Discrete Mathematics, exercise sheet 9

1. (3 points) Show that if graph G has at least 11 nodes, then it is not possible that both G and the complement of G are planar.

2. (3 points) Find a graph G on 8 nodes such that neither G nor the complement of G is planar.

3. a) (2 points) Let $(P, \mathcal{L})$ be a projective plane with order n, and let $A \subseteq P$ be a set of points such that any three points of A are not collinear. Show that $|A| \leq n + 2$.
b) (4 points) If n is odd, show that $|A| \leq n + 1$.

4. (3 points) Let P be a finite set and let $\mathcal{L}$ be a system of subsets of P satisfying conditions
(i), Any two distinct sets $L_1, L_2 \in \mathcal{L}$ intersect in exactly one element, i.e. $|L_1 \cap L_2| = 1$.
(ii) For any two distinct elements $p_1, p_2 \in P$,there exists exactly one set $L \in \mathcal{L}$ such that $p_1 \in L$ and $p_2 \in L$.
(iii'): There exist at least two distinct lines $L_1, L_2 \in \mathcal{L}$ having at least 3 points each.
Prove that any such $(P, \mathcal{L})$ is a finite projective plane.

Definition
In a graph $G = (V, E)$, a *stable set* is a subset C of V such that no pair of vertices in C is connected with an edge. An *edge cover* is a subset F of E such that for each vertex v there exists $e \in F$ where v is an endpoint of e. Note that an edge cover can exist only if G has no isolated vertices.
$\alpha(G) := \max\{|C| : C \text{ is a stable set}\}$,
$\tau(G) := \min\{|W| : W \text{ is a vertex cover}\}$,
$\nu(G) := \max\{|M| : M \text{ is a matching}\}$,
$\rho(G) := \min\{|F| : F \text{ is an edge cover}\}$.

5. (3 points) Prove that if $G = (V, E)$ is a graph without isolated vertices, then

$$\alpha(G) + \tau(G) = |V| = \nu(G) + \rho(G)$$

6. (4 points) Is it possible to arrange 8 bus routes in a city so that
(i) if any single route is removed (doesn't operate, say) then any stop can still be reached from any other stop, with at most one change, and
(ii) if any two routes are removed, then the network becomes disconnected?

7. For handing in. (7 points) Prove that the Fano plane is the only projective plane of order 2 (i.e. any projective plane of order 2 is isomorphic to it. Define an isomorphism of set systems first).