Discrete Mathematics, exercise sheet 9
Solutions

1. (3 points) Show that if graph G has at least 11 nodes, then it is not possible that both G and the complement of G are planar.

Solution: The number of nodes in G is $n \geq 11$.The union of G and $\overline{G}$ is K_n, so they have $n(n-1)/2$ edges together. Suppose (for contradiction) that both G and $\overline{G}$ are planar. Therefore they have at most $3n-6$ edges, thus $n(n-1)/2 \leq 6n-12$. After rearranging, $n^2 - 13n + 24 \leq 0$. This function on n is an upwards parabola, so n cannot be greater than the bigger root of the quadrastic equation $n \leq (13+\sqrt{169-96})/2 < 11$, contradiction.

2. (3 points) Find a graph G on 8 nodes such that neither G nor the complement of G is planar.

Solution: Let G be a $K_{3,3}$ and two isolated vertices. Then the complement of G contains K_5 as a subgraph.

3. a) (2 points) Let $(P, \mathcal{L})$ be a projective plane with order n, and let $A \subseteq P$ be a set of points such that any three points of A are not collinear. Show that $|A| \leq n+2$.
b) (4 points) If n is odd, show that $|A| \leq n+1$.

Solution: **a)** Take a $p \in A$, there are $n+1$ lines through p and by the pigeonhole principle, we can select only one other point of each of them. Together with p, the number of selected points is $\leq n+2$.
b) Suppose that $A \subseteq P$ is a set of points such that any three points of A are not collinear and $|A| = n+2$. Take a $p \in A$, and denote the $n+1$ lines through p as $L_1, L_2 \ldots L_{n+1}$ From the previous part, we know that Now look only at lines $L_1, L_2 \ldots L_n$. A contains exactly one "non-p" points of each of these lines: denote them as $q_1 \ldots\ldots q_n$. ($q_i \in L_i$, $q_i \in A, q_i \neq p$.) Points q_i and q_j $(1 \leq i < j \leq n)$ define a line, and this line has an intersection with L_{n+1}. Call this point r_{ij}. This point cannot be in A because that would mean 3 selected points on one line. We call the points we can get this way *forbidden* points. For any $1 \leq i, j, k \leq n$ $r_{ij} \neq r_{ik}$. (If $q_i q_j$ and $q_i q_k$ defined the same intersection point, q_i q_j and q_k would lie on one line.)
It is possible that $r_{ij} = r_{lk}$ if i, j, l, k are four different numbers. Thus one forbidden points can belong to $\lfloor \frac{n}{2} \rfloor$ point pairs. The number of forbidden points on line L_{n+1} is at least $\frac{\binom{n}{2}}{\lfloor \frac{n}{2} \rfloor}$.
If n is odd,

$$\frac{\binom{n}{2}}{\lfloor \frac{n}{2} \rfloor} = \frac{\binom{n}{2}}{\frac{n-1}{2}} = n$$

On line L_{n+1}, p is selected and every other point is forbidden, so we cannot find a fitting point on the last line.

4. (3 points) Let P be a finite set and let $\mathcal{L}$ be a system of subsets of P satisfying conditions
(i), Any two distinct sets $L_1, L_2 \in \mathcal{L}$ intersect in exactly one element, i.e. $|L_1 \cap L_2| = 1$.
(ii) For any two distinct elements $p_1, p_2 \in P$,there exists exactly one set $L \in \mathcal{L}$ such that $p_1 \in L$ and $p_2 \in L$.
(iii'): There exist at least two distinct lines $L_1, L_2 \in \mathcal{L}$ having at least 3 points each.
Prove that any such $(P, \mathcal{L})$ is a finite projective plane.

Solution: Parts (i) and (ii) are exactly waht we had if the definition of a projective plane. We need to show that this system contains 4 points in general position. We know there exist at least two distinct lines $L_1, L_2 \in \mathcal{L}$ having at least 3 points each. This 2 lines have and intersection point, call it p_0. There are at least 2 other points on line L_1, denote them as p_1, p_2, and there are at least 2 other points on line L_2, denote them as p_3, p_4. We claim that p_1, p_2, p_3, p_4 are in a general position. Suppose there is a line L' that contains 3 of these 4 points. Then L' contains either p_1, p_2 or p_3, p_4. Two points define only one line, so L' is either L_1 or L_2. Without loss of generality, we can say that $L' = L_1$ and it contains p_1, p_2, p_3. But then p_3 is on both L_1 and L_2. There is only one intersection point, so $p_3 = p_0$. Contradiction, because p_3 and p_0 are different points.

Definition
In a graph $G = (V, E)$, a *stable set* is a subset C of V such that no pair of vertices in C is connected with an

edge. An *edge cover* is a subset F of E such that for each vertex v there exists $e \in F$ where v is an endpoint of e. Note that an edge cover can exist only if G has no isolated vertices.
$\alpha(G) := \max\{|C| : C \text{ is a stable set}\}$,
$\tau(G) := \min\{|W| : W \text{ is a vertex cover}\}$,
$\nu(G) := \max\{|M| : M \text{ is a matching}\}$,
$\rho(G) := \min\{|F| : F \text{ is an edge cover}\}$.

5. (3 points) Prove that if $G = (V, E)$ is a graph without isolated vertices, then

$$\alpha(G) + \tau(G) = |V| = \nu(G) + \rho(G)$$

Note: This is Gallai's theorem.

Solution: U is a stable set if and only if $V \setminus U$ is a vertex cover. The first equality follows directly from this statement.
To see the second equality, first let M be a matching of size $\nu(G)$. For each of the $|V| - 2|M|$ vertices v not covered by M, add to M an edge covering v. We obtain an edge cover F of size $|M|+(|V|-2|M|) = |V|-|M|$. Hence $\rho(G) \leq |F| = |V| - |M| = |V| - \mu(G)$.
Second, let F be an edge cover of size $\rho(G)$. Choose from each component of the graph (V, F) one edge, to obtain a matching M. As (V, F) has at least $|V| - |F|$ components, we have $\mu(G) \geq |M| \geq |V| - |F| = |V| - \rho(G)$.
(Any graph (V, E) has at least $|V| - |E|$ components, this can be shown by induction on $|E|$: adding any edge reduces the number of components by at most one.)

6. (4 points) Is it possible to arrange 8 bus routes in a city so that
(i) if any single route is removed (doesn't operate, say) then any stop can still be reached from any other stop, with at most one change, and
(ii) if any two routes are removed, then the network becomes disconnected?

Solution: Yes. Draw 8 lines in the plane in general position (no 2 parallel, no 3 intersecting at a common point). Let the intersections represent stops and the lines bus routes.

7. **For handing in.** (7 points) Prove that the Fano plane is the only projective plane of order 2 (i.e. any projective plane of order 2 is isomorphic to it. Define an isomorphism of set systems first).