Discrete Mathematics, exercise sheet 10

Reminder: In a block design: The town has v inhabitants; they organize b clubs; every club has the same number of members, say k; everybody belongs to exactly r clubs, and for any pair of citizens, there are exactly λ clubs where both of them are members.
We know that $bk = vr$, $\lambda(v-1) = r(k-1)$, and $b \geq v$.

1. (3 points) Prove that if we color the points of the Fano plane with 2 colors, there will be line where all three points have the same color.

Solution: Color the point with red and blue. We call a 2-coloring "good" if every line contains both red and blue points. Suppose for contradiction that there exists a *good* coloring on the Fano plane. The red points have to cover all the lines, (and same for the blue points). If there is only 1 red point, it covers three lines, if there are 2 red points, they cover 5 lines (every point is on 3 lines and these 2 points have one line in common). By symmetry, coloring with 5 red and 2 blue, or 6 red and 1 blue lines cannot be *good* either.
Suppose there are 3 red and 4 blue points. If the 3 red point are on one line, the coloring is not *good*, since there is an all red line. So the 3 red points are not collinear. They cover the three lines passing though any pair of the 3 red points, and 3 more lines since every point has degree 3. In total, the red points covered 6 lines, so there has to be an all blue line. Therefore the Fano plane cannot have any *good* coloring.

2. (2 points) In a town, there are 924 clubs, and every club has 21 members. Every 2 people can meet each other in exactly 2 clubs. How many inhabitants are in this town? One person is a member of how many clubs?

Solution: $bk = vr$, so $924 \cdot 21 = vr$.
$\lambda(v-1) = r(k-1)$, so $2(v-1) = 20r$. From this $v = 10r + 1$ Putting this into the first equation $924 \cdot 21 = 19404 = (10r+1)r = 10r^2 + r$. Solving this quadratic equation we get that ther are 441 people, and everyone is a member of 44 clubs.

3. (4 points) In a town, the clubs form a block design and every club has a badge. On a big event, everyone from the town is present, and everyone wears a badge of a club he/she is a member of. (Each person wears only one badge.) Is it always possible that everyone wears different badges?

Solution: First, we need that there are enough different badges, at least as many as citizens. That is, $b \geq v$. This is indeed guaranteed by Fisher's Inequality.
We assign a bipartite graph to our block design. represents the people (this side has v points); the upper set of points represents the clubs (this side has b points). We connect point a to point X if citizen a is a member in club X. Choose a subset A of citizens, $|A| = n$, the set of clubs that someone from A is a member of is $\Gamma(A)$. $|\Gamma(A)| = m$. We want to use Hall's theorem, we claim that $m \geq n$.
Every citizen node has degree r, every club node has k. All the edges from A are also edges from $\Gamma(A)$, therefore $nr \leq mk$.
We know from a lemma that $bk = vr$, and $b \geq v$, therefore $k \leq r$. So $mk \leq mr$.
Therefore $nr \leq mk \leq mr$, $nr \leq mr$ thus $n \leq m$. From Hall's theorem, everyone can wear a different badge.

4. (2 points) Show that in a block design with $k = 3$ and $\lambda = 1$, the residue of v divided by 6 is 1 or 3.

Solution: We learned that $bk = vr$ and $\lambda(v-1) = r(k-1)$. For this special case, $3b = vr$ and $v - 1 = 2r$. And hence $r = \frac{v-1}{2}$ and $b = \frac{v(v-1)}{6}$ The numbers r and b must be integers, v is an odd number, so if we divide it by 6, the remainders can be 1, 3, or 5. Furthermore, v can not be of the form $6j + 5$, because then $b = \frac{(6j+5)(6j+4)}{6} = 6j^2 + 9j + 3 + \frac{1}{3}$ which is not an integer.
Block desings like these are called *Steiner systems*

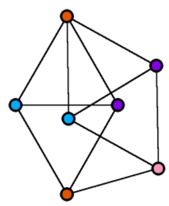
5. (3 points) Can you create a block design with the following parameters? $v = 13, k = 3, \lambda = 1$.

Solution: Yes, it is possible.
If we can partition the egdes of a complete graph on 13 nodes to disjoint triangles, we get the desired block design.

Take 13 points on a cycle, that form a regular 13-gon. Number the nodes from 0 to 13. One of the clubs is the triangle $\{0, 1, 4\}$, another club is triangle $\{0, 2, 7\}$. Select all the triangles we get by rotating $\{0, 1, 4\}$ and $\{0, 2, 7\}$ around the center of the cycle. These are also clubs. We claim we got a partition of K_{13} into triangles.
We say two points have distance k if they are k steps away from each other on the cycle. Here, distance 6 is the same as distance 7. Note that the sides of triangle $\{0, 1, 4\}$ have distance 1, 3, and 4, and the sides of triangle $\{0, 2, 7\}$ have distance 2, 5, and 6. With the rotation method, every edge is incuded in exactly one traingle.

6. (3 points) We color the points of the $\mathbb{R}^2$ plane with 3 colors. Show that there are two points such that their distance is 1, and they have the same color.

Solution: Suppose we can color the plane with 3 colors such that two points of distance 1 always have different color. If we build two equilateral unit triangles together, the opposite points (two points that have distance $\sqrt{3}$) should have the same color. The following picture shows, that there two points that should have the same color by the previous logic, but their distance is 1. Contradiction, we cannot color the plane with 3 colors.



7. **For handing in.** (8 points) In a group, everyone has 3 friends. (We assume that friendship is mutual.) If A and B are not friends, there is exactly one person in the group that they are both friends with. If A and B are friends, then they do not have a common friend in the group. Is this situation possible? If it is possible, how many people are in the group?

Solution: Represent it with a graph, the nodes are the people, two nodes are connected by and edge if the two endpoints are friends. The situation in the problem is possible, an example is the Petersen graph. Let n be the number of nodes, and e the number of edges. We know $n = 10$ is possible, and we want to show that this is the only possible size.
Let us count the number of "cherries" in the graph. If A and B are not friends, there is exactly one person in the group that they are both friends with, if they are friends, zero. Thus the number of cherries is $\binom{n}{2} - e$ (counted by the legs of the cherry). On the other hand, the graph is 3-regular, so the number of cherries is $3n$ (counted by the head of the cherry).
$\binom{n}{2} - e = 3n$
Since the graph is 3-regular, $e = 3n/2$.
$\binom{n}{2} - \frac{3n}{2} = 3n$
$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{9n}{2}$
$n - 1 = 9$
$n = 10$. There are 10 people in the group.

Second solution: From the conditions, there are no C_3 or C_4 in the graph. Person A has three friends B, C, D. They cannot be friends of each other, and cannot have a common friend who is not A, so they each have 2 new friends: E, F, G, H, I, J. This gives $1 + 3 + 6$ people, so there has to be at least 10 people in the group. We can connect E, F, G, H, I, J to each other in a way that satisfies all the conditions, so we get a good construction (which is isomorphic to the Petersen graph). Suppose there are more than 10 people. Then the 11^{th} person cannot be friends with A, B, C or D (they are "full", already have 3 friends). The 11^{th} person and A are not friends and do not have a common friend. Contradiction.