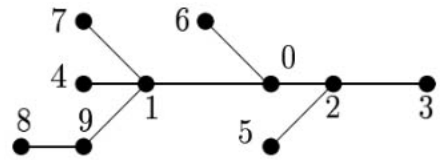
Discrete Mathematics, exercise sheet 7
Solutions

1. (2 points) Give the Prüfer code of the following tree:



Solution: The two-line Prüfer code is

3	4	5	2	6	7	8	9	1
2	1	2	0	0	1	9	1	0

Therefore the "short" Prüfer code is $2, 1, 2, 0, 0, 1, 9, 1$.

2. (2 points) Select the value of x such that $1, 1, 5, x, 6, 6$ is a Prüfer code of a tree, in which every degree is odd. Give the tree as well.

Solution: Only $x = 5$ can be good, otherwise 5 appears in the code only once, thus the degree of node with label 5 is 2. Choosing $x = 5$ is actually good, we get a tree with three wides with degree 3 and all the other nodes have degree 1. (I omit the picture now.)

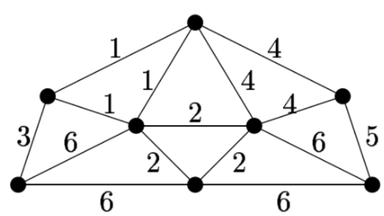
3. (2 points) Show that if a tree has a k-degree node, then it has at least k leaves. Is the reverse statement true?

Solution: From every edge of the node with degree k, start a path. All of this paths must end in a leaf, and since the graph does not contain a cycle, the path cannot merge, all these endpoints are different.

4. (2 points) How many trees are there on n labelled nodes, that have at least 3 leaves?

Solution: A tree has at least 3 leaves if and only if it is not a path. Using Cayley's formula and an earlier problem, the answer is $n^{n-2} - n!/2$, if $n \geq 2$. (If $n = 1$, there are no such trees.)

5. (2 points) Find a minimum cost spanning tree of this graph. How many minimum cost spanning trees are there?



Solution: The costs of the edges in the spanning tree are 1, 1, 2, 2, 3, 4, 5, the total cost is 18. From the three edges with cost 1, we can choose any two. Same for the edges with cost 2. The edge with cost 3 is unique. From the edges with cost 4, to avoid cycles, we can choose one of two possibilities. The edge with cost 5 is unique again, and we do not choose any of the cost 6 edges. Therefore the number of possibilies is $\binom{3}{2}\binom{3}{2}\binom{2}{1} = 18$.

6. (2 points) G is a simple graph, its vertices are labelled with $1, 2, \ldots, 100$. Nodes i and j are connected by an edge in G if and only if $|i - j| \leq 2$. Does G contain an Eulerian circuit or an Eulerian walk?

Solution: G is connected. Nodes 2 and 99 have degree 3. Every other node has even degree (two or four). Therefore there is an Eulerian walk in the graph. (It starts in 2 and ends is 99 or vice versa.) There is no Eulerian circuit in the graph.

7. (2 points) Is there a graph on 10 nodes that contains an Eulerian circuit and the sum of the degrees is 34?

Solution: Yes, there is. For example, if the degree sequence is 2 2 2 4 4 4 4 4 4 4.

8. (1+1 point) **a)** Find a graph, where every degree is even, and it does not contain an Eulerian circuit. **b)** Find a graph that is not conneted, and contains an Eulerian circuit.

Solution: This problem is about the importance of connectivity and isolated nodes. **a)** Two disjoint cycles. **b)** Take a connected graph with an Eulerian circuit and add some isolated nodes. (Example: A cycle and one isolated node.)

9. (4 points) In a group everyone knows 4 other people. (We assume that acquaintance is mutual.) Show that they can sit down around some round tables in a way that everyone knows his/her two neighbors.

Solution: The problem is the following: in a 4-regular graph G, find a spanning subgraph H, sucht that every component of H is a cycle. Suppose that G is connected. Then it has an Eulerian circuit. $2|E| = \sum_{v \in V} d(v) = 4|V|$, thus $|E| = 2|V|$. Following the Eulerian circuit, color the edges of the graph red and blue in an alternating way. Since the graph has an even number of edges, the alternating coloring is kept even for the starting node. Let H be the subgraph formed by the red edges. Every degree in H is two, so it is an union of disjoint cycles.
If G is not connected, we do the previous method for each connected component.

10. (5 points) A government wants to connect cities with roads, (i. e. they want to build a spanning tree). Optimists and pessimists win in unpredictable order. This means that sometimes they build the cheapest line that does not create a cycle with those lines already constructed; sometimes they mark the most expensive lines "impossible" until they get to a line that cannot be marked impossible without disconnecting the network, and then they build it. Prove that they still end up with an optimal cost spanning tree.

Solution: See in the Lecture notes.

11. **For handing in.** (6 points) Tree T has 17 nodes and the degree of each node is either 1 or 4. After Alice added some edges to this graph, it has an Eulerian circuit. At least how many edges did she add?

Solution: 6 edges. Let k be the number of nodes with degree 4. The tree has 16 edges, so the sum of the degrees is $\sum_{v \in V} d(v) = 4k + (17 - k) = 32$. We get that $k = 5$. The tree has 12 nodes with odd degree. By adding 6 edges, Alice can achieve that every degree of the graph is even, thus it contains an Eulerian circuit.