Discrete Mathematics, week 3
solutions

For problems 1-4, each subproblem is worth 1 point.

1. **a)** There is a building with n floors (counting the ground floor as well).
How many ways can we paint the levels to red, yellow or blue?
b) What happens if two consecutive levels cannot have the same color?

Solution: a, 3^n (independent choice for every floor)
b, $3 \cdot 2^{n-1}$ starting from the ground floor, going up, we can choose from 3 colors for the ground floor, and from 2 colors for every floor above it, since the color of the lower neighbor is forbidden.

2. **a)** How many ways can a lion, a pengiun, a tiger and a polar bear stand in a row?
b) What if we have one more lion?
c) What if we have yet one more lion?
d) We have 4 lions, 2 tigers and 3 polar bears. (We do not distinguish between animals of the same species.)

Solution: **a)** 4!; **b)** 5!/2: differentiating between the lions, it would be 5!, in the end we divide by two; **c)** 6!/3! similarly, we counted each ordering 3! times; **d)** $\frac{9!}{4!\cdot 2!\cdot 3!}$ permutation with repetition.

3. On a 8×8 chessboard, how many ways can we place
a) one black and one white stones; **b)** two white stones;
c) one black, one white, and one green stone; **d)** three white stones;
e) three black and four white stones?

Solution: **a)** $64 \cdot 63$ (independent decisions); **b)** $64 \cdot 63/2$ (divide the solution of **a** by 2); **c)** $64 \cdot 63 \cdot 62$; **d)** $\frac{64\cdot 63\cdot 62}{3!}$; **e)** $\frac{64\cdot\dots\cdot 58}{3!\cdot 4!}$.

4. How many 8-digit numbers are there? How many 8-digit number are there with the following property:
a) the consecutive digits are different.
b) it does not contain the digit 5.
c) it contains the digit 5.
d) there are two digits that are the same (there may be more).
e) there are two *consecutive* digits that are the same (there may be more).
f) there are exactly two digits that are the same.
g) there are exactly two *consecutive* digits that are the same.

Solution: There are $9 \cdot 10^7$ eight-digit numbers, the first digit cannot be zero.
a) 9^8; **b)** $8 \cdot 9^7$; **c)** all minus bad ones: $9 \cdot 10^7 - 8 \cdot 9^7$;
d) $9 \cdot 10^7 - 9 \cdot 9 \cdot 8 \cdot 7 \cdot \dots \cdot 3$;
e) $9 \cdot 10^7 - 9^8$.
f) We can also solve it by case analysis. A tricky proof: Select the place of the two digits that are the same. Then from left to right write the digits, when we reach the first selected place write the same digit to the second place as well. The solution is $\binom{8}{2} \cdot 9 \cdot 9 \cdot 8 \cdot 7 \cdot \ldots \cdot 4$.
g) First version: the number has all different digits, except for two, which are consecutive. Take all possible 7-digit numbers with all differetn digits, and double one of the digits: $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 7$
Second version: Non-consecutive digits are allowed to be the same. Take a 7-digit numbers where the consecutive digits are different (similar to subproblem **a)**) then double one digit. This gives $7 \cdot 9^7$ as a solution.

5. (2 points) Alice goes the florist, and would like to buy 7 flowers. The shop has roses, tulips and carnations. How many ways can she buy 7 flowers? (We do not distinguish between flowers of the same type.)

Solution: This is the same as the chocolate problem, or the equation problem. Place the 7 flowers and 2 separators (anywhere, we don't have to buy ≥ 1 from each) this gives us $\binom{9}{2} = \frac{9\cdot 8}{2} = 36$ variations. With $n = 3, k = 7$ this is $\binom{n+k-1}{k} = \binom{9}{7} = \binom{9}{2} = 36$.

6. (2 points) George is in Manhattan, and he wants to walk from the corner of 8^{th} Avenue and 42^{nd} Street to the corner of 11^{th} Avenue and 57^{th} Street. He wants to walk one of the shortest possible paths. How many ways can he do it? (The streets and avenues form a grid.)

Solution: (Looking at the map, with approximate directions) This walk involves 3 corners (Avenues) going west, and 15 corners (Streets) going north. In total he takes 18 „steps" and 3 of them is going west. This gives $\binom{18}{3} = \frac{18 \cdot 17 \cdot 16}{3!} = 816$ possibilities.

7. (3 points) How many ways can n boys and n girls stand in a line, if two boys cannot stand next to each other, and two girls cannot stand next to each other?

Solution: $2 \cdot (n!)^2$, they must stand in a boy-girl-boy-girl alternating order. First we decide if a boy or a girl starts the row. Among the boys there are $n!$ possibilities, among the girls, also $n!$.

8. (5 points) 13 green, 15 gray, and 17 red chameleons live in Madagascar. They always meet in pairs. They are easily frightened, if two chameleons of different colors meet, they get so frightened that they both switch to the third color. Is it possible, that after some time, all of them acquire the same color?

Solution: Look at the number of chameleons of each color modulo 3. They represent 3 different residue classes. When two chamelaeons meet, two numbers decrease by one, one increases by 2, so modulo 3 they all change the same way. In other words, the difference between two color classes modulo 3 does not change. Therefore, we cannot reach 0, 0, 45 at any time.

9. For handing in.
a) (4 points) How many ways can we place 8 rooks on a 8×8 chessboard, such that no pair of them can capture each other? (A rook can capture an another rook if they are in the same row or in the same coloumn.)
b) (6 points) What if they are not allowed to capture each other and the arrangement of the pieces should be centrally symmetric to the center of the chessboard?

Solution: a, Every row and every coloumn has exactly one rook. Putting down the rooks row by row we have 8! possibilities.
b, By symmetry, the place of the rook in the first row determines the place of the rook on the last row. Then we can place the second rook in 6 places. The total number of possibilities is $8 \cdot 6 \cdot 4 \cdot 2$.