Discrete Mathematics, week 2

1. (1 point each)

a) Show that a largest element is always maximal.

b) Find an example of a poset with a maximal element but no largest element.

c) Find a poset having no smallest element and no minimal element either, but possessing a largest element.

2. (2 points)
Consider the set $\{1, 2, \ldots, n\}$ ordered by the divisibility relation. What is the maximum possible number of elements of a set $X \subseteq \{1, 2, \ldots, n\}$ that is ordered linearly by the relation? (such a set X is called a chain)

3. (3 points)
Prove that a relation R on a set X satisfies $R \cap R^{-1} = \Delta_X$ if and only if R is reflexive and antisymmetric.

4. (3 points)
Let X, Y, Z be finite sets, let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be relations, and let A_R and A_S be their adjacency matrices, respectively. (A_R has $|X|$ rows and $|Y|$ coloumns and $a_{Rij} = 1$ if $x_i R y_j$ otherwise 0.) Their matrix product is $A_R A_S$. Discover and describe the connection of the composed relation $R \circ S$ to the matrix product $A_R A_S$.

5. (for handing in, 8 points)
Let R be a relation on a set X such that there is no finite sequence of elements x_1, x_2, … x_k of X satisfying $x_1 R x_2$, $x_2 R x_3, \ldots, x_{k-1} R x_k, x_k R x_1$ (we say that such an R is acyclic). Prove that there exists an ordering $\preceq$ on X such that $R \subseteq \preceq$. You may assume that X is finite if this helps.