Discrete Mathematics **Comprehensive Exam**

16.07.2019. (100 pts, 120 mins)

Using any written material, calculators or mobile phones is not allowed. Please turn off your phone. Use only paper and pen.
You can use any theorems or statements from the lecture (without proof) if you state them properly. Except if the exercise is to prove that theorem.
Grading: minimum points needed for each grade

1.0	1.3	1.7	2.0	2.3	2.7	3.0	3.3	3.7	4.0	5.0
90	84	78	72	66	60	55	50	45	40	0

---

1. (5+5 points)
a) How many five digit numbers are there with 4 even and 1 odd digits?
b) How many six digit numbers are there such that all of its digits are different, and it has 4 even and 2 odd digits?

Solution:
a) The number cannot start with zero. If it starts with an even digit: $\binom{4}{1} \cdot 4 \cdot 5^4$
If it starts with an odd digit: $1 \cdot 5^5$
In total: $4 \cdot 4 \cdot 5^4 + 5^5 = 5^4(16+5) = 10000 + 3125 = 13125$

b) If it starts with an even digit: $\binom{5}{2} \cdot 4 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4$
We can place the 2 odd digits in two places out of the last 5 in $\binom{5}{2}$ ways. For the first digit we can choose from 4 possibilities (zero is not allowed) for the second even digit we can choose from 4 possibilites again (five minus the digit that was used for the first digit) for the next even digit we can choose from 3 possibiliteis and so on. For the odd digits: $5 \cdot 4$ possibilities.
If it starts with an odd digit: $\binom{5}{1} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4$
In total: $\binom{5}{2} \cdot 4 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 + \binom{5}{1} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 = 5!(160+100) = 19200 + 12000 = 31200.$

Second solution Calculate all numbers, then substract those that start with zero.
a) $\binom{5}{1} \cdot 5^5 - \binom{4}{1} \cdot 5^4 = 5^6 - 4 \cdot 5^4 = 5^4 \cdot (25-4) = 5^4 \cdot 21$
b) $(\binom{6}{2} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4) - (\binom{5}{2} \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4) = 5!(15 \cdot 5 \cdot 4 - 10 \cdot 4) = 5! \cdot 260 = 31200$

2. (3+8+3 points)
a) What is the definition of a tree?
b) Prove that if an undirected finite graph has n vertices and $n-1$ edges and it is connected, then it is a tree.
c) How many trees are there on n labelled vertices? (without proof)

Solution: **a)** A graph is a *tree* if it is connected and does not contain a cycle.
b) If the graph G contains a cycle, remove an edge of the cycle. Repeat this until we cannot remove edges anymore. From a Theorem in the Lecture, "A graph is a tree if and only if it is connected, but deleting any of its edges results in a disconnected graph." Therefore we reached a tree (which is a spanning tree T of G). The spanning tree has $n-1$ edges, thus $T = G$, G is a tree.
c) Cayley's formula: For every positive integer n, the number of trees on n labelled vertices is n^{n-2}.

3. (4+14 points)
a) Describe Kruskal's algorithm.
b) Prove that Kruskal's algorithm always gives a minimum cost spanning tree.

Solution:

a) List the edges in the increasing order of cost: $c(e_1) \le c(e_2) \le \ldots c(e_m)$.
At the start of the algorithm $T = (V, \emptyset)$
For $i = 1 \ldots m$
If $T + e_i$ does not contain a cycle, let $T := T + e_i$. (Otherwise skip that edge, and T is unchanged)

b)
Suppose the Kruskal algorithm gives T, and there is a spanning tree H such that $c(H) < c(T)$.
Let us imagine the process of constructing T, and the step when we first pick an edge that is not an edge of H. Let e be this edge. If we add e to H, we get a cycle C. This cycle is not fully contained in T, so it has an edge f that is not an edge of T. If we add the edge e to H and then delete f, we get a (third) tree H'. (Why is H' a tree? We removed and edge of cycle C, so H' is still connected, and it has $n-1$ edges.) We want to show that H' is at most as expensive as H. This clearly means that e is at most as expensive as f. Suppose (by indirect argument) that f is cheaper than e.
Now comes a crucial question: Why didn't the optimistic government select f instead of e at this point in time? The only reason could be that f was ruled out because it would have formed a cycle C' with the edges of T already selected. But all these previously selected edges are edges of H, since we are inspecting the step when the first edge not in H was added to T. Since f itself is an edge of H, it follows that all edges of C' are edges of H, which is impossible, since H is a tree. This contradiction proves that f cannot be cheaper than e and hence H cannot be cheaper than H'.

So we replace H by this tree H' that is not more expensive. In addition, the new tree H' has the advantage that it coincides with T in more edges, since we deleted from H an edge not in T and added an edge in T. This implies that if H' is different from T and we repeat the same argument again and again, we get trees that are not more expensive than H, and coincide with T in more and more edges. Sooner of later we must end up with T itself, proving that T was no more expensive than H.

4. (10 points)
State the Max flow - Min cut theorem. (without proof)

Solution:

We have a directed graph $D = (V, E)$, $s, t \in V$ are special vertices (source and target) and there is a non-negative function $c : E \to \mathbb{R}_+$, called the capacity. We call $N := (D, s, t, c)$ a network.
A function $f : E \to \mathbb{R}_+$ is a (feasible) flow, if it satisfies:

- $0 \le f(ij) \le c(ij)$ for every $ij \in E$
- Kirchhoff's law: $\sum_{j:ji \in E} f(ji) = \sum_{j:ij \in E} f(ij)$ for every vertex $i \in V$ such that $i \ne s$ and $i \ne t$.

The value of a flow is $v(f) = \sum_{j:sj \in E} f(sj) - \sum_{js: \in E} f(js)$.
This is the amount of flow leaving s minus the amount of flow entering s.
The capacity of the cut defined by A, B, where $A, B \subseteq V$, $A \cap B = \emptyset$, $A \cup B = V$ $s \in A$ and $t \in B$ is

$$c(A, B) = \sum_{ij \in E, i \in A, j \in B} c(ij)$$

Theorem 1 (Max Flow Min Cut, Ford- Fulkerson)**.** *In every network, the maximum value of a flow equals the minimum capacity of a cut.*

$$\max_{f \text{ is a flow}} v(f) = \min_{A,B \text{ is an } s\bar{t} \text{ cut}} c(A, B)$$

5. (14 points)

How many ways are there to seat n married couples at a round table with $2n$ chairs in such a way that no husband and wife sit next to each other? (Solutions that can be transformed to each other via rotation or reflection still count as different solutions.)

Solution: Define A_i as the set of all ways of seating in which the i^{th} couple is adjacent. Use the inclusion-exclusion principle

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \ldots (-1)^{n-1}|A_1 \cap A_1 \cap \ldots A_n|$$

The total number of possible seatings, without the condition in the problem, is $(2n)!$.

$|A_i| = 2\cdot(2n)\cdot(2n-2)!$ The i^{th} couple can sit on $2n$ "double seats", and the two of them can sit in 2 possible orders, and the remaining $2n-2$ people can sit in $(2n-2)!$ ways.

$|A_i \cap A_j| = 2^2 \cdot (2n)\cdot(2n-3)\cdot(2n-4)! = 2^2\cdot(2n)\cdot(2n-3)!$ The i^{th} couple can sit on $2n$ "double seats" then the i^{th} couple can sit on $2n-3$ double seats, and the remaining $2n-4$ people can sit in $(2n-4)!$ ways. Inside the couples they can switch places 2^2 ways.

For k couples: imagine if a couple sits next to each other, we merge them together into "one person". Then, we need to sit down $2n-k$ people, that can be done in $(2n-k)!$ ways. But, actually, the seats are numbered $1, 2, 3\ldots 2n$, and if a couple sat on $2n$ and 1 then this does not show up in the previous counting. This way we get additional $k\cdot(2n-k-1)!$ possibilies: there are k possibilities which couple sits at seats $2n, 1$ and $(2n-k-1)!$ ways to put the other people/pairs.

Thus, $|A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}| = 2^k\cdot((2n-k)!+k\cdot(2n-k-1)!) = 2^k\cdot(2n-k-1)!((2n-k)+k) = 2^k\cdot(2n-k-1)!\cdot 2n$. The number of good seatings is

$$(2n)! - \left|\bigcup_{i=1}^{n} A_i\right| = (2n)! - \left(\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\cdot 2^k\cdot(2n-k-1)!\cdot 2n\right)$$

6. (10 points)

At a dance party, 10 boys and 10 girls are present. We want to organize rounds, in each round 10 pairs dance. Any boy and any girl dance with each other only once. Four rounds already happened. Show that we can organize the participants into pairs for the remaining 6 rounds.

Solution:

We can create a bipartite graph, there is an edge between a boy and a girl if they have not danced yet. In this graph every node has degree 6. Use a theorem from the lecture:

Theorem 2. *If every node of a bipartite graph has the same degree $d \ge 1$, then it contains a perfect matching.*

We found a perfect matching, that is one round. Remove this matching, now every degree is 5, we find a matching again. Repeat this step until the graph is empty.

7. (12 points)
There are 10 red balls, 10 blue balls, and 10 green balls.
In how many different ways can you pick 16 balls such that there is at least one ball for each color?

Solution: $a(x) = (x + x^2 + x^3 + \cdots + x^{10})(x + x^2 + x^3 + \cdots + x^{10})(x + x^2 + x^3 + \cdots + x^{10})$. The answer is the coefficient of x^{16} in this product.

$$a(x) = x^3 \left(\frac{1 - x^{10}}{1 - x}\right)^3 = x^3 \frac{1}{(1 - x)^3}(1 - 3x^{10} + 3x^{20} - x^{30})$$

From the generalized binomial theorem, $\frac{1}{(1-x)^3} = \binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 \ldots$
Therefore, $a(x) = (\binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 \ldots)x^3(1 - 3x^{10} + 3x^{20} - x^{30})$

We are looking for x^{16}, it appears in $\binom{15}{2}x^{13} \cdot x^3$ and in $\binom{5}{2}x^3 \cdot x^3 \cdot (-3x^{10})$ thus the coefficient of x^{16} in $a(x)$ is $\binom{15}{2} - 3\binom{5}{2} = 105 - 3 \cdot 10 = 75$. We can select the balls in 75 ways.

8. (12 points)
The inhabitants of a town like to form clubs. Every club has the same size. Each citizen A must behave "equally" toward citizens B and C, so A must meet B in the same number of clubs as she meets C.
Show that this implies that all the people are members of the same number of clubs.

Solution: The town has v inhabitants. Every club has k members, and any two people meet in λ clubs. (We assume that $k \geq 2$).
Citizen A is a member of r_A clubs. She has $(v-1)\lambda$ "meetings", because she meets any of the other $v-1$ citizens λ times. On the other hand, in a given club, she meets $k-1$ people, therefore $r_A = \frac{(v-1)\lambda}{k-1}$. We get the same result for any citizen, so everyone is the member of the same number of clubs.