Discrete Mathematics, exercise sheet 12
Solutions

Definitions:
Let D be a digraph and $c : A \to \mathbb{R}$. A **potential** is a function $\pi : V \to \mathbb{R}$. We say that π is **feasible** (with respect to c) if $\pi(v) - \pi(u) \leq c(e)$ for every $e \in [u, v]_D$. (We can also write $e = uv$ if there are no parallel edges.)

A cost function $c : A \to \mathbb{R}$ is called **conservative** is there is no negative cost directed cycle.

1. (1 point) Let $\pi : V \to \mathbb{R}$ be everywhere 0, that is, $\pi(v) = 0$ for every $v \in V$. When is this a feasible potential?

Solution: If the cost function is nonnegative: $c(uv) \geq 0$ for every $uv \in A$.

2. (2 points) Show that if a feasible potential exist for a given c, then a nonnegative feasible potential also exists.

Solution: Let $k := \max_{v \in V} |\pi(v)|$ and $\pi'(v) = \pi(v) + k$ for every $v \in V$. This is still a feasible potential, since $(\pi(v) - k) - (\pi(u) - k) = \pi(v) - \pi(u) \leq c(e)$.

3. (1+1+2+2 points) Let D be a digraph and $c : A \to \mathbb{R}$ is a conservative cost function, π_1 and π_2 are feasible potentials. Show that:

- $\pi_1 + 4$ is also a feasible potential.
- $\frac{\pi_1+\pi_2}{2}$ and $\frac{3\pi_1+4\pi_2}{7}$ are feasible potentials.
- $\min(\pi_1, \pi_2)$ is a feasible potential. What about $\max(\pi_1, \pi_2)$?
- $\lfloor \pi_1 \rfloor$ is a feasible potential if c is integer valued. Is it true for $\lceil \pi_1 \rceil$?

Solution:

- same as in Problem 2.
- $\frac{3\pi_1(v)+4\pi_2(v)}{7} - \frac{3\pi_1(u)+4\pi_2(u)}{7} = \frac{3}{7}(\pi_1(v) - \pi_1(u)) + \frac{4}{7}(\pi_2(v) - \pi_2(u)) \leq \frac{3}{7}c(uv) + \frac{4}{7}c(uv) = c(uv)$
- Look at a fixed u and v. Suppose that $\pi_1(u) = \min(\pi_1(u), \pi_2(u))$.

  Then $\min(\pi_1(v), \pi_2(v)) - \min(\pi_1(u), \pi_2(u)) = \min(\pi_1(v), \pi_2(v)) - \pi_1(u) \leq \pi_1(v) - \pi_1(u) \leq c(uv)$. So it is still a feasible potential.

  A similar argument works for the maximum: Suppose that $\pi_1(v) = \min(\pi_1(v), \pi_2(v))$.
  $\max(\pi_1(v), \pi_2(v)) - \max(\pi_1(u), \pi_2(u)) = \pi_1(v) - \max(\pi_1(u), \pi_2(u)) \leq \pi_1(v) - \pi_1(u) \leq c(uv)$.

- $\pi_1(v) - \pi_1(u) \leq c(uv)$, by reorganizing the sides $\pi(v) \leq c(uv) + \pi(u)$.
  $\lfloor \pi_1(v) \rfloor \leq \lfloor c(uv) + \pi(u) \rfloor$ Since c is integer valued, $\lfloor \pi_1(v) \rfloor \leq c(uv) + \lfloor \pi(u) \rfloor$ and this is what we wanted. The same works for $\lceil \pi_1 \rceil$.

4. (3 points) Let D be a digraph, $s, t \in V$ and $c : A \to \mathbb{R}$ is a conservative cost function. We will call an arc $a \in A$ *beautiful* if there is a minimum cost directed $s \to t$ path containing a. Show that if path P is an $s \to t$ path and all of its arcs are beautiful, then P is a cheapest path.

Solution: From Gallai's theorem, there is a π feasible potential for c. From Duffin's theorem,

$$\min\{\tilde{c}(P) : P \text{ is an } s \to t \text{ path}\} = \max\{\pi(t) - \pi(s) : \pi \text{ is a feasible potential}\}$$

Take the optimal feasible potential π from Duffin's theorem, and work with that. We will call an edge uv an "tight edge" if $\pi(v) - \pi(u) = c(e)$.
For any path P' with vertices $s = v_0, v_1, v_2 \ldots t = v_k$,
$c(P') = \sum_{i=0}^{k-1} c(v_i v_{i+1}) \geq \sum_{i=0}^{k-1} (\pi(v_{i+1}) - \pi(v_i)) = \pi(t) - \pi(s)$ If P' is a cheapest path, all of its edges are tight edges. Therefore every beautiful edge is tight, and if we build a path from tight edges, it will be a cheapest path.

5. (2 points) Let D be a digraph $s, t \in V$ and $c : A \to \mathbb{R}$ is a cost function, but it is not everywhere nonnegative. We pick a constant k and make a new *nonnegative* cost function, $c^+(a) = c(a) + k$ for every $a \in A$. Using Dijkstra's algorithm with cost function c^+ do we always get a cheapest $s \to t$ path with respect to the original cost?

Solution: No. For example, D has a path of length 2 with edge weights $-10, -10$ and a path of length 3 with edge weights $-10, -10, -9$, if we add $k = 10$ to them, we have weights $0, 0$ versus $0, 0, 1$. Dijkstra's algorithm picks $0 + 0$ but in the original graph, $-10, -10, -9$ is the cheapest path.

6. (3 points) Let D be a digraph $s, t \in V$ and $S, H \subseteq V$ are $s\bar{t}$ sets with minimal outdegree. Show that $S \cup H$ and $S \cap H$ are $s\bar{t}$ sets with minimal outdegree as well.

Solution: Let k be the outdegree of a $s\bar{t}$ set in D.
We can show that $d^+(S) + d^+(H) \geq d^+(S \cup H) + d^+(S \cap H)$ by looking at all the possible edges that leave S, H, $S \cap H$ or $S \cup H$.
$k + k = d^+(S) + d^+(H) \geq d^+(S \cup H) + d^+(S \cap H) \geq k + k$. There has to be equality thoughout, so $S \cup H$ and $S \cap H$ also have outdegree k.

Menger's theorem (undirected, vertex-version)
Let G be a finite undirected graph and s and t two nonadjacent vertices. Then the size of the minimum vertex cut for s and t (the minimum number of vertices, distinct from s and t, whose removal disconnects s and t) is equal to the maximum number of pairwise internally vertex-disjoint paths from s to t.

7. **For handing in.** (8 points) Prove Hall's theorem from the undirected vertex-version of Menger's theorem.

Solution: Let G be a bipartite graph $G = (A, B; E)$. If there is a matching covering A then for every $X \subseteq A$, $|\Gamma(X)| \geq |X|$.
We will use Menger for the other direction of Hall's theorem. Add two vertices to the graph: s and t. Connect s to every vertex in A, and connect t to every vertex in B. Denote the graph we get this way by G'. Let C be a minimum vertex cut for s and t in G'. Now $\Gamma(A \setminus C) \subseteq B \cap C$ because there cannot be an edge between $A \setminus C$ and $B \setminus C$.
Suppose that for every $X \subseteq A$, $|\Gamma(X)| \geq |X|$.
Then, $|C| = |A \cap C| + |B \cap C| \geq |A \cap C| + |\Gamma(A \setminus C)| \geq |A \cap C| + |A \setminus C)| = |A|$ The size of the minimu vertex cut is at least $|A|$. Using Menger's theorem, there are $|A|$ internally vertex-disjoint paths from s to t. Removing s and t from these paths, we get a matching of size $|A|$ in the original graph, i.e. a matching that covers side A.