Discrete Mathematics, exercise sheet 11
Solutions

1. (3 points) For natural numbers $m \leq n$, we define a Latin $m \times n$ rectangle as a rectangular table with m rows and n columns with entries chosen from the set $\{1, 2, ..., n\}$ and such that no row or column contains the same number twice. Count the number of all possible Latin $2 \times n$ rectangles.

Solution: $n! \times$ (the number of permutations with no fixed point).
We learned earlier that the number of permutations with no fixed point is $n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$.
So, in total the number of all possible Latin $2 \times n$ rectangles is $n! \cdot n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$.

2. Define a *liberated square* of order n as an $n \times n$ table with entries belonging to the set $\{1, 2, ..., n\}$. Orthogonality of liberated squares is defined in the same way as for Latin squares. For a given number t, consider the following two conditions:
(i) There exist t mutually orthogonal Latin squares of order n.
(ii) There exist $t + 2$ mutually orthogonal liberated squares of order n.
(a) (2 points) Prove that (i) implies (ii).
(b) (4 points) Prove that (ii) implies (i).

Solution: **(a)** To given t orthogonal Latin squares, add one square with all the entries of the i^{th} row equal to i, $i = 1, 2, ..., n$, and one square having all the entries j in the j^{th} column, $j = 1, 2, ..., n$.
(b) In order that a liberated square be orthogonal to another, it has to contain each $i \in \{1, 2, ..., n\}$ exactly n times. Permute entries of the given $t+2$ orthogonal liberated squares (the same permutation in each square) in such a way that the first square has all numbers i in the i^{th} row, $i = 1, ..., n$. Then permute entries inside each row (again, in the same way for all squares) so that the second square has all the j in the j^{th} column. Check that the remaining t squares are Latin.

3. Let X be a finite set and let $\mathcal{M}$ be a system of subsets of X. Suppose that each set in $\mathcal{M}$ has exactly k elements. A 2-coloring a set-system means we color the elements with 2 colors in a way that none of the sets in $\mathcal{M}$ is monochromatic. Let $m(k)$ be the smallest number of sets in a system $\mathcal{M}$ that is not 2-colorable.

(3 points) Prove that $m(4) \geq 15$, i.e. that any system of 14 4-tuples can be 2-colored (distinguishing two cases according to the total number of points.)

Solution: Similar to the proof where we showed that $m(3) \geq 7$.
Case 1: $|X| \leq 14$. If needed, add some nodes, now we have exactly 14 nodes. Color 7 of them white, 7 of them red. There are $\binom{14}{7} = 3432$ such colorings. For a given quadruple, there are $2\binom{10}{3}$ colorings that makes them monochoromatic. (Color this 4 points white, and from the remaining 10, 3 points are white. Same for red.)
For every quadruple, the probability that it is monochormatic is $\frac{2\binom{10}{3}}{\binom{14}{7}}$ The probability that at least one of the 14 quadruples is monochromatic is at most $14 \cdot \frac{2\binom{10}{3}}{\binom{14}{7}} = \frac{14 \cdot 120 \cdot 2}{3432} = \frac{3360}{3432} < 1$. We use the probabilistic method, there has to be a 2-coloring is the set system.
Case 2: $|X| > 14$.
We say that x and y are connected if there exists a set $M \in \mathcal{M}$ containing both x and y. If x and y are points that are not connected, we define a new set system $(X', \mathcal{M}')$ arising by "gluing" x and y together. The points x and y are replaced by a single point z, and we put z into all sets that previously contained either x or y. If a "glued" set system is 2-colorable, then the original is also 2-colorable.
We claim there are 2 points that are not connected. Every quadruple makes 6 point-pairs connected. There are at most $14 \cdot 6$ connected pairs, and the total number of pairs is at least $\binom{15}{2}$. Since $14 \cdot 6 < \binom{15}{2}$, so there are 2 points that are not connected. Do the gluing steps until we reach $|X| = 14$.

Note: this solution also works if the two cases are $|X| \leq 13$ and $|X| > 13$.

4. (3 points) We have 27 fair coins and one counterfeit coin, which looks like a fair coin but is a bit heavier. Show that one needs at least 4 weighings to determine the counterfeit coin. We have no calibrated weights, and in one weighing we can only find out which of two groups of some k coins each is heavier, assuming that if both groups consist of fair coins only the result is an equilibrium.

Solution: Each weighing has 3 possible outcomes, and hence 3 weighings can only distinguish one among 27 possibilities.

5. (5 points) We toss a fair coin n times. What is the expected number of *runs*? Runs are consecutive tosses with the same result. For instance, the toss sequence HHHTTHTH has 5 runs. (HHH, TT, H, T, H). (Tip: It is better to count boundaries between runs.)

Solution: It's better to count boundaries among runs. The probability that a given position between two tosses is a boundary is $\frac{1}{2}$. The first toss can be anything. For any of the next $n-1$ tosses, there is a $\frac{1}{2}$ change we start a next run.
Let X be the random variable that counts the number of changes, and let A_i be the event that there is a change in the i^{th} gap. I_{A_i} is an indicator random variable. $I_{A_i} = 1$ if there is a change, and 0 if not. By the additivity of expected value,

$$E[X] = \sum_{i=1}^{n-1} E[I_{A_i}] = \sum_{i=1}^{n-1} P(A_i) = (n-1)\frac{1}{2} = \frac{n-1}{2}$$

There are one more runs than boundaries, therefore the expected number of the number of runs is $1 + \frac{n-1}{2} = \frac{n+1}{2}$.