Discrete Mathematics **Comprehensive Exam 2**

17.09.2019. (100 pts, 120 mins)

Using any written material, calculators or mobile phones is not allowed. Please turn off your phone.
Use only paper and pen.
You can use any theorems or statements from the lecture (without proof) if you state them properly. Except if the exercise is to prove that theorem.

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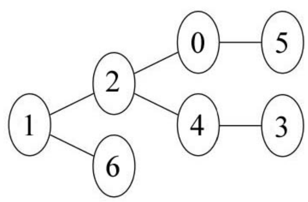
1. (4+6+6 points)
Alice wants to send postcards to 12 friends. In the shop there are only 3 kinds of postcards. In how many ways can she send the postcards, if
a) there is a large number of each kind of postcard, and she wants to send one card to each friend;
b) there is a large number of each kind of postcard, and she is willing to send one or more postcards to each friend (but no one should get two identical cards);
c) the shop has only 4 of each kind of postcard, and she wants to send one card to each friend?

Solution: **a)** 3 kinds of cards, and I can choose freely in each step: 3^{12}
b) every friend receives a nonempty set of the 3 cards: 7 options for each friend, so 7^{12} in total.
c) permutation with repetition: she has 12 cards to send to 12 people, but 4+4+4 them are identical. $\frac{12!}{4!^3}$

2. (6+8 points)
a) Give the Prüfer code of the following tree.



b) Draw the tree with the Prüfer code 5 2 1 5 0 1 3.

Solution: **a)** 4 2 0 1 2
b) From the decoding method, the long Prüfer code is
4 6 2 7 5 8 1 3
5 2 1 5 0 1 3 0
From this we can draw the tree. Consider the path 4-5-0-3-1-2-6 and add two more edges: 5-7 and 1-8.

3. (10 points)
Let G be a simple graph, such that each vertex of G has degree 3. Show that if the edges of G cannot have a proper edge-coloring with 3 colors, then G does not have a Hamiltonian cycle.

Solution: First, since every degree is 3, and we know that the sum of the degrees (here $3n$) is twice the number of the edges, the graph has an even number of vertices.
Suppose the graph contains a Hamiltonian cycle, this cycle has an even lenght. Color the edges of the cycle red and blue in an alternating way. Since G is 3-regular, if we remove the Hamiltonian cycle from the graph, the remaining part is a perfect matching, color the edges of this matching green. This way we got a 3-coloring of the edges.

4. (15 points)
Prove Mirsky's theorem: For any finite partially ordered set P, $\omega(P)$ equals the minimum number of antichains into which the poset may be partitioned.
(Here, $\omega(P)$ denotes the maximum length of a chain in the poset P.)

Solution:
In such a partition, every two elements of the longest chain must go into two different antichains, so the number of antichains is always greater than or equal to the height.
We want to show that $\omega(P)$ antichains are enough. For any $x \in X$, define $l(x)$ as the size of the longest chain whose greatest element is x. Define A_i as $A_i := \{x \in X : l(x) = i\}$.
$A_1 \cup \cdots \cup A_{\omega(P)}$ is a partition of X into $\omega(P)$ mutually disjoint sets.
We show that very A_i is an antichain. Suppose that A_i is not an antichain, exists two points x, $y \in A_i$ so that $x < y$. Take the longes chain to x, and add y. This is a chain of lenght $l(x)+1$ whose greatest element is y. This implies $l(x) < l(y)$, contradiction.

5. (15 points)
In how many ways can we get exactly 25 points when rolling four 12-sided dice?

Solution: Take the generating function $a(x) = (x + x^2 + x^3 + \cdots + x^{12})^4$. The answer is the coefficient of x^{25} in this product.

$$a(x) = x^4 \left(\frac{1-x^{12}}{1-x}\right)^4 = x^4 \frac{1}{(1-x)^4}(1 - 4x^{12} + 6x^{24} - 4x^{36} + x^{48})$$

From the generalized binomial theorem, $\frac{1}{(1-x)^4} = \binom{3}{4} + \binom{4}{3}x + \binom{5}{3}x^2 + \binom{6}{3}x^3 \ldots$
Therefore, $a(x) = (\binom{3}{3} + \binom{4}{3}x + \binom{5}{3}x^2 + \binom{6}{3}x^3 + \ldots)x^4(1 - 4x^{12} + 6x^{24} - 4x^{36} + x^{48})$

We are looking for x^{25}, it appears in $\binom{24}{3}x^{21} \cdot x^4$ and in $\binom{12}{3}x^9 \cdot x^4 \cdot (-4x^{12})$ thus the coefficient of x^{25} in $a(x)$ is $\binom{24}{3} - 4\binom{12}{3}$.
(With numbers, $\binom{24}{3} - 4\binom{12}{3} = 2024 - 4 \cdot 220 = 1144$.)

6. (15 points)
Let $G = (A, B; E)$ be a bipartite graph with the same number of nodes on both sides. Suppose that every nonempty subset $X \subseteq A$ has at least $|X| - 1$ neighbors. Prove that G contains a matching that matches up all but one node on each side.

Solution: Add one more vertex to side B and connect this vertex to eveny vertex on side A. Then every every nonempty subset $X \subseteq A$ has at least $|X|$ neighbors. Use Hall's theorem, there is a matching covering side A. Remove the edge in the matching that contains the newly added vertex, we still have an matching that covers all but one nodes on side A. Since originally the two sizes has the name number of nodes, this matching also covers all but one nodes on the original side B.

7. (15 points)
The inhabitants of a town form clubs. Every club has exactly 3 people, and every two people meet in exactly one club. If the town has v inhabitants, show that the residue of v divided by 6 is 1 or 3.

Solution: We can show that every person is in the same number of clubs. Everyone should meet all the $v-1$ other inhabitants, and in one club you can meet two other people, so every person a member of $r = \frac{v-1}{2}$ clubs. This is a block design with $k = 3$, $\lambda = 1$. Let b be the number of the clubs.
We learned that $bk = vr$ and $\lambda(v-1) = r(k-1)$. For this special case, $3b = vr$ and $v - 1 = 2r$. And hence $r = \frac{v-1}{2}$ and $b = \frac{v(v-1)}{6}$. The numbers r and b must be integers, thus v is an odd number, so if we divide it by 6, the remainders can be 1, 3, or 5. Furthermore, v can not be of the form $6j+5$, because then $b = \frac{(6j+5)(6j+4)}{6} = 6j^2 + 9j + 3 + \frac{1}{3}$ which is not an integer.