Discrete Mathematics, exercise sheet 4

For problems 1-2, each subproblem is worth 1 point, except for 1/e.

1. In a German lottery, players are required to choose six main numbers between 1 and 49 plus an additional number, known as the Superzahl, between 0 and 9. To win the jackpot, a player must match all seven numbers, but prizes are available for matching as few as two main numbers plus the Superzahl.
a) What is the probability of getting all seven numbers right?
b) What is the probability of getting 6 numbers right, but not the Superzahl?
c) What is the probability of getting the 6 numbers right? (We don't care about the Superzahl)
d) What is the probability getting exactly 5 numbers of the main 6 right? (We don't care about the Superzahl)
e) (2 p) What is the probability getting at least 3 numbers of the main 6 right, and getting the Superzahl wrong?
Solution:

$$a, \quad \frac{1}{\binom{49}{6}} \cdot \frac{1}{10} \quad \text{and} \quad \binom{49}{6} = 13983816, \text{ so the probability is } \frac{1}{139838160}$$

$$b, \quad \frac{1}{\binom{49}{6}} \cdot \frac{9}{10} \qquad c, \quad \frac{1}{\binom{49}{6}}$$

$$d, \quad \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}}$$

$$e, \quad \left(\frac{1}{\binom{49}{6}} + \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}} + \frac{\binom{6}{4}\binom{43}{2}}{\binom{49}{6}} + \frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}}\right) \cdot \frac{9}{10}$$

2. There are 10 red, 20 yellow and 40 green balls in a box. With closed eyes, at least how many balls should we pick, to surely have
a) one yellow ball?
b) three balls with different colors?
c) three balls of the same color?
d) 15 balls of the same color?
e) two green balls that were drawn right after each other?
Solution: **a)** $10 + 40 + 1 = 51$; **b)** $40 + 20 + 1 = 61$; **c)** $2 + 2 + 2 + 1 = 7$; **d)** $10 + 14 + 14 + 1 = 39$; **e)** $2 \cdot 10 + 2 \cdot 20 + 2 = 62$ (10 green-red, 20 green-yellow and two more).

3. (1 point) Show that

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$$

Solution:

$$\frac{n-k}{k+1}\binom{n}{k} = \frac{n-k}{k+1} \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(k+1)!(n-k-1)!} = \binom{n}{k+1}$$

4. (4 points) Prove the following equality

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

Solution: Let us choose n elements out of $\{1, 2, 3, \ldots 2n\}$. If we chose k elements from the fist half, then we have to choose $n-k$ elements of the second half.

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

5. (2 points) How many ways can we cover a $2 \times n$ "chessboard" with 1×2 dominoes?

Solution: If $n = 1$: 1 way, if $n = 2$: two ways, if $n = 3$: three ways. We will show that the number of possibilities is F_{n+1}. Use induction. Take a $2 \times n$ "chessboard". If the last domino is vertical, we should fill the remaining $2 \times n - 1$ place with dominoes. We know from the induction hypothesis that this can be done F_n ways. If the last domino is horizontal, actually there has to be two horizontal dominoes. The remaining $2 \times n - 2$ part can be filled F_{n-1} ways. In total, we can fill it $F_n + F_{n-1} = F_{n+1}$ ways.

6. (3 points) Show that the product of n consecutive positive integers is always divisible by $n!$.

Solution: $\frac{m(m-1)\cdots(m-n+1)}{n!} = \binom{m}{n}$, which is an integer.

7. (2 points) How many ways can we choose three different numbers from the set $\{1, 2, 3, \ldots 100\}$ in a way that the sum of these three numbers is divisible by 3?

Solution: Either the three numbers have all different residues modulo 3, or all the same. In $\{1, 2, 3, \ldots 100\}$ there are 33 numbers having residue 0, 34 numbers have residue 1, and 33 numbers have residue 2. The total number of possibilities is $\binom{33}{3} + \binom{34}{3} + \binom{33}{3} + 33 \cdot 34 \cdot 33$.

8. For handing in. (7 points)

Prove that for the Fibonacci numbers $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ for every $n \geq 0$.

Solution: Use induction. For $n = 0$, the statement is true, since $0 = F_0 = F_2 - 1 = 1 - 1$

Induction hypothesis: $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$

Then, for $n + 1$, $F_0 + F_1 + F_2 + \cdots + F_n + F_{n+1} = F_{n+1} + F_{n+2} - 1 = F_{n+3} - 1$. Done.