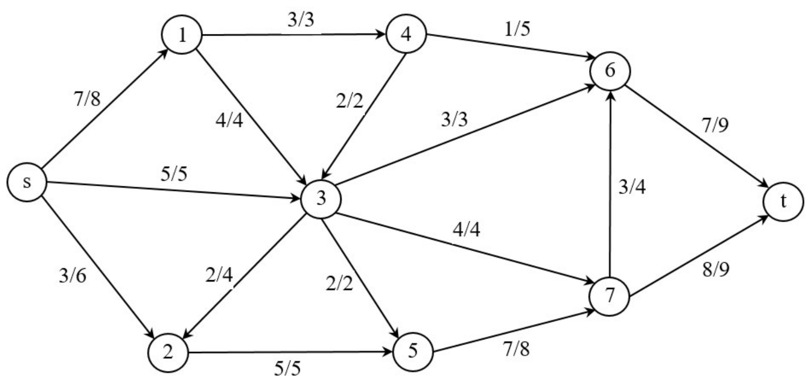
Discrete Mathematics, exercise sheet 13

1. (4 points) Consider the following network. We want to find the maximum flow from s to t. In the following picture, on the arcs, the first number shows the flow $f(ij)$ and the second number shows the capacity. For example, on arc $3 \to 2$ the capacity is 4, i.e. $c(v_3v_2) = 4$, and 2 units flow on it i.e. $f(v_3v_2) = 2$.



a) Find an augmenting path from s to t. What are the forward edges and what are the backward edges? With how many units can we increase the value of the flow?
b) After this one augmenting step, is the flow optimal? If yes, find the minimum cut. If not, find all the remaining steps of the Ford-Fulkerson algorithm.

2. (5 points) Prove Kőnig's theorem (in a biparite graph, size of the maximum matching = size of the mininum vertex cover) from the Max-flow Min-cut theorem.

3. (3 points) $D = (V, E)$ is a directed graph, $s, t \in V$ and $c_1, c_2, \dots, c_k$ are capacity functions on the edges. ($c_i : E \to \mathbb{R}_+$ for every i) Create an algorithm to decide whether there exists a $s\bar{t}$ cut that is a minimum cut for each of these capacity functions.

4. (2 points) How many ways are there to distribute 10 identical balls among 2 boys and 2 girls, if each boy should get at least one ball and each girl should get at least 2 balls? Express the answer as a coefficient of a suitable power of x in a suitable product of polynomials.

5. (2 points) Find the probability that we get exactly 12 points when rolling 3 dice.

6. (4 points) Find generating functions for the following sequences (express them in a closed form, without infinite series):
a) $0, 0, 0, 0, -6, 6, -6, 6, -6, \dots$
b) $1, 0, 1, 0, 1, 0, \dots$
c) $1, 2, 1, 4, 1, 8, \dots$
d) $1, 1, 0, 1, 1, 0, 1, 1, 0, \dots$

7. *This time, there is no hand-in problem.*