipartite graph with degrees 3,3,3,3,3,3,3,3,3,3,5,6,6? (These can be distributed in the two classes of nodes arbitrarily.)

Solution: If the two classes are A and B, $\sum_{v\in A}d(v)=\sum_{v\in B}d(v)$. No matter how we allocate the degrees, on one side, the sum of the degrees is divisible by 3, on the other side, it is not.

62. (2 points) An island is inhabited by six tribes. They are on good terms and split up the island between them, so that each tribe has a hunting territory of 100 square miles. The whole island has an area of 600 miles. The tribes decide that they all should choose new totems. They decide that each tribe should pick one of the six species of tortoise that live on the island. They want to pick different totems, and totem for each tribe should occur somewhere on their territory. The areas where the different species of tortoises live don't overlap, and they have they same area - 100 square miles. (Of course, the way the tortoises divide up the islands may be entirely different from the way way the tribes do.) Show that such a selection of totems is always possible.

Solution: Create a bipartite graph $G=(A,B,E)$ with the six tribes on one side, the six tortoises of the other side. Tribe a and tortoise b are connected by an edge, if their territories overlap. Take a subset of the tribes, $X\subseteq A$. The total territory of them is $|X|\cdot 100$ square miles. This cannot be covered with less than $|X|$ tortoise territories, therefore $|\Gamma(X)|\geq|X|$. Using Hall's theorem, the graph has a perfect matching.

63. **For handing in.** (10 points) Determine the number of pieces into which a circle is divided if n points on its circumference are joined by all possible chords. The chords are in a general position, no three of them goes thought the same point.

Solution: For small numbers, the number of regions is 1, 2, 4, 8, 16, 31, 57, 99, 163, 256...
Statement: For $n\geq 4$, the number of regions is $\binom{n}{4}+\binom{n}{2}+1$.
Taking all n vertices on the cycle and all the intersections inside the cycle, we get a planar graph.
The number of nodes is $|V|=n+\binom{n}{4}$. We use problem 9 from exercise sheet 6, the number of intersections of the diagonals of a convex n-sided polygon is $\binom{n}{4}$.
Every inner node has degree 4. Every outer node has degree $n+1$.
$2|E|=\sum d(v)=4\binom{n}{4}+n(n+1)$
$|E|=2\binom{n}{4}+\frac{n(n+1)}{2}=2\binom{n}{4}+\binom{n}{2}+n$
Using the Euler formula, the number of faces is $|E|+2-|V|$. Excluding the infinite face, the number of regions is $|E|+1-|V|=2\binom{n}{4}+\binom{n}{2}+n+1-(n+\binom{n}{4})=\binom{n}{4}+\binom{n}{2}+1$.

See: http://mathworld.wolfram.com/CircleDivisionbyChords.html

9 Exercise sheet 9

64. (3 points) Show that if graph G has at least 11 nodes, then it is not possible that both G and the complement of G are planar.

Solution: The number of nodes in G is $n \geq 11$.The union of G and $\overline{G}$ is K_n, so they have $n(n-1)/2$ edges together. Suppose (for contradiction) that both G and $\overline{G}$ are planar. Therefore they have at most $3n-6$ edges, thus $n(n-1)/2 \leq 6n-12$. After rearranging, $n^2 - 13n + 24 \leq 0$. This function on n is an upwards parabola, so n cannot be greater than the bigger root of the quadrastic equation $n \leq (13+\sqrt{169-96})/2 < 11$, contradiction.

65. (3 points) Find a graph G on 8 nodes such that neither G nor the complement of G is planar.

Solution: Let G be a $K_{3,3}$ and two isolated vertices. Then the complement of G contains K_5 as a subgraph.

66. a) (2 points) Let $(P, \mathcal{L})$ be a projective plane with order n, and let $A \subseteq P$ be a set of points such that any three points of A are not collinear. Show that $|A| \leq n+2$.
b) (4 points) If n is odd, show that $|A| \leq n+1$.

Solution: **a)** Take a $p \in A$, there are $n+1$ lines through p and by the pigeonhole principle, we can select only one other point of each of them. Together with p, the number of selected points is $\leq n+2$.
b) Suppose that $A \subseteq P$ is a set of points such that any three points of A are not collinear and $|A| = n+2$. Take a $p \in A$, and denote the $n+1$ lines through p as $L_1, L_2 \ldots L_{n+1}$ From the previous part, we know that Now look only at lines $L_1, L_2 \ldots L_n$. A contains exactly one "non-p" points of each of these lines: denote them as $q_1 \ldots\ldots q_n$. ($q_i \in L_i$, $q_i \in A, q_i \neq p$.) Points q_i and q_j $(1 \leq i < j \leq n)$ define a line, and this line has an intersection with L_{n+1}. Call this point r_{ij}. This point cannot be in A because that would mean 3 selected points on one line. We call the points we can get this way *forbidden* points. For any $1 \leq i, j, k \leq n$ $r_{ij} \neq r_{ik}$. (If $q_i q_j$ and $q_i q_k$ defined the same intersection point, q_i q_j and q_k would lie on one line.)
It is possible that $r_{ij} = r_{lk}$ if i, j, l, k are four different numbers. Thus one forbidden points can belong to $\lfloor \frac{n}{2} \rfloor$ point pairs. The number of forbidden points on line L_{n+1} is at least $\frac{\binom{n}{2}}{\lfloor \frac{n}{2} \rfloor}$.
If n is odd,

$$\frac{\binom{n}{2}}{\lfloor \frac{n}{2} \rfloor} = \frac{\binom{n}{2}}{\frac{n-1}{2}} = n$$

On line L_{n+1}, p is selected and every other point is forbidden, so we cannot find a fitting point on the last line.

67. (3 points) Let P be a finite set and let $\mathcal{L}$ be a system of subsets of P satisfying conditions
(i), Any two distinct sets $L_1, L_2 \in \mathcal{L}$ intersect in exactly one element, i.e. $|L_1 \cap L_2| = 1$.
(ii) For any two distinct elements $p_1, p_2 \in P$,there exists exactly one set $L \in \mathcal{L}$ such that $p_1 \in L$ and $p_2 \in L$.
(iii'): There exist at least two distinct lines $L_1, L_2 \in \mathcal{L}$ having at least 3 points each.
Prove that any such $(P, \mathcal{L})$ is a finite projective plane.

Solution: Parts (i) and (ii) are exactly waht we had if the definition of a projective plane. We need to show that this system contains 4 points in general position. We know there exist at least two distinct lines $L_1, L_2 \in \mathcal{L}$ having at least 3 points each. This 2 lines have and intersection point, call it p_0. There are at least 2 other points on line L_1, denote them as p_1, p_2, and there are at least 2 other points on line L_2, denote them as p_3, p_4. We claim that p_1, p_2, p_3, p_4 are in a general position. Suppose there is a line L' that contains 3 of these 4 points. Then L' contains either p_1, p_2 or p_3, p_4. Two points define only one line, so L' is either L_1 or L_2. Without loss of generality, we can say that $L' = L_1$ and it contains p_1, p_2, p_3. But then p_3 is on both L_1 and L_2. There is only one intersection point, so $p_3 = p_0$. Contradiction, because p_3 and p_0 are different points.

Definition
In a graph $G = (V, E)$, a *stable set* is a subset C of V such that no pair of vertices in C is connected with an edge. An *edge cover* is a subset F of E such that for each vertex v there exists $e \in F$ where v is an endpoint of e. Note that an edge cover can exist only if G has no isolated vertices.

$\alpha(G) := \max\{|C| : C \text{ is a stable set}\}$,
$\tau(G) := \min\{|W| : W \text{ is a vertex cover}\}$,
$\nu(G) := \max\{|M| : M \text{ is a matching}\}$,
$\rho(G) := \min\{|F| : F \text{ is an edge cover}\}$.

68. (3 points) Prove that if $G = (V, E)$ is a graph without isolated vertices, then

$$\alpha(G) + \tau(G) = |V| = \nu(G) + \rho(G)$$

Note: This is Gallai's theorem.

Solution: U is a stable set if and only if $V \setminus U$ is a vertex cover. The first equality follows directly from this statement.
To see the second equality, first let M be a matching of size $\nu(G)$. For each of the $|V| - 2|M|$ vertices v not covered by M, add to M an edge covering v. We obtain an edge cover F of size $|M|+(|V|-2|M|) = |V|-|M|$. Hence $\rho(G) \leq |F| = |V| - |M| = |V| - \mu(G)$.
Second, let F be an edge cover of size $\rho(G)$. Choose from each component of the graph (V, F) one edge, to obtain a matching M. As (V, F) has at least $|V| - |F|$ components, we have $\mu(G) \geq |M| \geq |V| - |F| = |V| - \rho(G)$.
(Any graph (V, E) has at least $|V| - |E|$ components, this can be shown by induction on $|E|$: adding any edge reduces the number of components by at most one.)

69. (4 points) Is it possible to arrange 8 bus routes in a city so that
(i) if any single route is removed (doesn't operate, say) then any stop can still be reached from any other stop, with at most one change, and
(ii) if any two routes are removed, then the network becomes disconnected?

Solution: Yes. Draw 8 lines in the plane in general position (no 2 parallel, no 3 intersecting at a common point). Let the intersections represent stops and the lines bus routes.

70. **For handing in.** (7 points) Prove that the Fano plane is the only projective plane of order 2 (i.e. any projective plane of order 2 is isomorphic to it. Define an isomorphism of set systems first).

Solution: Isomorphism: Let $(P_1, \mathcal{L}_1)$ and $(P_2, \mathcal{L}_2)$ be two projective planes, such that $|P_1| = |P_2|$, $|\mathcal{L}_1| = |\mathcal{L}_2|$ there exist a bijection f between P_1 and P_2 and a bijection g between $\mathcal{L}_1$ and $\mathcal{L}_2$ such that for every $p_i \in P_1$ and $L_i \in \mathcal{L}_1$, $p_i \in L_i$ if and only if $f(p_i) \in g(L_i)$.

10 Exercise sheet 10

Reminder: In a block design: The town has v inhabitants; they organize b clubs; every club has the same number of members, say k; everybody belongs to exactly r clubs, and for any pair of citizens, there are exactly λ clubs where both of them are members.
We know that $bk = vr$, $\lambda(v - 1) = r(k - 1)$, and $b \geq v$.

71. (3 points) Prove that if we color the points of the Fano plane with 2 colors, there will be line where all three points have the same color.

Solution: Color the point with red and blue. We call a 2-coloring "good" if every line contains both red and blue points. Suppose for contradiction that there exists a *good* coloring on the Fano plane. The red points have to cover all the lines, (and same for the blue points). If there is only 1 red point, it covers three lines, if there are 2 red points, they cover 5 lines (every point is on 3 lines and these 2 points have one line in common). By symmetry, coloring with 5 red and 2 blue, or 6 red and 1 blue lines cannot be *good* either.
Suppose there are 3 red and 4 blue points. If the 3 red point are on one line, the coloring is not *good*, since there is an all red line. So the 3 red points are not collinear. They cover the three lines passing though any pair of the 3 red points, and 3 more lines since every point has degree 3. In total, the red points covered 6 lines, so there has to be an all blue line. Therefore the Fano plane cannot have any *good* coloring.

72. (2 points) In a town, there are 924 clubs, and every club has 21 members. Every 2 people can meet each other in exactly 2 clubs. How many inhabitants are in this town? One person is a member of how many clubs?

Solution: $bk = vr$, so $924 \cdot 21 = vr$.
$\lambda(v-1) = r(k-1)$, so $2(v-1) = 20r$. From this $v = 10r + 1$ Putting this into the first equation $924 \cdot 21 = 19404 = (10r + 1)r = 10r^2 + r$. Solving this quadratic equation we get that ther are 441 people, and everyone is a member of 44 clubs.

73. (4 points) In a town, the clubs form a block design and every club has a badge. On a big event, everyone from the town is present, and everyone wears a badge of a club he/she is a member of. (Each person wears only one badge.) Is it always possible that everyone wears different badges?

Solution: First, we need that there are enough different badges, at least as many as citizens. That is, $b \geq v$. This is indeed guaranteed by Fisher's Inequality.
We assign a bipartite graph to our block design. represents the people (this side has v points); the upper set of points represents the clubs (this side has b points). We connect point a to point X if citizen a is a member in club X. Choose a subset A of citizens, $|A| = n$, the set of clubs that someone from A is a member of is $\Gamma(A)$. $|\Gamma(A)| = m$. We want to use Hall's theorem, we claim that $m \geq n$.
Every citizen node has degree r, every club node has k. All the edges from A are also edges from $\Gamma(A)$, therefore $nr \leq mk$.
We know from a lemma that $bk = vr$, and $b \geq v$, therefore $k \leq r$. So $mk \leq mr$.
Therefore $nr \leq mk \leq mr$, $nr \leq mr$ thus $n \leq m$. From Hall's theorem, everyone can wear a different badge.

74. (2 points) Show that in a block design with $k = 3$ and $\lambda = 1$, the residue of v divided by 6 is 1 or 3.

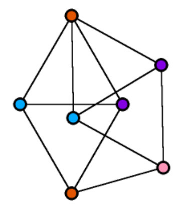
Solution: We learned that $bk = vr$ and $\lambda(v-1) = r(k-1)$. For this special case, $3b = vr$ and $v - 1 = 2r$. And hence $r = \frac{v-1}{2}$ and $b = \frac{v(v-1)}{6}$ The numbers r and b must be integers, v is an odd number, so if we divide it by 6, the remainders can be 1, 3, or 5. Furthermore, v can not be of the form $6j + 5$, because then $b = \frac{(6j+5)(6j+4)}{6} = 6j^2 + 9j + 3 + \frac{1}{3}$ which is not an integer.
Block desings like these are called *Steiner systems*

75. (3 points) Can you create a block design with the following parameters? $v = 13, k = 3, \lambda = 1$.

Solution: Yes, it is possible.
If we can partition the egdes of a complete graph on 13 nodes to disjoint triangles, we get the desired block design.
Take 13 points on a cycle, that form a regular 13-gon. Number the nodes from 0 to 13. One of the clubs is the triangle $\{0, 1, 4\}$, another club is triangle $\{0, 2, 7\}$. Select all the triangles we get by rotating $\{0, 1, 4\}$ and $\{0, 2, 7\}$ around the center of the cycle. These are also clubs. We claim we got a partition of K_{13} into triangles.
We say two points have distance k if they are k steps away from each other on the cycle. Here, distance 6 is the same as distance 7. Note that the sides of triangle $\{0, 1, 4\}$ have distance 1, 3, and 4, and the sides of triangle $\{0, 2, 7\}$ have distance 2, 5, and 6. With the rotation method, every edge is incuded in exactly one traingle.

76. (3 points) We color the points of the $\mathbb{R}^2$ plane with 3 colors. Show that there are two points such that their distance is 1, and they have the same color.

Solution: Suppose we can color the plane with 3 colors such that two points of distance 1 always have different color. If we build two equilateral unit triangles together, the opposite points (two points that have distance $\sqrt{3}$) should have the same color. The following picture shows, that there two points that should have the same color by the previous logic, but their distance is 1. Contradiction, we cannot color the plane with 3 colors.



77. For handing in. (8 points) In a group, everyone has 3 friends. (We assume that friendship is mutual.) If A and B are not friends, there is exactly one person in the group that they are both friends with. If A and B are friends, then they do not have a common friend in the group. Is this situation possible? If it is possible, how many people are in the group?

Solution: Represent it with a graph, the nodes are the people, two nodes are connected by and edge if the two endpoints are friends. The situation in the problem is possible, an example is the Petersen graph. Let n be the number of nodes, and e the number of edges. We know $n = 10$ is possible, and we want to show that this is the only possible size.
Let us count the number of "cherries" in the graph. If A and B are not friends, there is exactly one person in the group that they are both friends with, if they are friends, zero. Thus the number of cherries is $\binom{n}{2} - e$ (counted by the legs of the cherry). On the other hand, the graph is 3-regular, so the number of cherries is $3n$ (counted by the head of the cherry).
$\binom{n}{2} - e = 3n$
Since the graph is 3-regular, $e = 3n/2$.
$\binom{n}{2} - \frac{3n}{2} = 3n$
$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{9n}{2}$
$n - 1 = 9$
$n = 10$. There are 10 people in the group.

Second solution: From the conditions, there are no C_3 or C_4 in the graph. Person A has three friends B, C, D. They cannot be friends of each other, and cannot have a common friend who is not A, so they each have 2 new friends: E, F, G, H, I, J. This gives $1 + 3 + 6$ people, so there has to be at least 10 people in the group. We can connect E, F, G, H, I, J to each other in a way that satisfies all the conditions, so we get a good construction (which is isomorphic to the Petersen graph). Suppose there are more than 10 people. Then the 11^{th} person cannot be friends with A, B, C or D (they are "full", already have 3 friends). The 11^{th} person and A are not friends and do not have a common friend. Contradiction.

11 Exercise sheet 11

78. (3 points) For natural numbers $m \le n$, we define a Latin $m \times n$ rectangle as a rectangular table with m rows and n columns with entries chosen from the set $\{1, 2, ..., n\}$ and such that no row or column contains the same number twice. Count the number of all possible Latin $2 \times n$ rectangles.

Solution: $n! \times$ (the number of permutations with no fixed point).
We learned earlier that the number of permutations with no fixed point is $n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$.
So, in total the number of all possible Latin $2 \times n$ rectangles is $n! \cdot n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$.

79. Define a *liberated square* of order n as an $n \times n$ table with entries belonging to the set $\{1, 2, ..., n\}$. Orthogonality of liberated squares is defined in the same way as for Latin squares. For a given number t, consider the following two conditions:
(i) There exist t mutually orthogonal Latin squares of order n.
(ii) There exist $t + 2$ mutually orthogonal liberated squares of order n.
(a) (2 points) Prove that (i) implies (ii).
(b) (4 points) Prove that (ii) implies (i).

Solution: **(a)** To given t orthogonal Latin squares, add one square with all the entries of the i^{th} row equal to i, $i = 1, 2, ..., n$, and one square having all the entries j in the j^{th} column, $j = 1, 2, ..., n$.
(b) In order that a liberated square be orthogonal to another, it has to contain each $i \in \{1, 2, ..., n\}$ exactly n times. Permute entries of the given $t+2$ orthogonal liberated squares (the same permutation in each square) in such a way that the first square has all numbers i in the i^{th} row, $i = 1, ..., n$. Then permute entries inside each row (again, in the same way for all squares) so that the second square has all the j in the j^{th} column. Check that the remaining t squares are Latin.

80. Let X be a finite set and let $\mathcal{M}$ be a system of subsets of X. Suppose that each set in $\mathcal{M}$ has exactly k elements. A 2-coloring a set-system means we color the elements with 2 colors in a way that none of the sets in $\mathcal{M}$ is monochromatic. Let $m(k)$ be the smallest number of sets in a system $\mathcal{M}$ that is not 2-colorable.

(3 points) Prove that $m(4) \geq 15$, i.e. that any system of 14 4-tuples can be 2-colored (distinguishing two cases according to the total number of points.)

Solution: Similar to the proof where we showed that $m(3) \geq 7$.
Case 1: $|X| \leq 14$. If needed, add some nodes, now we have exactly 14 nodes. Color 7 of them white, 7 of them red. There are $\binom{14}{7} = 3432$ such colorings. For a given quadruple, there are $2\binom{10}{3}$ colorings that makes them monochoromatic. (Color this 4 points white, and from the remaining 10, 3 points are white. Same for red.)
For every quadruple, the probability that it is monochormatic is $\frac{2\binom{10}{3}}{\binom{14}{7}}$ The probability that at least one of the 14 quadruples is monochromatic is at most $14 \cdot \frac{2\binom{10}{3}}{\binom{14}{7}} = \frac{14 \cdot 120 \cdot 2}{3432} = \frac{3360}{3432} < 1$. We use the probabilistic method, there has to be a 2-coloring is the set system.
Case 2: $|X| > 14$.
We say that x and y are connected if there exists a set $M \in \mathcal{M}$ containing both x and y. If x and y are points that are not connected, we define a new set system $(X', \mathcal{M}')$ arising by "gluing" x and y together. The points x and y are replaced by a single point z, and we put z into all sets that previously contained either x or y. If a "glued" set system is 2-colorable, then the original is also 2-colorable.
We claim there are 2 points that are not connected. Every quadruple makes 6 point-pairs connected. There are at most $14 \cdot 6$ connected pairs, and the total number of pairs is at least $\binom{15}{2}$. Since $14 \cdot 6 < \binom{15}{2}$, so there are 2 points that are not connected. Do the gluing steps until we reach $|X| = 14$.

Note: this solution also works if the two cases are $|X| \leq 13$ and $|X| > 13$.

81. (3 points) We have 27 fair coins and one counterfeit coin, which looks like a fair coin but is a bit heavier. Show that one needs at least 4 weighings to determine the counterfeit coin. We have no calibrated weights, and in one weighing we can only find out which of two groups of some k coins each is heavier, assuming that if both groups consist of fair coins only the result is an equilibrium.

Solution: Each weighing has 3 possible outcomes, and hence 3 weighings can only distinguish one among 27 possibilities.

82. (5 points) We toss a fair coin n times. What is the expected number of *runs*? Runs are consecutive tosses with the same result. For instance, the toss sequence HHHTTHTH has 5 runs. (HHH, TT, H, T, H). (Tip: It is better to count boundaries between runs.)

Solution: It's better to count boundaries among runs. The probability that a given position between two tosses is a boundary is $\frac{1}{2}$. The first toss can be anything. For any of the next $n-1$ tosses, there is a $\frac{1}{2}$ change we start a next run.
Let X be the random variable that counts the number of changes, and let A_i be the event that there is a change in the i^{th} gap. I_{A_i} is an indicator random variable. $I_{A_i} = 1$ if there is a change, and 0 if not. By the additivity of expected value,

$$E[X] = \sum_{i=1}^{n-1} E[I_{A_i}] = \sum_{i=1}^{n-1} P(A_i) = (n-1)\frac{1}{2} = \frac{n-1}{2}$$

There are one more runs than boundaries, therefore the expected number of the number of runs is $1 + \frac{n-1}{2} = \frac{n+1}{2}$.

12 Exercise sheet 12

Definitions:
Let D be a digraph and $c : A \to \mathbb{R}$. A **potential** is a function $\pi : V \to \mathbb{R}$. We say that π is **feasible** (with respect to c) if $\pi(v) - \pi(u) \leq c(e)$ for every $e \in [u, v]_D$. (We can also write $e = uv$ if there are no parallel edges.)

A cost function $c : A \to \mathbb{R}$ is called **conservative** is there is no negative cost directed cycle.

83. (1 point) Let $\pi : V \to \mathbb{R}$ be everywhere 0, that is, $\pi(v) = 0$ for every $v \in V$. When is this a feasible potential?

Solution: If the cost function is nonnegative: $c(uv) \geq 0$ for every $uv \in A$.

84. (2 points) Show that if a feasible potential exist for a given c, then a nonnegative feasible potential also exists.

Solution: Let $k := \max_{v \in V} |\pi(v)|$ and $\pi'(v) = \pi(v) + k$ for every $v \in V$. This is still a feasible potential, since $(\pi(v) + k) - (\pi(u) + k) = \pi(v) - \pi(u) \leq c(e)$.

85. (1+1+2+2 points) Let D be a digraph and $c : A \to \mathbb{R}$ is a conservative cost function, π_1 and π_2 are feasible potentials. Show that:

- $\pi_1 + 4$ is also a feasible potential.
- $\frac{\pi_1 + \pi_2}{2}$ and $\frac{3\pi_1 + 4\pi_2}{7}$ are feasible potentials.
- $\min(\pi_1, \pi_2)$ is a feasible potential. What about $\max(\pi_1, \pi_2)$?
- $\lfloor \pi_1 \rfloor$ is a feasible potential if c is integer valued. Is it true for $\lceil \pi_1 \rceil$?

Solution:

- same as in Problem 2.
- $\frac{3\pi_1(v)+4\pi_2(v)}{7} - \frac{3\pi_1(u)+4\pi_2(u)}{7} = \frac{3}{7}(\pi_1(v) - \pi_1(u)) + \frac{4}{7}(\pi_2(v) - \pi_2(u)) \leq \frac{3}{7}c(uv) + \frac{4}{7}c(uv) = c(uv)$
- Look at a fixed u and v. Suppose that $\pi_1(u) = \min(\pi_1(u), \pi_2(u))$.

  Then $\min(\pi_1(v), \pi_2(v)) - \min(\pi_1(u), \pi_2(u)) = \min(\pi_1(v), \pi_2(v)) - \pi_1(u) \leq \pi_1(v) - \pi_1(u) \leq c(uv)$. So it is still a feasible potential.

  A similar argument works for the maximum: Suppose that $\pi_1(v) = \min(\pi_1(v), \pi_2(v))$.
  $\max(\pi_1(v), \pi_2(v)) - \max(\pi_1(u), \pi_2(u)) = \pi_1(v) - \max(\pi_1(u), \pi_2(u)) \leq \pi_1(v) - \pi_1(u) \leq c(uv)$.

- $\pi_1(v) - \pi_1(u) \leq c(uv)$, by reorganizing the sides $\pi(v) \leq c(uv) + \pi(u)$.
  $\lfloor \pi_1(v) \rfloor \leq \lfloor c(uv) + \pi(u) \rfloor$ Since c is integer valued, $\lfloor \pi_1(v) \rfloor \leq c(uv) + \lfloor \pi(u) \rfloor$ and this is what we wanted. The same works for $\lceil \pi_1 \rceil$.

86. (3 points) Let D be a digraph, $s,t \in V$ and $c : A \to \mathbb{R}$ is a conservative cost function. We will call an arc $a \in A$ *beautiful* if there is a minimum cost directed $s \to t$ path containing a. Show that if path P is an $s \to t$ path and all of its arcs are beautiful, then P is a cheapest path.

Solution: From Gallai's theorem, there is a π feasible potential for c. From Duffin's theorem,

$$\min\{\tilde{c}(P) : P \text{ is an } s \to t \text{ path}\} = \max\{\pi(t) - \pi(s) : \pi \text{ is a feasible potential}\}$$

Take the optimal feasible potential π from Duffin's theorem, and work with that. We will call an edge uv an "tight edge" if $\pi(v) - \pi(u) = c(e)$.
For any path P' with vertices $s = v_0, v_1, v_2 \dots t = v_k$,
$c(P') = \sum_{i=0}^{k-1} c(v_i v_{i+1}) \geq \sum_{i=0}^{k-1} (\pi(v_{i+1}) - \pi(v_i)) = \pi(t) - \pi(s)$ If P' is a cheapest path, all of its edges are tight edges. Therefore every beautiful edge is tight, and if we build a path from tight edges, it will be a cheapest path.

87. (2 points) Let D be a digraph $s,t \in V$ and $c : A \to \mathbb{R}$ is a cost function, but it is not everywhere nonnegative. We pick a constant k and make a new *nonnegative* cost function, $c^+(a) = c(a) + k$ for every $a \in A$. Using Dijkstra's algorithm with cost function c^+ do we always get a cheapest $s \to t$ path with respect to the original cost?

Solution: No. For example, D has a path of length 2 with edge weights $-10, -10$ and a path of length 3 with edge weights $-10, -10, -9$, if we add $k = 10$ to them, we have weights $0, 0$ versus $0, 0, 1$. Dijkstra's algorithm picks $0 + 0$ but in the original graph, $-10, -10, -9$ is the cheapest path.

88. (3 points) Let D be a digraph $s,t \in V$ and $S, H \subseteq V$ are $s\bar{t}$ sets with minimal outdegree. Show that $S \cup H$ and $S \cap H$ are $s\bar{t}$ sets with minimal outdegree as well.

Solution: Let k be the outdegree of a $s\bar{t}$ set in D.
We can show that $d^+(S) + d^+(H) \geq d^+(S \cup H) + d^+(S \cap H)$ by looking at all the possible edges that leave S, H, $S \cap H$ or $S \cup H$.
$k + k = d^+(S) + d^+(H) \geq d^+(S \cup H) + d^+(S \cap H) \geq k + k$. There has to be equality thoughout, so $S \cup H$ and $S \cap H$ also have outdegree k.

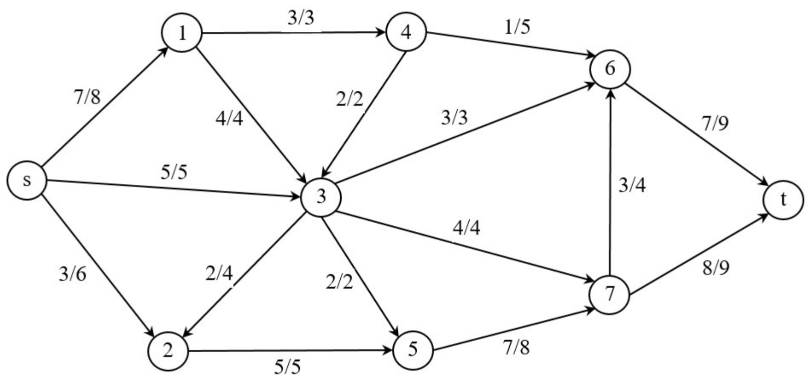
Menger's theorem (undirected, vertex-version)
Let G be a finite undirected graph and s and t two nonadjacent vertices. Then the size of the minimum vertex cut for s and t (the minimum number of vertices, distinct from s and t, whose removal disconnects s and t) is equal to the maximum number of pairwise internally vertex-disjoint paths from s to t.

89. **For handing in.** (8 points) Prove Hall's theorem from the undirected vertex-version of Menger's theorem.

Solution: Let G be a bipartite graph $G = (A, B; E)$. If there is a matching covering A then for every $X \subseteq A$, $|\Gamma(X)| \geq |X|$.
We will use Menger for the other direction of Hall's theorem. Add two vertices to the graph: s and t. Connect s to every vertex in A, and connect t to every vertex in B. Denote the graph we get this way by G'. Let C be a minimum vertex cut for s and t in G'. Now $\Gamma(A \setminus C) \subseteq B \cap C$ because there cannot be an edge between $A \setminus C$ and $B \setminus C$.
Suppose that for every $X \subseteq A$, $|\Gamma(X)| \geq |X|$.
Then, $|C| = |A \cap C| + |B \cap C| \geq |A \cap C| + |\Gamma(A \setminus C)| \geq |A \cap C| + |A \setminus C)| = |A|$ The size of the minimu vertex cut is at least $|A|$. Using Menger's theorem, there are $|A|$ internally vertex-disjoint paths from s to t. Removing s and t from these paths, we get a matching of size $|A|$ in the original graph, i.e. a matching that covers side A.

13 Exercise sheet 13

90. (4 points) Consider the following network. We want to find the maximum flow from s to t. In the following picture, on the arcs, the first number shows the flow $f(ij)$ and the second number shows the capacity. For example, on arc $3 \to 2$ the capacity is 4, i.e. $c(v_3v_2) = 4$, and 2 units flow on it i.e. $f(v_3v_2) = 2$.



a) Find an augmenting path from s to t. What are the forward edges and what are the backward edges? With how many units can we increase the value of the flow?
b) After this one augmenting step, is the flow optimal? If yes, find the minimum cut. If not, find all the remaining steps of the Ford-Fulkerson algorithm.

Solution: **a)** The only augmenting path is s-2-3-4-6-t. Forward edges: $s \to 2$, $4 \to 6$, $6 \to t$. Backward edges: $3 \to 2$, $4 \to 3$. We can improve the flow with 2 units.
b) After this step, the flow is optimal. The value of the flow is now $7 + 5 + 5 = 17$. Minumum cut: $A = \{s, 1, 2\}$, $B = \{3, 4, 5, 6, 7, t\}$. The capacity of this cut is $3 + 4 + 5 + 0 + 5 = 17$. Another minimum cut: $A = \{s, 1, 2, 3\}$, $B = \{4, 5, 6, 7, t\}$.

91. (5 points) Prove Kőnig's theorem (in a biparite graph, size of the maximum matching = size of the mininum vertex cover) from the Max-flow Min-cut theorem.

Solution: Let G be a bipartite graph $G = (U, V; E)$. It is easy to see, that if we have a matching of size ν, we need at least ν nodes to cover every edge, thus $\tau \geq \nu$.
Add two vertices to the graph: s and t. Connect s to every vertex in U, and connect t to every vertex in V. Direct the edges from s to U, U to V and V to t. Denote the graph we get this way by $D' = (V', E')$. The capacity of every edge is 1. Since capacities are integers, we can find an integer valued maximum flow.
Let S, T be a minimum cut in this network. ($s \in S$ and $t \in T$) The set of directed edges in this (directed) cut is $Cut(S) = \{ev \in E' : u \in S, v \in T\}$. If there is an uv edge such that $u \in U \cap S$ and $v \in V \setminus S$, move vertex v to S. $S := S \cup \{v\}$. This way, the edge uv is not in the cut anymore. (There may be another wv edges that also leave the cut) Edge vt enters the cut. This way, the number of edges in the cut cannot increase, thus the capacity of the cut cannot increase.
Repeat this step until all the neighbors of $U \cap S$ are in S. Now, the capacity of the cut is $k = |U \setminus S| + |V \cap S|$ and $(U \setminus S) \cup (V \cap S)$ covers all the edges in G.
From the max flow min cut theorem, there is a flow of value k with 0-1 values on the edges. Using the edges with flow value 1 between U and V, we get a matching of size k.
Therefore, we found size k a vertex cover and a size k matching in the original graph, thus $\tau = \nu$.

Second solution: Add two vertices to the graph: s and t. Connect s to every vertex in U, and connect t to every vertex in V. Direct the edges from s to U, U to V and V to t. The capacity of every edge from s or to t 1. The capacity is M for all the edges between U and V, where M is a large integer (it is enough if $M > |V|$.) Let S, T be a minimum capacity cut in this network. ($s \in S$ and $t \in T$). Because of the large capacities on the middle edges, there is no uv edge such that $u \in U \cap S$ and $v \in V \setminus S$. The rest of the proof is the same as in the previous solution.

92. (3 points) $D = (V, E)$ is a directed graph, $s, t \in V$ and $c_1, c_2, \ldots, c_k$ are capacity functions on the edges. ($c_i : E \to \mathbb{R}_+$ for every i) Create an algorithm to decide whether there exists a $s\bar{t}$ cut that is a minimum cut for all of these capacity functions.

Solution: Let $c := c_1 + c_2 + \cdots + c_k$. Run the Ford- Fulkerson algorithm $k + 1$ times, for each of the c_i

capacities and also for c as a capacity function.
Let A, B be the minimal cut for c. For any c_i

$$c_i(A,B) \geq \min_{A'B' \text{ is an } st \text{ cut}} c_i(A',B')$$

$$c(A,B) = \sum_{i=1}^{k} c_i(A,B) \geq \sum_{i=1}^{k} \min_{A'B' \text{ is an } st \text{ cut}} c_i(A',B')$$

If the capacity of the minimal cut for c equals the sum of the capacities of the minimal cuts for each c_i, i.e. $c(A,B) = \sum_{i=1}^{k} \min_{A'B' \text{ is an } st \text{ cut}} c_i(A',B')$ then the cut we are looking for exists: (A,B) is a cut like that. It it is not equal, no such cut exists. It is clear that (A,B) cannot be the good cut in this case. If some other (C,D) cut is minimal for each of the c_i capacities, then if $c(A,B) \geq c(C,D)$, so (C,D) would be minimal for capacity c as well.

93. (2 points) How many ways are there to distribute 10 identical balls among 2 boys and 2 girls, if each boy should get at least one ball and each girl should get at least 2 balls? Express the answer as a coefficient of a suitable power of x in a suitable product of polynomials.

Solution: $(x+x^2+x^3+\dots+x^5)(x+x^2+x^3+\dots+x^5)(x^2+x^3+\dots+x^6)(x^2+x^3+\dots+x^6)$ The answer is a coefficient of x^{10} in this product. We may also use $(x+x^2+x^3+\dots)(x+x^2+x^3+\dots)(x^2+x^3+\dots)(x^2+x^3+\dots)$ instead.

$a(x) = (x+x^2+x^3+\dots)(x+x^2+x^3+\dots)(x^2+x^3+\dots)(x^2+x^3+\dots) = x^6(1+x+x^2+x^3+\dots)^4 = x^6\frac{1}{(1-x)^4}$
From the generalized binomial theorem, $\frac{1}{(1-x)^4} = \binom{3}{3} + \binom{4}{3}x + \binom{5}{3}x^2 + \binom{6}{3}x^3 + \binom{7}{3}x^4 \dots$ So the coefficient we are looking for is $\binom{7}{3} = 35$.

Alternative solution, without generating functions: Give $1+1+2+2$ balls to the boys and girls. We are left with 4 balls that we want to share among 4 people. This can be done by placing 3 separators between 4 objects, so in $\binom{7}{3}$ ways.

94. (2 points) Find the probability that we get exactly 12 points when rolling 3 dice.

Solution:
$a(x) = (x+x^2+x^3+\dots+x^6)(x+x^2+x^3+\dots+x^6)(x+x^2+x^3+\dots+x^6)$. The answer is the coefficient of x^{12} in this product.

$$a(x) = x^3\left(\frac{1-x^6}{1-x}\right)^3 = x^3\frac{1}{(1-x)^3}(1-3x^6+3x^{12}-x^{18})$$

From the generalized binomial theorem, $\frac{1}{(1-x)^3} = \binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 \dots$
Therefore, $a(x) = (\binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 \dots)x^3(1-3x^6+3x^{12}-x^{18})$

We are looking for x^{12}, it appears in $\binom{11}{2}x^9 \cdot x^3$ and in $\binom{5}{2}x^3 \cdot x^3 \cdot (-3x^6)$ thus the coefficient of x^{12} in $a(x)$ is $\binom{11}{2} - 3\binom{5}{2} = 55 - 3 \cdot 10 = 25$.
There are $6^3 = 216$ ways to roll 3 dice, so the probability that the sum is 12 is $\frac{25}{216}$.

95. (4 points) Find generating functions for the following sequences (express them in a closed form, without infinite series):

a) $0, 0, 0, 0, -6, 6, -6, 6, -6, \ldots$

b) $1, 0, 1, 0, 1, 0, \ldots$

c) $1, 2, 1, 4, 1, 8, \ldots$

d) $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

Solution: **a)**

$$6(x^5 - x^4)\frac{1}{1-x^2} = -6x^4\frac{1}{1+x}$$

b)

$$\frac{1}{1-x^2}$$

c)

$$\frac{\frac{1}{1-2x^2} - 1}{x} + \frac{1}{1-x^2} = \frac{1}{x-2x^3} - \frac{1}{x} + \frac{1}{1-x^2} = \frac{2x}{1-2x^2} + \frac{1}{1-x^2}$$

d)

$$(1+x)\frac{1}{1-x^3}$$