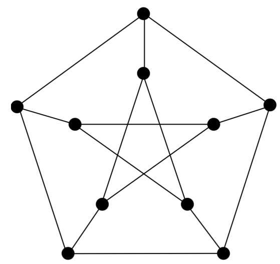
Discrete Mathematics, exercise sheet 8
Solutions

1. (2 points) Is the complement of the cycle of length 6 (C_6) a planar graph?

Solution: Yes, it is planar.

2. (3 points) Show that the Petersen graph is not planar.



Solution: Suppose that it has a planar drawing. The Petersen graph does not contain a cycle of lenght 3 or 4, so every country has at least 5 sides. From this, $e \geq \frac{5f}{2}$. We know that $v = 10$ and $e + 15$. From the Euler formula, $f + v = e + 2$, $f + 10 = 15 + 2$, so $f + 7$. This contradicts $e \geq \frac{5f}{2}$, therefore the graph cannot be planar.
Second solution: We can find a subdivision of $K_{3,3}$ as a subgraph of the Petersen graph.

3. (2 points each) **a)** Show that the edges of the Petersen graph cannot be colored with 3 colors.
b) Show that the Petersen graph does not have a Hamiltonian cycle, but deleting any vertex, the remaining graph has a Hamiltonian cycle.

Solution: **a)** Color the edges of the outer cycle with 3 colors: red, blue, red, blue, green. This defines the colors of the 5 edges that connect the outer cycle and the inner cycle, 3 of them is colored green. We can see that we cannot use the color green in the inner cycle, so we should color a cycle of lenght 5 with 2 colors. That is not possible.
b) Suppose the graph contains a Hamiltonian cycle, this is a cycle of lenght 10. Color the edges of the cycle red and blue in an alternating way. Since the Petersen graph is 3-regular, if we remove the Hamiltonian cycle from the graph, the remaining part is a perfect matching, color the edges of this matching green. This way we got a 3-coloring of the edges. From part **a)** we know that this is not possible.

4. (4 points) A group of musicians are traveling. Everyone has 3 enemies in the group. Show that they can be divided to sit on two buses in a way that everyone has at most one enemy who is traveling on the same bus as him.

Solution: As a graph theory problem: we have a finite, 3-regular graph. We want to find a partition of the vertices, $V = A \cup B, A \cap B = \emptyset$, such that every $v \in A$ has at most one neighbor in A, and every $v \in B$ has at most one neighbor in B. There are finitely many partitions, choose one where the number of edges between A and B is maximal.
Suppose for contradiction that (w.l.o.g.) there is an $x \in A$ such that x has at least two neighbors in A, move this node to B. $A := A \setminus \{x\}$, $B := B \cup \{x\}$. With this step, the number of edges between A and B increases. But it was already maximal. Contradiction.

5. (4 points) A *regular polyhedron* is a (3 dimensional) polyhedron whose faces are identical regular polygons. All side lengths are equal, and all angles are equal. In every vertex the same number of faces meet.
Using Euler's Formula, show that only five convex regular polyhedra exist.

Solution: For a regular polytope, the resulting topological planar graph has the same degree, d, of each vertex (where $d \geq 3$), and each face has the same number, $k \geq 3$, of vertices on its boundary.
Let us denote the number of vertices of the considered graph $G = (V, E)$ by n, the number of edges by e, and the number of faces by f. First we use the equation $\sum_{v \in V} d(v) = 2|E|$ which in our case specializes to $dn = 2e$. Similarly, $kf = 2e$.

Using Euler's formula

$$e+2=n+f=\frac{2e}{d}+\frac{2e}{k}$$

Divide by $2e$.

$$\frac{1}{2}+\frac{1}{e}=\frac{1}{d}+\frac{1}{k}$$

Hence if both d and k are known, the other parameters n, e, and f are already determined uniquely. Min $(d,k)=3$, for otherwise $\frac{1}{d}+\frac{1}{k}\leq\frac{1}{2}$.
For $d=3$, if $k\geq 6$, then $\frac{1}{d}+\frac{1}{k}\leq\frac{1}{2}$. Therefore we get $k\in\{3,4,5\}$. Hence one of the following possibilities must occur:

d	k	n	e	f
3	3	4	6	4
3	4	8	12	6
3	5	20	30	12
4	3	6	12	8
5	3	12	30	20

6. (2 points) Is there a bipartite graph with degrees 3,3,3,3,3,3,3,3,3,3,5,6,6? (These can be distributed in the two classes of nodes arbitrarily.)

Solution: If the two classes are A and B, $\sum_{v\in A}d(v)=\sum_{v\in B}d(v)$. No matter how we allocate the degrees, on one side, the sum of the degrees is divisible by 3, on the other side, it is not.

7. (2 points) An island is inhabited by six tribes. They are on good terms and split up the island between them, so that each tribe has a hunting territory of 100 square miles. The whole island has an area of 600 miles. The tribes decide that they all should choose new totems. They decide that each tribe should pick one of the six species of tortoise that live on the island. They want to pick different totems, and totem for each tribe should occur somewhere on their territory. The areas where the different species of tortoises live don't overlap, and they have they same area - 100 square miles. (Of course, the way the tortoises divide up the islands may be entirely different from the way way the tribes do.) Show that such a selection of totems is always possible.

Solution: Create a bipartite graph $G=(A,B,E)$ with the six tribes on one side, the six tortoises of the other side. Tribe a and tortoise b are connected by an edge, if their territories overlap. Take a subset of the tribes, $X\subseteq A$. The total territory of them is $|X|\cdot 100$ square miles. This cannot be covered with less than $|X|$ tortoise territories, therefore $|\Gamma(X)|\geq|X|$. Using Hall's theorem, the graph has a perfect matching.

8. **For handing in.** (10 points) Determine the number of pieces into which a circle is divided if n points on its circumference are joined by all possible chords. The chords are in a general position, no three of them goes thought the same point.

Solution: For small numbers, the number of regions is 1, 2, 4, 8, 16, 31, 57, 99, 163, 256...
Statement: For $n\geq 4$, the number of regions is $\binom{n}{4}+\binom{n}{2}+1$.
Taking all n vertices on the cycle and all the intersections inside the cycle, we get a planar graph.
The number of nodes is $|V|=n+\binom{n}{4}$. We use problem 9 from exercise sheet 6, the number of intersections of the diagonals of a convex n-sided polygon is $\binom{n}{4}$.
Every inner node has degree 4. Every outer node has degree $n+1$.
$2|E|=\sum d(v)=4\binom{n}{4}+n(n+1)$
$|E|=2\binom{n}{4}+\frac{n(n+1)}{2}=2\binom{n}{4}+\binom{n}{2}+n$
Using the Euler formula, the number of faces is $|E|+2-|V|$. Excluding the infinite face, the number of regions is $|E|+1-|V|=2\binom{n}{4}+\binom{n}{2}+n+1-(n+\binom{n}{4})=\binom{n}{4}+\binom{n}{2}+1$.