Discrete Mathematics, exercise sheet 5

1. (3 points) The inclusion-exclusion principle states the following: For finite sets $A_1, A_2 \dots A_n$:

$$\bigcup_{i=1}^{n} A_{i} = \sum_{i=1}^{n} |A_{i}| - \sum_{1 \le i < j \le n} |A_{i} \cap A_{j}| + \sum_{1 \le i < j < k \le n} |A_{i} \cap A_{j} \cap A_{k}| - \dots (-1)^{n-1} |A_{1} \cap A_{1} \cap \dots A_{n}|$$

Prove this statement with induction. (So, suppose we already know that the statement is true for n-1 sets, and using this, prove for n sets.)

2. (3 points) Express the following sum in a closed form.

$$\binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}4 + \dots + \binom{n}{n}2^n$$

3. (2 points) Prove that F_n and F_{n-1} are relative primes. (F_n is the n^{th} Fibonacci number.)

4. (2 points) There are 350 farmers in a large region. 260 of them farm beetroot, 100 farm potatoes, 70 farm radish, 40 farm beetroot and radish, 40 farm potatoes and radish, and 30 farm beetroot and potatoes. All of them farm something out of these three vegetables.

Determine the number of farmers that farm beetroot, potatoes, and radish.

5. (4 points) There is a necklace with n beads and one big asymptric jewel. (The jewel is needed so that every bead is identifiable, we can say "this is the third bead to the left from the jewel"). The beads are colored with k possible colors. Neighboring beads must have different colors. Every bead has 2 neighbors. (The big jewel does not count as a neighbor and it is not colored.)

How many different ways can we color the necklace?

6. (5 points) A convex polygon with n sides is cut into triangles by connecting vertices with non-crossing line segments (polygon triangulation). The number of triangles formed is n - 2. How many different ways can this be achieved? (Solutions that can be transformed to each other via rotation of reflection still count as different solutions.)

7. For handing in. (8 points)

Let $x_1, x_2 \dots x_{100}$ be integers. Prove that there exist integers i and j such that $1 \le i \le j \le 100$ and

$$\sum_{k=i}^{j} x_k \text{ is divisible by 100.}$$