Discrete Mathematics, exercise sheet 6
Solutions

1. (2 points) In a group of 8 people, some of them shake hands. Is it possible that everyone shaked hands with a different number of people?

Solution: Everyone had at least 0, at most 7 handshakes. It is not possible that someone shaked hands with everyone and someone else with no one. So, by the pigeonhole principle there has to be two people with the same number of handshakes.

2. (2 points) In a simple, connected graph on 6 vertices, the degrees of 5 vertices are 1, 2, 3, 4, 5 respectively. What may be the degree of the 6^{th} vertex?

Solution: Let us call the 5 vertices with known degree $v_1, v_2, \dots v_5$, where $d(v_i) = i$. The degree of v_6 is unknown. Node v_5 is connected by an edge to every node, so the only neighbor of v_1 is v_5. Node v_4 is connected to very node except v_1. Therefore the two neighbors of v_2 are v_5 and v_4.
For v_6, we know that it is connected by an edge to v_5 and v_4 and not connected to v_1 and v_2. The same is true for v_3. Since the degree of v_3 is 3, v_3 is connected by an edge to v_6, therefore the degree of v_6 is 3.

Second solution: Since the sum of the degrees is even, the missing number has to be odd: 1, 3, or 5. Use the first half of the first solution. For v_6, we know that it is connected by an edge to v_5 and v_4 and not connected to v_1 and v_2, this rules out the degree being 5 or 1.

3. (2 points) Draw all simple graphs on 4, 5, or 6 vertixes that are isomorphic to their complement.
(The *complement* of a graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are connected by an edge if and only if they are not connected by an edge in G.
Two graphs are *isomorphic* if there exists a one-to-one correpondence between the nodes of the first graph and the nodes of the second graph so that two nodes in the first graph that are connected by an edge correspond to nodes in the second graph that are connected by an edge, and vice versa.)

Solution:
$n = 4$ A path with 4 nodes is a good solution.
$n = 5$ A cycle of length 5 is good, and there is another solution: a triangle plus 2 edges: $v_1v_2, v_2v_3, v_3v_1, v_1v_3, v_2v_5$
$n = 6$ It is not possible. K_6 has $\binom{6}{2} = 15$ edges, half of the edges should be in G, half of them in the complement. This cannot be done with an odd umber of edges.

4. (1 point each) Is there a simple graph where the degrees of the vertices are
a) 3, 3, 3, 2, 2, 2, 1, 1, 1; **b)** 6, 6, 5, 4, 4, 3, 2, 2, 1;
c) 7, 7, 7, 6, 6, 6, 5, 5, 5; **d)** 1, 3, 3, 4, 5, 6, 6?

Solution: **a** is possible.
b is impossible beacause the sum of the degrees is odd.
c is possible, if we have a solution for **a**, the complement of that graph is a solution for **c**.
d is impossible. The graph is simple and has 7 nodes. Therefore two nodes are connected to every other node, there cannot be a node with degree 1.

5. (2 points) In a simple graph, vertex v has an odd degree. Prove there is a path from v to another vertex with odd degree.

Solution: Let v be an odd degree node, and let G' be the connected component containing v. The sum of the degrees in G' is even, so there has to be another node w with odd degree. Since they are in the same connected component, there is a path from v to w.

6. (3 points) Characterize the graphs with the following property: any two edges have a common endpoint.

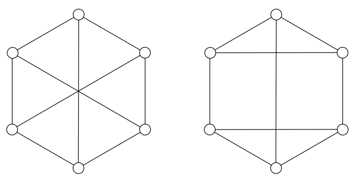
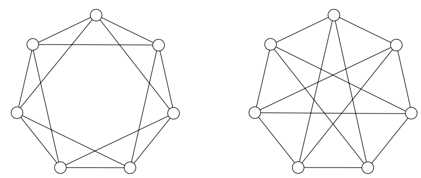
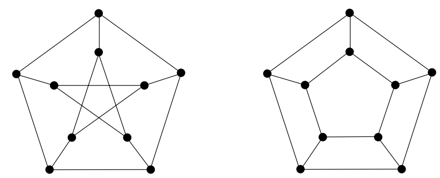
Definition: In graph theory, a star S_k is the complete bipartite graph $K_{1,k}$: a tree with one internal node and k leaves.

Solution: First, consider the case where the graph is simple. If the graph has at most 2 edges, the answer is

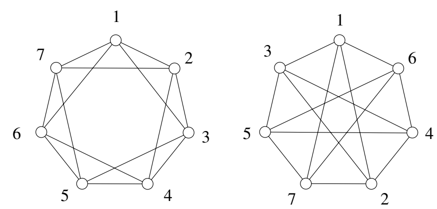
a path of length 1 or 2. If the graph has at least 3 edges: Let $e = \{u, v\}$ and $f = \{u, z\}$ be two edges of the graph (their common node is u). A third edge can either be edge $\{v, z\}$, or contain u and another vertex. If the three edges form a triangle we cannot add any more edges, if they form a star, we can add more edges, all of the new edges should contain u, so the result is still a star.
Therefore, the answer is, this graph is either a star or a cycle of length 3.

If the graph is not simple, we can add parallel edges to any edge and if the graph was a star, we can add loops to its internal node.

7. (3 points) Which pairs of graphs are isomorphic?

a)  **b)**  **c)** 

Solution: **a)** Not isomorphic, the right-hand side graph contains a triangle, but the left-hand side graph does not.
b) Yes, they are isomorphic, see the picture.
c) Not isomorphic. The left is the Petersen graph, and it does not contain a cycle of length 4, while the right graph contains a cycle of length 4.



8. (2 points) Draw the tree with the Prüfer code 4 3 0 1 1 3.

Solution: We recreate the two-line "long" Prüfer code

2	4	5	6	7	1	3
4	3	0	1	1	3	0

The columns are the edges of the tree.

9. (5 points) At most how many intersections do the diagonals of a convex n-sided polygon have?

Solution: Any two intersecting diagonals cover 4 nodes, and 4 nodes gives exacly one pair of intersenting diagonals. (Other intersection points fall outside of the convex polygon). Therefore there is a one-to-one correspondence between sets of 4 nodes and intersection points. The number of intersections is $\binom{n}{4}$.

10. For handing in. How many trees are there on n labeled vertices such that
a) (3 points) the degree of each node is at most 2.
b) (5 points) the node with label 1 has degree 1.

Solution: **a)** A tree where each degree is at most 2 is a path. There are $n!$ possibilities to list n vertices in a row, but this way we counted each path twice (from left to right, from right to left) thus the number of labeled paths is $n!/2$.
b) removing the node with label 1, we get a labeled tree on $n-1$ nodes. Using the Cayley formula, there are $(n-1)^{n-3}$ such trees. We can reconnect node with label 1 to any of the other nodes, so in total there are $(n-1)(n-1)^{n-3} = (n-1)^{n-2}$ possibilities.