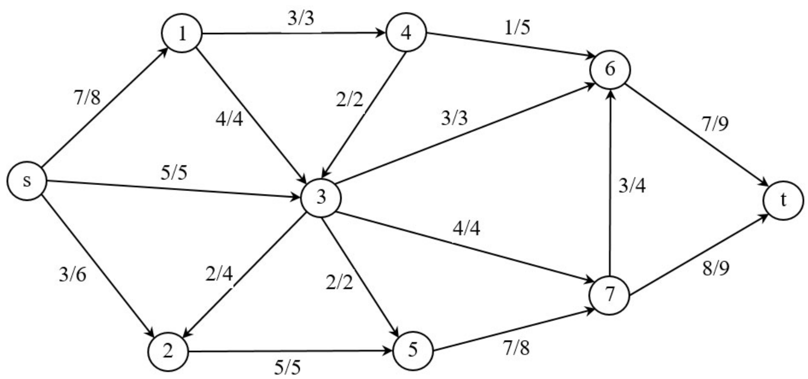
Discrete Mathematics, exercise sheet 13
Solutions

1. (4 points) Consider the following network. We want to find the maximum flow from s to t. In the following picture, on the arcs, the first number shows the flow $f(ij)$ and the second number shows the capacity. For example, on arc $3 \to 2$ the capacity is 4, i.e. $c(v_3v_2) = 4$, and 2 units flow on it i.e. $f(v_3v_2) = 2$.



a) Find an augmenting path from s to t. What are the forward edges and what are the backward edges? With how many units can we increase the value of the flow?
b) After this one augmenting step, is the flow optimal? If yes, find the minimum cut. If not, find all the remaining steps of the Ford-Fulkerson algorithm.

Solution: **a)** The only augmenting path is s-2-3-4-6-t. Forward edges: $s \to 2$, $4 \to 6$, $6 \to t$. Backward edges: $3 \to 2$, $4 \to 3$. We can improve the flow with 2 units.
b) After this step, the flow is optimal. The value of the flow is now $7 + 5 + 5 = 17$. Minumum cut: $A = \{s, 1, 2\}$, $B = \{3, 4, 5, 6, 7, t\}$. The capacity of this cut is $3 + 4 + 5 + 0 + 5 = 17$. Another minimum cut: $A = \{s, 1, 2, 3\}$, $B = \{4, 5, 6, 7, t\}$.

2. (5 points) Prove Kőnig's theorem (in a biparite graph, size of the maximum matching = size of the mininum vertex cover) from the Max-flow Min-cut theorem.

Solution: Let G be a bipartite graph $G = (U, V; E)$. It is easy to see, that if we have a matching of size ν, we need at least ν nodes to cover every edge, thus $\tau \geq \nu$.
Add two vertices to the graph: s and t. Connect s to every vertex in U, and connect t to every vertex in V. Direct the edges from s to U, U to V and V to t. Denote the graph we get this way by $D' = (V', E')$. The capacity of every edge is 1. Since capacities are integers, we can find an integer valued maximum flow.
Let S, T be a minimum cut in this network. ($s \in S$ and $t \in T$) The set of directed edges in this (directed) cut is $Cut(S) = \{ev \in E' : u \in S, v \in T\}$. If there is an uv edge such that $u \in U \cap S$ and $v \in V \setminus S$, move vertex v to S. $S := S \cup \{v\}$. This way, the edge uv is not in the cut anymore. (There may be another wv edges that also leave the cut) Edge vt enters the cut. This way, the number of edges in the cut cannot increase, thus the capacity of the cut cannot increase.
Repeat this step until all the neighbors of $U \cap S$ are in S. Now, the capacity of the cut is $k = |U \setminus S| + |V \cap S|$ and $(U \setminus S) \cup (V \cap S)$ covers all the edges in G.
From the max flow min cut theorem, there is a flow of value k with 0-1 values on the edges. Using the edges with flow value 1 between U and V, we get a matching of size k.
Therefore, we found size k a vertex cover and a size k matching in the original graph, thus $\tau = \nu$.

Second solution: Add two vertices to the graph: s and t. Connect s to every vertex in U, and connect t to every vertex in V. Direct the edges from s to U, U to V and V to t. The capacity of every edge from s or to t 1. The capacity is M for all the edges between U and V, where M is a large integer (it is enough if $M > |V|$.) Let S, T be a minimum capacity cut in this network. ($s \in S$ and $t \in T$). Because of the large capacities on the middle edges, there is no uv edge such that $u \in U \cap S$ and $v \in V \setminus S$. The rest of the proof is the same as in the previous solution.

3. (3 points) $D = (V, E)$ is a directed graph, $s, t \in V$ and $c_1, c_2, \dots, c_k$ are capacity functions on the edges. ($c_i : E \to \mathbb{R}_+$ for every i) Create an algorithm to decide whether there exists a $s\bar{t}$ cut that is a minimum cut for all of these capacity functions.

Solution: Let $c := c_1 + c_2 + \dots + c_k$. Run the Ford- Fulkerson algorithm $k+1$ times, for each of the c_i capacities and also for c as a capacity function.
Let A, B be the minimal cut for c. For any c_i

$$c_i(A, B) \geq \min_{A'B' \text{ is an } st \text{ cut}} c_i(A', B')$$

$$c(A, B) = \sum_{i=1}^{k} c_i(A, B) \geq \sum_{i=1}^{k} \min_{A'B' \text{ is an } st \text{ cut}} c_i(A', B')$$

If the capacity of the minimal cut for c equals the sum of the capacities of the minimal cuts for each c_i, i.e. $c(A, B) = \sum_{i=1}^{k} \min_{A'B' \text{ is an } st \text{ cut}} c_i(A', B')$ then the cut we are looking for exists: (A, B) is a cut like that. It it is not equal, no such cut exists. It is clear that (A, B) cannot be the good cut in this case. If some other (C, D) cut is minimal for each of the c_i capacities, then if $c(A, B) \geq c(C, D)$, so (C, D) would be minimal for capacity c as well.

4. (2 points) How many ways are there to distribute 10 identical balls among 2 boys and 2 girls, if each boy should get at least one ball and each girl should get at least 2 balls? Express the answer as a coefficient of a suitable power of x in a suitable product of polynomials.

Solution: $(x+x^2+x^3+\dots+x^5)(x+x^2+x^3+\dots+x^5)(x^2+x^3+\dots+x^6)(x^2+x^3+\dots+x^6)$ The answer is a coefficient of x^{10} in this product. We may also use $(x+x^2+x^3+\dots)(x+x^2+x^3+\dots)(x^2+x^3+\dots)(x^2+x^3+\dots)$ instead.

$a(x) = (x+x^2+x^3+\dots)(x+x^2+x^3+\dots)(x^2+x^3+\dots)(x^2+x^3+\dots) = x^6(1+x+x^2+x^3+\dots)^4 = x^6 \frac{1}{(1-x)^4}$
From the generalized binomial theorem, $\frac{1}{(1-x)^4} = \binom{3}{3} + \binom{4}{3}x + \binom{5}{3}x^2 + \binom{6}{3}x^3 + \binom{7}{3}x^4 \dots$ So the coefficient we are looking for is $\binom{7}{3} = 35$.

Alternative solution, without generating functions: Give $1+1+2+2$ balls to the boys and girls. We are left with 4 balls that we want to share among 4 people. This can be done by placing 3 separators between 4 objects, so in $\binom{7}{3}$ ways.

5. (2 points) Find the probability that we get exactly 12 points when rolling 3 dice.

Solution:
$a(x) = (x + x^2 + x^3 + \dots + x^6)(x + x^2 + x^3 + \dots + x^6)(x + x^2 + x^3 + \dots + x^6)$. The answer is the coefficient of x^{12} in this product.

$$a(x) = x^3 \left(\frac{1 - x^6}{1 - x}\right)^3 = x^3 \frac{1}{(1-x)^3}(1 - 3x^6 + 3x^{12} - x^{18})$$

From the generalized binomial theorem, $\frac{1}{(1-x)^3} = \binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 \dots$
Therefore, $a(x) = (\binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 \dots)x^3(1 - 3x^6 + 3x^{12} - x^{18})$

We are looking for x^{12}, it appears in $\binom{11}{2}x^9 \cdot x^3$ and in $\binom{5}{2}x^3 \cdot x^3 \cdot (-3x^6)$ thus the coefficient of x^{12} in $a(x)$ is $\binom{11}{2} - 3\binom{5}{2} = 55 - 3 \cdot 10 = 25$.
There are $6^3 = 216$ ways to roll 3 dice, so the probability that the sum is 12 is $\frac{25}{216}$.

6. (4 points) Find generating functions for the following sequences (express them in a closed form, without infinite series):

a) $0, 0, 0, 0, -6, 6, -6, 6, -6, \ldots$

b) $1, 0, 1, 0, 1, 0, \ldots$

c) $1, 2, 1, 4, 1, 8, \ldots$

d) $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

Solution: **a)**

$$6(x^5 - x^4)\frac{1}{1-x^2} = -6x^4\frac{1}{1+x}$$

b)

$$\frac{1}{1-x^2}$$

c)

$$\frac{\frac{1}{1-2x^2} - 1}{x} + \frac{1}{1-x^2} = \frac{1}{x-2x^3} - \frac{1}{x} + \frac{1}{1-x^2} = \frac{2x}{1-2x^2} + \frac{1}{1-x^2}$$

d)

$$(1+x)\frac{1}{1-x^3}$$