Discrete Mathematics, exercise sheet 11

1. (3 points) For natural numbers $m \le n$, we define a Latin $m \times n$ rectangle as a rectangular table with m rows and n columns with entries chosen from the set $\{1, 2, ..., n\}$ and such that no row or column contains the same number twice. Count the number of all possible Latin $2 \times n$ rectangles.

2. Define a *liberated square* of order n as an $n \times n$ table with entries belonging to the set $\{1, 2, ..., n\}$. Orthogonality of liberated squares is defined in the same way as for Latin squares. For a given number t, consider the following two conditions:
(i) There exist t mutually orthogonal Latin squares of order n.
(ii) There exist $t + 2$ mutually orthogonal liberated squares of order n.
(a) (2 points) Prove that (i) implies (ii).
(b) (4 points) Prove that (ii) implies (i).

3. Let X be a finite set and let $\mathcal{M}$ be a system of subsets of X. Suppose that each set in $\mathcal{M}$ has exactly k elements. A 2-coloring a set-system means we color the elements with 2 colors in a way that none of the sets in $\mathcal{M}$ is monochromatic. Let $m(k)$ be the smallest number of sets in a system $\mathcal{M}$ that is not 2-colorable.

(3 points) Prove that $m(4) \ge 15$, i.e. that any system of 14 4-tuples can be 2-colored (distinguishing two cases according to the total number of points.)

4. (3 points) We have 27 fair coins and one counterfeit coin, which looks like a fair coin but is a bit heavier. Show that one needs at least 4 weighings to determine the counterfeit coin. We have no calibrated weights, and in one weighing we can only find out which of two groups of some k coins each is heavier, assuming that if both groups consist of fair coins only the result is an equilibrium.

5. (5 points) We toss a fair coin n times. What is the expected number of *runs*? Runs are consecutive tosses with the same result. For instance, the toss sequence HHHTTHTH has 5 runs. (HHH, TT, H, T, H). (Tip: It is better to count boundaries between runs.)

6. *This time, there is no hand-in problem.*