Discrete Mathematics, exercise sheet 5
solutions

1. (3 points) The inclusion-exclusion principle states the following: For finite sets $A_1, A_2 \dots A_n$:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \dots (-1)^{n-1}|A_1 \cap A_1 \cap \dots A_n|$$

Prove this statement with induction. (So, suppose we already know that the statement is true for $n-1$ sets, and using this, prove for n sets.)

2. (3 points) Express the following sum in a closed form.

$$\binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}4 + \dots + \binom{n}{n}2^n$$

Solution: Use the binomial theorem: $\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k \cdot 1^{n-k} = (2+1)^n = 3^n$

3. (2 points) Prove that F_n and F_{n-1} are relative primes. (F_n is the n^{th} Fibonacci number.)

Solution: Suppose F_n and F_{n-1} are not relative primes. This means there is a $p > 1$, $p|F_n$ and $p|F_{n-1}$. Since $F_{n-2} = F_n - F_{n-1}$, $p|F_{n-2}$. Therefore F_{n-1} and F_{n-2} are not relative primes either. After some steps, we reach that $F_1 = 1$ and $F_2 = 1$ are not relative primes. But they are. Contradiction.

4. (2 points) There are 350 farmers in a large region. 260 of them farm beetroot, 100 farm potatoes, 70 farm radish, 40 farm beetroot and radish, 40 farm potatoes and radish, and 30 farm beetroot and potatoes. All of them farm something out of these three vegetables.
Determine the number of farmers that farm beetroot, potatoes, and radish.

Solution: Let x be the number of people who farm all three. Use the inclusion exclusion prociple.

$$\left|\bigcup_{i=1}^{3} A_i\right| = \sum_{i=1}^{3} |A_i| - \sum_{1\le i<j\le 3} |A_i \cap A_j| + |A_1 \cap A_2 \cap A_2|$$

$$350 = 260 + 100 + 70 - (40 + 40 + 30) + x$$

$$350 = 430 - 110 + x$$

$$350 = 320 + x$$

$$x = 30$$

Therefore, 30 farmers farm beetroot, potatoes, and radish.

5. (4 points) There is a necklace with n beads ($n \ge 2$) and one big assymetric jewel. (The jewel is needed so that every bead is identifiable, we can say "this is the third bead to the left from the jewel"). The beads are colored with k possible colors. Neighboring beads must have different colors. Every bead has 2 neighbors. (The big jewel does not count as a neighbor and it is not colored.)
How many different ways can we color the necklace?

Solution: The number of good colorings is $(k-1)^n + (-1)^n(k-1)$.
We will use the inclusion-exclusion principle. The total number of coloring (without any restriction) is k^n.
Let the beads be $b_1, b_2, \dots b_n$. Let A_i be the set of all coloring where b_i and b_{i-1} have the same color.
$|A_i| = k^{n-1}$. (k possible colors for the pair, k^{n-2} for everything else.)
$|A_i \cap A_j| = k^{n-2}$. The colors of 2 beads are determined by their left neighbors, and we are free to choose the colors of the other beads. With this reasoning we can see that $|\bigcap_{i\in I} A_i| = k^{|I|}$ for $1 \le |I| \le n-1$.
$|\bigcap_{i=1}^{n} A_i| = k$ (and not 1). The colors of all beads are the same, so we have k possibilities.
The number of good colorings is $k^n - n\cdot k^{n-1} + \binom{n}{2}\cdot k^{n-2} - \binom{n}{2}\cdot k^{n-3} + \dots + (-1)^{n-1}\binom{n}{n-1}\cdot k + (-1)^{n-1}\binom{n}{n}\cdot k$
This equals $k^n - n\cdot k^{n-1} + \binom{n}{2}\cdot k^{n-2} - \binom{n}{2}\cdot k^{n-3} + \dots + (-1)^{n-1}\binom{n}{n-1}\cdot k + (-1)^{n-1}\binom{n}{n}\cdot 1 + (-1)^{n-1}\binom{n}{n}\cdot (k-1)$

Using the binomial theorem, the solution is $(k-1)^n + (-1)^n(k-1)$.

(We can also check our result for small examples. If $k = 1, n \geq 2$, then there is no good coloring, the answer is 0. If $k = 2$, and n is even, we have 2 good colorings, if n is odd, then 0.)

Second solution: We use induction. Let $P(n, k)$ denote the number of good colorings of a necklace with n beads and k possible colors. For $n = 2$, $P(2, k) = k(k-1)$, which satisfies the formula $(k-1)^n+(-1)^n(k-1) = (k-1)^2 + (-1)^2(k-1) = (k-1)^2 + (k-1) = k(k-1)$.
Take a chain with n beads. The neighboring beads must have different colors, but the two endpoints only have one neighbor, so the number of good colorings is $k(k-1)^{n-1}$. We can choose from k colors for the first bead, and from $k-1$ for all the others, the color of the previous bead is forbidden. If the two endpoint have different colors, this is a good coloring for the necklace as well. If the two endpoint have same color, merge them into one, and we get a good coloring for a necklace with $n-1$ beads.
Therefore, $k(k-1)^{n-1} = P(n, k) + P(n-1, k)$. Using the induction hypothesis for $n-1$,
$P(n, k) = k(k-1)^{n-1} - P(n-1, k) = k(k-1)^{n-1} - ((k-1)^{n-1} + (-1)^{n-1}(k-1)) =$
$k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1) = k(k-1)^{n-1} - (k-1)^{n-1} + (-1)^n(k-1) = (k-1)^n + (-1)^n(k-1)$.

6. (5 points) A convex polygon with n sides is cut into triangles by connecting vertices with non-crossing line segments (polygon triangulation). The number of triangles formed is $n-2$.
How many different ways can this be achieved? (Solutions that can be transformed to each other via rotation of reflection still count as different solutions.)

Solution: The number of different ways that this can be achieved is the Catalan number C_{n-2}. For example: triangle: 1 way, quadrilateral: 2 ways, pentagon: 5 ways.
We know that $C_0 = 1$ and $C_{n+1} = \sum_{i=0}^{n} C_i\, C_{n-i}$ for $n \geq 0$. ($C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14\ldots$).
Let T_n denote the number of triangulations of an n-gon, We can see that $T_n = C_{n-2}$ is correct for $n = 3, 4, 5$.
We will use induction. Induction step: Take a convex polygon with $n+2$ sides. The vertices are $v_0, v_1 \ldots v_{n+1}$.
The side $v_0 v_{n+1}$ should be included in a triangle, the third node of the traingle can be any one of $v_1 \ldots v_n$.
If it is v_i, removing this triangle, we need to find a triangulation of $v_0 v_1 \ldots v_i$ (a polygon with $i+1$ sides) and $v_i v_{i+1} \ldots v_{n+1}$ (a polygon with $n-i+2$ sides). In the degenerate cases when $i = 1$ or $i = n$, we should "triangulate a segment", here the number of solutions equals the T_{n+1}, so we can define T_2 as 1, which satisfies $T_2 = C_0$. Using this recursion and the induction hypothesis,

$$T_{n+2} = \sum_{i=1}^{n} T_{i+1}\, T_{n-i+2} = \sum_{i=1}^{n} C_{i-1}\, C_{n-i} = \sum_{j=0}^{n-1} C_j\, C_{n-j-1} = C_n$$

7. For handing in. (8 points)
Let $x_1, x_2 \ldots x_{100}$ be integers. Prove that there exist integers i and j such that $1 \leq i \leq j \leq 100$ and

$$\sum_{k=i}^{j} x_k \text{ is divisible by } 100.$$

Solution:
Let s_i be $s_i = \sum_{k=1}^{i} x_k$ for every $1 \leq i \leq 100$. If there is an i such that $100|s_i$, we are done, $100| \sum_{k=1}^{i} x_k$.
If none of $s_1 \ldots s_{100}$ is divisible by 100, only 99 residue classes are possible, so by the pigeonhole principle there is an i and j such that $s_i \equiv s_j \pmod{100}$.
$\sum_{k=1}^{i} x_k \equiv \sum_{k=1}^{j} x_k \pmod{100}$.
We can suppose without loss of generality that $j > i$.
$100| \sum_{k=1}^{j} x_k - \sum_{k=1}^{i} x_k$
$100| \sum_{k=i+1}^{j} x_k$. So we found a good sum.