Discrete Mathematics, exercise sheet 12

Definitions:
Let D be a digraph and $c : A \to \mathbb{R}$. A **potential** is a function $\pi : V \to \mathbb{R}$. We say that π is **feasible** (with respect to c) if $\pi(v) - \pi(u) \leq c(e)$ for every $e \in [u, v]_D$.

A cost function $c : A \to \mathbb{R}$ is called **conservative** is there is no negative cost directed cycle.

1. (1 point) Let $\pi : V \to \mathbb{R}$ be everywhere 0, that is, $\pi(v) = 0$ for every $v \in V$. When is this a feasible potential?

2. (2 points) Show that if a feasible potential exist for a given c, then a nonnegative feasible potential also exists.

3. (1+1+2+2 points) Let D be a digraph and $c : A \to \mathbb{R}$ is a conservative cost function, π_1 and π_2 are feasible potentials. Show that:

- $\pi_1 + 4$ is also a feasible potential.
- $\frac{\pi_1+\pi_2}{2}$ and $\frac{3\pi_1+4\pi_2}{7}$ are feasible potentials.
- $\min(\pi_1, \pi_2)$ is a feasible potential. What about $\max(\pi_1, \pi_2)$?
- $\lfloor \pi_1 \rfloor$ is a feasible potential if c is integer valued. Is it true for $\lceil \pi_1 \rceil$?

4. (3 points) Let D be a digraph, $s, t \in V$ and $c : A \to \mathbb{R}$ is a conservative cost function. We will call an arc $a \in A$ *beautiful* if there is a minimum cost directed $s \to t$ path containing a. Show that if path P is an $s \to t$ path and all of its arcs are beautiful, then P is a cheapest path.

5. (2 points) Let D be a digraph $s, t \in V$ and $c : A \to \mathbb{R}$ is a cost function, but it is not everywhere nonnegative. We pick a constant k and make a new *nonnegative* cost function, $c^+(a) = c(a) + k$ for every $a \in A$. Using Dijkstra's algorithm with cost function c^+ do we always get a cheapest $s \to t$ path with respect to the original cost?

6. (3 points) Let D be a digraph $s, t \in V$ and $S, H \subseteq V$ are $s\bar{t}$ sets with minimal outdegree. Show that $S \cup H$ and $S \cap H$ are $s\bar{t}$ sets with minimal outdegree as well.

Menger's theorem (undirected, vertex-version)
Let G be a finite undirected graph and s and t two nonadjacent vertices. Then the size of the minimum vertex cut for s and t (the minimum number of vertices, distinct from s and t, whose removal disconnects s and t) is equal to the maximum number of pairwise internally vertex-disjoint paths from s to t.

7. **For handing in.** (8 points) Prove Hall's theorem from the undirected vertex-version of Menger's theorem.