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# Structure of generalized stable matchings

Ph.D. thesis

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# Contents

# Chapter 1

# Introduction

The history of stable marriages started in 1962 with the paper of Gale and Shapley. There they described a scenario, where $n$ men and $n$ women are given, and each of them has a preference order on the members of the other gender. We want to arrange marriages between them, in such a way that there is no blocking pair. A blocking pair means a couple of a man and woman who are not married to each other, but they would both leave their current spouses to be with each other. Gale and Shapley proved [24] that there always exists a stable marriage scheme, and it can be found with the so-called deferred acceptance algorithm.

Since then, many papers were published on stable matching, and some books to summarize the results: Knuth [36], Gusfield and Irving [26], Roth Sotomayer [44] and the most recent by Manlove [37].

The output of the Gale-Shapley algorithm is a man-optimal solution, that is, each man receives the best possible partner he could find in any of the stable marriage schemes. If we change the role of the genders in the algorithm, we get a woman-optimal stable matching. In these stable marriage schemes, one side of the marriage market receives the best and the other side the worst possible partners. An observation attributed to Conway generalizes man and woman optimality. It states that stable marriages form a complete lattice for the partial order defined by the men. That is, if $S_1$ and $S_2$ are two stable marriage schemes and each man picks the better out of his partners in $S_1$ and $S_2$, then these choices determine another stable marriage scheme. Similarly, if men choose their less preferred partners, we also get a stable marriage scheme.

The second well-known model is for college admissions, where we seek a stable arrangement of applicants to universities. Every student can go to only one college, but colleges can accommodate multiple students. These types of models are often referred as many-to-one matchings. Another application is allocating medical residents to hospitals. The National Resident Matching Program (NRMP), also called the Match, which largely follows the Gale-Shapley algorithm has been in use in the USA since 1952, ten years prior to the Gale-Shapley paper.

The observation of the lattice property of stable solutions also applies to colleges and students. It is important for both the marriage and the college models that each agent on the market has strict preference orders. If we allow ties in the preference orders, then there are three well-known extensions of stability: we can talk about weakly stable, strongly stable and superstable solutions.

In Hungary, the college admission system is a little different from the above model. Each student submits a set of applications to different colleges and declares a linear preference order over these applications. Each college has a strict quota on the number of admissible students. A score is assigned to each application based on entrance exams. After all this information is known, each college declares a score limit, and each student is accepted at the first college on her preference list, where her score is not below the appropriate limit. These score limits have to be stable; that is, no college receives more students than its quota. Moreover, each college would receive more students than its quota if it lowers its score limit, while the other ones keep theirs. An admission scheme that fulfills the properties above is called a "score-stable" solution. In this model, many of students may have the exact same score at a given college, so the preference lists of colleges contain ties. In the traditional many-to-one model, the colleges had strict preference orders.

Our models can also be described with choice functions, as in the paper of Kelso-Crawford [34]. For an agent in the matching market, if we offer her any given set of possible partners, she will pick her most preferred subset. In this way her preferences define a set-function, the so-called choice function. Choice functions can have useful properties such as substitutability and path independence.

A result of Blair in [11] generalizes Conway's observation, by proving that if both sides of the matching market have so-called substitutable and IRC choice functions, then stable solutions form a lattice under a natural partial order. Here IRC is an abbreviation for "irrelevance of rejected contracts". It seems that in the literature, most of the practical, interesting stability notions involve the irrelevance of rejected contracts. Sometimes authors define a choice function with a preference order over all possible subsets, and this implies IRC.

However, the Hungarian college entrance mechanism is an important example of this, which outputs a stable solution even though the choice functions are certainly not IRC. We shall generalize Blair's theorem for models involving non-IRC choice functions. It turns out that if we drop the IRC property, then it is not clear what is exactly a stable solution. For this reason, we study four kinds of stabilities: dominating stability, three-stability (which is defined by a three-partition of the contract set), four-stability (which comes from a four-partition of the contract set) and score-stability. This last notion is also generalized to so-called "loser-free" choice functions, allowing us to work with more flexible models describing diverse market situations, like company-worker admissions, with no strict preference ordering on the company's side.

We compare the above four different stability notions. We shall examine the connections between the definitions in regard to the path-independent property. Aygün and Sönmez [6] showed that if $\mathcal{F}$, $\mathcal{G}$ are substitutable and IRC choice functions, then three-stable and dominating stable sets are equivalent, but if $\mathcal{F}$ and $\mathcal{G}$ are not IRC, then none of them implies the other. We extend this by considering the other two stabilities (four-stability and score-stability), as well.

It is possible to generalize choice functions in an even more abstract way. All the subsets of a given set form a (distributive) lattice for the partial order defined by one set containing another. For a choice function $\mathcal{F}$, $\mathcal{F}(X) \subseteq X$ for every $X$. We can define a function over any lattice, and call it a choice function if $\mathcal{F}(x) \leq x$ for every $x$. Important notions on choice functions can be generalized to this setting, and we obtain some interesting results in this way.

A theorem by Sands, Sauer and Woodrow [45] says that two partially ordered sets $P_1$ and $P_2$ on the same ground set admit a common antichain $K$ dominating every element not in $K$. By domination, we mean that for every $e \notin K$, there is an $x \in K$ such that $x$ is greater than $e$ in least one of the two partial orders. Aharoni, Berger and Gorelik [2] generalized this to a weighted setting. In a given poset, every element has a demand and a weight, (both are integers or real numbers) and an element is dominated if there is a chain above it such that the sum of weights on the chain is at least the demand of this element. With the help of choice functions over lattices, we can generalize the Aharoni-Berger-Gorelik theorem even more.

In the second half of the thesis, we study supply chain structures, which can be used to model contract relationships between firms. In our model, firms have heterogeneous preferences over bilateral contracts with other firms. Contracts may encode many dimensions of a relationship including the quantity of a good being traded, time of delivery, quality, and price. The universe of possible relationships between firms is described by a *contract network* – a multi-sided matching market in which firms form downstream contracts to sell outputs and upstream contracts to buy inputs.

We focus on the existence and structure of stable outcomes in decentralized, real-world matching markets. In the production networks that we consider, stable outcomes play the role of an equilibrium concept and may serve as a reasonable prediction of the outcome of market interactions [23]. We find a general result: any contract network has an outcome that satisfies a natural extension of *pairwise stability* in the marriage market [24]. Our model of matching markets includes many previous models of matching with contracts, including many-to-one and many-to-many matching markets.

We build on the seminal contribution by Ostrovsky [38], who introduced a matching model of *supply chains*. In a supply chain, there are agents who only supply inputs (e.g. farmers); agents who only buy final outputs (e.g. consumers); while the rest of the agents are intermediaries who buy inputs and sell outputs (e.g. supermarkets). All agents are partially ordered along the supply chain: downstream (upstream) firms cannot sell to (buy from) firms upstream (downstream) i.e. the contract network is *acyclic*. His key assumption about the market, which we retain in this thesis, was that firms' choice functions over contracts satisfy *same-side substitutability* and *cross-side complementarity* (Hatfield and Kominers [31] later called these conditions *full substitutability*). This assumption requires that firms view any downstream or any upstream contracts as substitutes, but any downstream and any upstream contract as complements.

While a supply chain may be a useful model of production in certain industries, in general, firms simultaneously supply inputs to *and* buy outputs from other firms (possibly through intermediaries). So, a contract network may contain a contract cycle. For example, the sectoral input-output network of the U.S. economy, illustrated by [1, Figure 3], shows that American firms are very interdependent and the contract network contains many cycles. Consider a coal mine that supplies coal to a steel factory. The factory uses coal to produce steel, which is an input for a manufacturing firm that sells mining equipment back to the mine. This creates a contract cycle. However, Hatfield and Kominers [31] showed that if a contract network has a contract cycle then set-stable outcomes may fail to exist. Our first result shows that checking whether an outcome is in fact set-stable is computationally hard. We then show that, even in the presence of contract cycles, outcomes that satisfy a weaker notion of stability, namely *trail stability* can still be found. A trail of contracts is a sequence of distinct contracts in which a seller (buyer) in one contract is a buyer (seller) in the subsequent one. We argue that trail stability is a useful and intuitive equilibrium concept for the analysis of matching markets in networks. Along a blocking trail, firms make unilateral offers to their neighbors while conditionally accepting a sequence of previous pairwise blocks. Firms can receive several offers along the trail. Trail-stable outcomes rule out any sequence of such consecutive pairwise blocks. Trail stability is equivalent to chain stability (and therefore to set stability under our assumptions) in acyclic contract networks and to pairwise stability in two-sided many-to-many matching markets with contracts.

In order to analyze properties of trail-stable outcomes, we introduce another stability notion, called *full trail stability*, which does not force intermediary firms to accept all the contracts along a trail, but rather only sign upstream/downstream pairs. We argue that this could also be seen as a useful stability notion for short-run contract relationships. But studying full trail stability also allows us to use familiar fixed-point theorems and other techniques from the matching literature. Fully trail-stable outcomes correspond to the fixed-points of an operator and form a particular lattice structure for *terminal agents*, who can sign only upstream or only downstream contracts. The lattice reflects the classic opposition-of-interests property of two-sided markets, but between terminal buyers and terminal sellers. In addition to this strong lattice property, we extend previous results on the existence of buyer- and seller-optimal stable outcomes, the rural hospitals theorem [43, 31], strategy-proofness [29, 31] as well as comparative statics on firm entry and exit [38, 30] that have only been studied in a supply-chain or two-sided setting under general choice functions. Fully trail-stable and trail-stable outcomes coincide under *separability*, a condition that ensures that decisions over certain pairs of upstream and downstream contracts are taken independently from others. We provide a complete description of the relationships between all stability notions we use here : set stability, chain stability, trail stability, full trail stability.

The majority of the results of this Thesis have been published (or submitted) in three papers. All papers are joint work with my supervisor, Tamás Fleiner, [19] [20] and the Trading Network paper is joint work with Tamás Fleiner, Alexander Teytelboym and Akihisa Tamura [22].

## 1.1 Preliminaries

In the stable marriage model, there is a set $M = \{m_1, \ldots, m_n\}$ of $n$ men and a set $W = \{w_1, \ldots, w_m\}$ of $m$ women, each of them having a strict preference order on the members of the other gender. In the very first model, there were an equal number of men and women and all of them ranked all members of the opposite gender. Here we consider a (less strict) model where the number of men ans women do not need to be equal and every agent finds a subset of the opposite sex acceptable, and they state a preference order on the set of acceptable people. We can represent this with an arbitrary bipartite graph $G = (M, W; E)$ where the nodes represent the men and women, they are partitioned into two color classes $M$ and $W$, and $E$—the set of edges in $G$—denotes the set of acceptable marriages. (A $(m, w)$ pair is an edge of the graph above if their marriage is a acceptable for both $m$ and $w$.

In other works in this area, the notion of *contract* is often used as a possible link between two given agents. We remark that in this thesis, *contract* and *edge* are synonyms, as they both describe a possible marriage or admission. Our model initially does not include money transfer or wages. However, we may allow multiple edges between two vertices of $G$ and thus we can model discrete prices on the contracts, since discrete monetary transfers are equivalent to the possibility of multiple contracts.

A given man $m$ has a preference order over all the women he finds acceptable. We will use the notation $w <_m w'$ if man $m$ prefers woman $w'$ to $w$. A subset of contracts is a *marriage scheme* if every person is married to at most one person. This corresponds to a *matching* in the graph $G$, (a subset $S$ of edges is a matching if no vertex of $G$ is adjacent to more than one edge in $S$). A matching $S \subseteq E$ can also be described as an involution $\mu : M \cup W \to M \cup W$ such that if $m$ and $w$ are married (that is, $(m, w) \in S$), then $\mu(m) = w$ and $\mu(w) = m$, and for an unmatched agent $a$, we define $\mu(a) = a$. We change the base set the preference order, and put the agent herself or himself into this set. For an agent $a \in M \cup W$, if $b$ is an acceptable spouse for $a$, then

$b >_a a$, if $b$ is unacceptable then $a >_a b$, which shows that $a$ prefers to stay single to marrying $b$.

For given marriage scheme a *blocking pair* is a man and a woman who are both either single or prefers the other one to their current spouse.A marriage scheme $S$ is called *stable* if there is no blocking pair.

With the notation above, a blocking pair is $(m, w) \notin S$, such that $m >_w \mu(w)$ and $w >_m \mu(m)$. Note that since we use the extended preference order, this condition includes one or both of them being single. In other words, a marriage scheme is stable if for every pair $(m, w) \notin S$, $\mu(m) >_m w$ or $\mu(w) >_w m$ holds.

We can define a partially ordered set $(\mathcal{S}, \geq_M)$, where $\mathcal{S}$ is the set of all possible stable matchings, and $\geq_M$ is a common partial order defined by the preferences of men. If $S$ and $S'$ are two different stable matchings, and $\mu$ and $\mu'$ are their corresponding involutions, $S \geq_M S'$ if $\mu(m_i) \geq_{m_i} \mu'(m_i)$ for all $m_i \in M$, and $S >_M S'$, if $S \geq_M S'$ and there exists a man $m_i$ such that $\mu(m_i) >_{m_i} \mu'(m_i)$. Similarly, there is another partial ordered set $(\mathcal{S}, \geq_W)$ where the ordering is defined by the women's preferences: for two stable matchings $S$ and $S'$ $S \geq_W S'$ if $\mu(w_i) \geq_{w_i} \mu'(w_i)$ for all $w_i \in W$. A well-known result is that a marriage scheme is unanimously better for men if and only if it is unanimously worse for women. (Here, *unanimously better for men* means that it is better or at least equally good for all men) This was first shown in the book of Knuth [36]:

**Theorem 1.1.1** (Knuth). *[36] If a stable matching is at least as good for as another from the point of view of each of the men, the second is at least as good as the first from the point of view of the women.*

With our notations, we can rephrase Theorem 1.1.1 as for two stable matchings $S$ and $S'$, $S \geq_M S'$ if and only if $S \leq_W S'$.

We call a stable matching $S$ *male-optimal* (*female-optimal*) if it is better for the men (women) than any other stable matching: $S \geq_M S'$ ($S \geq_W S'$) for every stable matching $S'$.

A stable matching $S$ is *male-pessimal* (*female-pessimal*) if $S \leq_M S'$ ($S \leq_W S'$) for every stable matching $S'$.

In [24], Gale and Shapley gave an algorithmic proof on the existence of a stable marriages. (In their algorithm they supposed that every marriage is acceptable, that is, this yields a complete bipartite graph.) This is the Gale–Shapley (or deferred acceptance) algorithm. The algorithm consists of rounds. In the first round, every man propose to the woman who is the first on his preference list. Each woman then considers all her suitors and keeps her most preferred one (but her acceptance is not final) and she refuses the others. In the next round, every man who does not currently have a partner proposes to the most-preferred woman to whom they have not yet proposed. Then the women choose again from the current suitors. The algorithm terminates if no new proposal occurs. This happens after at most $const \cdot n^2$ steps, since every man proposes to every women at most once.

**Theorem 1.1.2** (Gale, Shapley). *[24] The outcome of the deferred acceptance algorithm is always a stable marriage scheme, moreover it is male-optimal and female-pessimal.*

**Example 1.1.3.** *Consider a marriage market with two men $m_1, m_2$, and two women $w_1, w_2$. The preferences are the following: $w_1 >_{m_1} w_2$, $w_2 >_{m_2} w_1$, $m_1 >_{w_2} m_2$, and $m_2 >_{w_1} m_1$, in other words, men $m_1$ and $m_2$ prefer women $w_1$ and $w_2$ respectively, and each women prefers the man who put her on the second place. Here, there are two possible stable marriage schemes: $S = \{m_1w_1, m_2w_2\}$ is male-optimal, female-pessimal, and $S' = \{m_1w_2, m_2w_1\}$ is is male-pessimal, female-optimal.*

Generally, there can be more than two stable matchings of course, and they show a more interesting structure than this little example. The following theorem appeared in the book of Knuth[36], where he attributes it to John Conway.

**Theorem 1.1.4** (Conway). *Assume that $S_1$ and $S_2$ are two stable matchings. Let every man choose the better of his partners in $S_1$ and $S_2$. In this way, we also get a stable matching.*

Denote the matching we obtain in this theorem by $S_1 \vee S_2$. Similarly, when the women choose their better partner, we get another stable matching which we denote by $S_1 \wedge S_2$. The consequence of this theorem is that the set of all stable marriages form a distributive lattice with respect to the partial order $\geq_M$ [36]. (We will give the detailed treatment of lattices later.)

In the following, we will look at models that are described not by strict preference orders but by choice functions. We shall see that "traditional" models nicely fit to this framework. Let us note that in a general scenario with some agents and preferences, we usually call arbitrary agent "she" (except when we talk about the marriage model).

A choice function is a set function which corresponds to an agent's preference profile. If $E$ is the set of all possible marriages or contracts available for this agent, for every subset of $E$ offered to her, she picks her favorite subset. More formally, a set function $\mathcal{F} : 2^E \to 2^E$ is called a *choice function* if $\mathcal{F}(A) \subseteq A$ holds for any subset $A$ of the ground set $E$.

We can define the direct sum of two choice functions: if $X \cap Y = \emptyset$ and $\mathcal{F}_1$ and $\mathcal{F}_2$ are two choice functions defined on the base sets $X$ and $Y$, i.e., $\mathcal{F}_1 : 2^X \to 2^X$ and $\mathcal{F}_2 : 2^Y \to 2^Y$ then we can define $\mathcal{F}$ as the direct sum of these two choice functions, $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, such that $\mathcal{F} : 2^{X \cup Y} \to 2^{X \cup Y}$ and for every $A \subseteq X \cup Y$, $\mathcal{F}(A) = \mathcal{F}_1(A \cap X) \cup \mathcal{F}_2(A \cap Y)$.
For example, on the graph of possible marriages, woman $w_1$ chooses from the possible marriages $w_1m_i$, and woman $w_2$ chooses from the marriages $w_2m_i$ so these two

marriage-sets are disjoint. The choice function we get as the direct sum of all women's choice functions chooses from all of the possible marriages.

For convenience, for a choice function $\mathcal{F}$, let $\overline{\mathcal{F}}(A) = A \setminus \mathcal{F}(A)$ denote the set of unselected elements. In the following, we list some important properties of set functions.

A set function, $\mathcal{F} : 2^E \to 2^E$, is *monotone* if $\mathcal{F}(A) \subseteq \mathcal{F}(B)$ whenever $A \subseteq B \subseteq E$ holds.
A set function, $\mathcal{F} : 2^E \to 2^E$, is *antitone* if $\mathcal{F}(A) \supseteq \mathcal{F}(B)$ whenever $A \subseteq B \subseteq E$ holds.
A choice function, $\mathcal{F} : 2^E \to 2^E$, is *substitutable* (sometimes called *comonotone*) if $\overline{\mathcal{F}}(A) \subseteq \overline{\mathcal{F}}(B)$ for any $A \subseteq B$ (that is, if $\overline{\mathcal{F}}$ is a monotone function).

Substitutability was originally defined by Kelso and Crawford [34] with prices, differently from our definition. It was shown in e.g., [28], that substitutability is equivalent to the property that if an agent chooses from an extended set of contracts, the set of rejected contracts expands.

There is an equivalent way to define substitutability. Suppose $A$ and $B$ are two subsets of $E$.

$$\text{If } A \subseteq B \text{ then } \mathcal{F}(B) \cap A \subseteq \mathcal{F}(A). \tag{1.1}$$

In other words, if from a larger set $B$, the agent accepted all contracts in $\mathcal{F}(B) \cap A$, she will continue to accept it from the smaller set of options, $A$.

**Statement 1.1.5.** *[17] For choice function $\mathcal{F}$, substitutability is equivalent to (1.1).*

*Proof.* If $\mathcal{F}$ is substitutable and $A \subseteq B$ then $\overline{\mathcal{F}}(A) \subseteq \overline{\mathcal{F}}(B)$ so $F(A) = A \setminus \overline{\mathcal{F}}(A) \supseteq A \setminus \overline{\mathcal{F}}(B) = A \cap \mathcal{F}(B)$.

If $\mathcal{F}(B) \cap A \subseteq \mathcal{F}(A)$ for every $A \subseteq B$ then $x \in \mathcal{F}(B) \cap A$ implies $x \in \mathcal{F}(A)$ so if a contract is rejected from $A$, it has to be rejected from $B$. $\square$

A set function $\mathcal{F} : 2^E \to 2^E$ satisfies the *Law of Aggregate Demand* if $|\mathcal{F}(A)| \leq |\mathcal{F}(B)|$ for every $A \subseteq B$ subsets of contracts.

This property also shows monotonicity in the preferences but in a different way than substitutability, here we only need that if a larger set of contracts are offered for an agent, she chooses a bigger cardinality set. Law of Aggregate Demand is often shortened to LAD in the Economics literature. This property was also called *increasing* in [17].

A *two-sided market* is a bipartite graph, where one of the sides (the men or the applicants) has a choice function $\mathcal{F}$ over the edges of the graph (i.e., over the contracts) and the other side (the women or the colleges) has a choice function $\mathcal{G}$ over the edges of the graph.

For agent $v$ ( *i.e.*, a vertex of $G = (M, W; E)$) let $E_v$ be the set of contracts involving $v$ ( *i.e.*, the edges from $v$).

**Example 1.1.6.** *Consider a marriage market, a bipartite graph $G = (M, W; E)$ where the edges represent the marriages acceptable for both parties, and men and women have a strict preference order over the edges incident to these agents. (This ordering is easily deduced from the original Gale–Shapley model, where agents have a strict preference order over the people they find acceptable.) For an arbitrary subset of the edges, $A \subseteq E$, every man $m$ covered by edge-set $A$ picks his favorite contract from $E(m)$. The set of these favorite edges make up $\mathcal{F}(A)$. Similarly, every man $w$ covered by edge-set $A$ picks her favorite contract from $E(w)$. The set of these favorite edges make up $\mathcal{G}(A)$. Thus this is a two-sided market.*

In a two-sided market, where the sides have the choice functions $\mathcal{F}$ and $\mathcal{G}$, a contract-set $A \subseteq E$ is *individually rational* (or *acceptable* ) if $\mathcal{F}(A) = A = \mathcal{G}(A)$.

If we want to increase or decrease a set with one element only, we will use the shorthand $A + x$ for $A \cup \{x\}$ and $A - x$ for $A \setminus \{x\}$.

Let us call a subset $A$ of all contracts an *outcome.* so, an outcome is a synonym to contract-set.

Outcome $A$ is $\mathcal{F}$*-independent* (or $\mathcal{F}$*-rational*), if $\mathcal{F}(A) = A$. This basically means that this set of contracts is acceptable for the side with choice function $\mathcal{F}$.
Let $W$ and $A$ be sets of contracts, we say that $A$ is $(W, \mathcal{F})$*-rational* if $A \subseteq \mathcal{F}(W \cup A)$ i.e., if choice function $\mathcal{F}$ chooses all contracts from set $A$ whenever the contract-set $A \cup W$ is offered. Particularly, for a single contract, we say that $e \in E$ is $(W, \mathcal{F})$-rational if $e \in \mathcal{F}(W + e)$. Note that $A$ is $(\emptyset, \mathcal{F})$-rational if and only if $A$ is $\mathcal{F}$-independent. Often when we talk about a given choice function $\mathcal{F}$, we simply call these sets $W$*-rational.*

Another well-known property in the literature is *Irrelevance of Rejected Contracts*, abbreviated to IRC. There are several ways to define this concept, so temporally we will use $\text{IRC}_1$ and $\text{IRC}_2$ to differentiate the two different definitions when necessary.

Choice function $\mathcal{F} : 2^E \to 2^E$ is *IRC*$_1$ if $\mathcal{F}(A) \subseteq B \subseteq A$ implies $\mathcal{F}(A) = \mathcal{F}(B)$ for any subsets $A$ and $B$ of $E$.

Irrelevance of Rejected Contracts appeared recently in the paper of Aygün and Sönmez [6] in the following form:

Contracts satisfy the Irrelevance of Rejected Contracts ($\text{IRC}_2$) for choice function $\mathcal{F}$ if for every $Y \subset X$, $\forall z \in X \setminus Y \quad z \notin \mathcal{F}(Y + z) \Rightarrow \mathcal{F}(Y) = \mathcal{F}(Y + z)$

We will show that if the contract-set is finite, these two definitions are the same, but if the contract set can be infinite, then $\text{IRC}_1$ is a stronger concept.

**Statement 1.1.7.** *If $\mathcal{F}$ is IRC$_1$ then it is IRC$_2$, even if the contract set is infinite.*

*Proof.* If we know that $\mathcal{F}$ is $\text{IRC}_1$, then $z \notin \mathcal{F}(Y + z)$ implies $\mathcal{F}(Y + z) \subseteq Y \subseteq (Y + z)$ therefore $\mathcal{F}(Y) = \mathcal{F}(Y + z)$ . □

**Statement 1.1.8.** *If the set of contracts is finite, then IRC$_1$ and IRC$_2$ are equivalent.*

*Proof.* As we have seen in Statement 1.1.9, $\mathrm{IRC}_1$ implies $\mathrm{IRC}_2$, even if the contract set is not finite.

If $\mathcal{F}$ satisfies $\mathrm{IRC}_2$ and there is a given $A$, $B$ such that $\mathcal{F}(A) \subseteq B \subseteq A$, then let $A \setminus B = \{z_1, z_2, \dots, z_k\}$. $\mathcal{F}(A) \subseteq (A - z_1) \subseteq A$, so for $Y = A - z_1$ we get that $z_1 \notin \mathcal{F}(Y + z_1) = \mathcal{F}(A)$, therefore $\mathcal{F}(A - z_1) = \mathcal{F}(A)$. Doing this step by step $\mathcal{F}(A) = \mathcal{F}(A - z_1) = \mathcal{F}(A \setminus \{z_1, z_2\}) = \dots = \mathcal{F}(B)$. □

**Example 1.1.9.** *For infinite contract sets, there exists a choice function which satisfies $IRC_2$ but not $IRC_1$. Let $E$ be the set of all positive integers. If $X \subseteq E$ is finite, let $\mathcal{F}(X) = X$, but if $X$ is an infinite set, let $\mathcal{F}(X)$ be the smallest element in $X$. It is easy to check that this is $IRC_2$. Let $A = \mathbb{Z}_+$ and $B = \{1, 2, 3\}$, so $\mathcal{F}(A) = \{1\} \subseteq B \subseteq A$ but $\mathcal{F}(A) \neq \mathcal{F}(B)$ thus $\mathcal{F}$ is not $IRC_1$.*

In the further sections, we will mainly deal with finite sets, but if a set is infinite, then IRC means $\mathrm{IRC}_1$.

A choice function $\mathcal{F} : 2^E \to 2^E$ is *path-independent* if $\mathcal{F}(A \cup B) = \mathcal{F}(\mathcal{F}(A) \cup B)$ holds for all subsets $A$ and $B$ of $E$.

**Lemma 1.1.10.** *[16] A choice function $\mathcal{F}$ is path-independent if and only if $\mathcal{F}$ is $IRC_1$ and substitutable.*

*Proof.* Suppose that $\mathcal{F}$ is path-independent, so $\mathcal{F}(A \cup B) = \mathcal{F}(\mathcal{F}(A) \cup B)$ for all subsets $A$ and $B$ of $E$. If $\mathcal{F}(A) \subseteq B \subseteq A$ then $\mathcal{F}(B) = \mathcal{F}(\mathcal{F}(A) \cup B) = \mathcal{F}(A \cup B) = \mathcal{F}(A)$. Therefore, $\mathcal{F}$ is $\mathrm{IRC}_1$.

If $\mathcal{F}$ is path-independent, let $A \subseteq B$ be two contract-sets. If a contract $x \in A$ is refused by $\mathcal{F}$, that is, $x \in A \setminus \mathcal{F}(A)$, note that $B = A \cup (B \setminus A)$, so $\mathcal{F}(B) = \mathcal{F}(A \cup (B \setminus A)) = \mathcal{F}(\mathcal{F}(A) \cup (B \setminus A))$ and therefore $x \notin \mathcal{F}(B)$. So $\mathcal{F}$ is substitutable.

For the opposite direction, if $\mathcal{F}$ is substitutable and $\mathrm{IRC}_1$ and $A, B$ are arbitrary subsets of $E$, then using property 1.1 for $A$ and $A \cup B$, we see that
$\mathcal{F}(A \cup B) = \mathcal{F}(A \cup B) \cap (A \cup B) \subseteq (\mathcal{F}(A \cup B) \cap A) \cup B \subseteq \mathcal{F}(A) \cup B \subseteq A \cup B$
Therefore, from $\mathrm{IRC}_1$, $\mathcal{F}(\mathcal{F}(A) \cup B) = \mathcal{F}(A \cup B)$ □

**Example 1.1.11.** *If $\mathcal{F}$ is not substitutable, then path-independence and $IRC_1$ are not necessarily equivalent. Let $E = \{a, b\}$ and $\mathcal{F}(\{a\}) = \mathcal{F}(\{b\}) = \emptyset$, $\mathcal{F}(\{a, b\}) = \{a, b\}$. This can be written as a preference ordering over sets: $\emptyset \prec \{a, b\}$ therefore it is $IRC_1$. But it is not path-independent, $\mathcal{F}(\{a, b\}) = \{a, b\} \neq \emptyset = \mathcal{F}(\mathcal{F}(\{a\}) \cup \{b\})$.*

In many papers, the choice function is defined by a strict preference order over all subsets of $E$, such that $\mathcal{F}(A)$ is the subset of $A$ first in this order.

Let us call a choice function $\mathcal{F}$ *linear order based* (LOB) if there exists an ordering $\prec$ of all subsets of $E$, such that there exists a best subset among all subsets of $A$. And this best subset is $\mathcal{F}(A)$, i.e., $\mathcal{F}(A) = \max_{\prec}\{X : X \subseteq A\}$.

In the following we will explore the connection between this definition and IRC. Note that the set of all contracts may be infinite, so it is not trivial that there exists a best one among all subsets of $A$ .

**Statement 1.1.12.** *If choice function $\mathcal{F}$ linear order based, then $\mathcal{F}$ is IRC$_1$ (and IRC$_2$ of course).*

*Proof.* If there is an ordering above all subsets, and every set has a most preferred subset, then $\mathcal{F}(A) \subseteq B \subseteq A$ means that the best of all $X \subseteq A$ is also in $B$. Since $B$ is a subset of $A$, the powerset of $B$ is a subset of the powerset of $A$, so $\mathcal{F}(A)$ is also the best choice in $B$, therefore $\mathcal{F}(A) = \mathcal{F}(B)$. So the choice function is IRC$_1$, and also IRC$_2$. □

**Example 1.1.13.** *[6] Statement 1.1.12 is not true in the opposite direction: there exists a choice function which is IRC$_1$, but there is no corresponding ordering on the subsets. For example, let $\mathcal{F}$ be the following choice function defined on three elements $\{a,\ b,\ c\}$:*
$\mathcal{F}(\emptyset) = \emptyset$
$\mathcal{F}(\{a\}) = \{a\}$
$\mathcal{F}(\{b\}) = \{b\}$
$\mathcal{F}(\{c\}) = \{c\}$
$\mathcal{F}(\{a, b\}) = \{a\}$
$\mathcal{F}(\{a, c\}) = \{c\}$
$\mathcal{F}(\{b, c\}) = \{b\}$
$\mathcal{F}(\{a, b, c\}) = \{a, b, c\}$

*This function satisfies $\mathcal{F}(A) \subseteq B \subseteq A \Rightarrow \mathcal{F}(A) = \mathcal{F}(B)$. Suppose that there is a good ordering. Since $\mathcal{F}(\{a, b\}) = a$, $\mathcal{F}(\{a, c\}) = c$ and $\mathcal{F}(\{b, c\}) = b$, the ordering should contain $a > b > c > a$, but this is not transitive.*

Note that this counterexample is not substitutable. Now, instead of IRC$_1$, we compare path-independence and subset-ordering.

**Theorem 1.1.14.** *If $\mathcal{F}$ is path-independent over a contract-set $E$, then $\mathcal{F}$ is linear order based.*

*Proof.* Suppose we have a path-independent choice function $\mathcal{F}$. We will define an appropriate preference ordering for it. By Lemma 1.1.10, $\mathcal{F}$ is also IRC$_1$. If for $\mathcal{F}(A) = B$ for some sets $A$ and $B$, then $\mathcal{F}(A) = B \subseteq A$, so from IRC$_1$, $\mathcal{F}(B) = B$, which means $\mathcal{F}$-independent sets can be chosen only. If $\mathcal{F}(A) \neq A$, then in our ordering $A$ is less preferred than the empty set, $A \prec \emptyset$. We do not bother defining the ordering between non-independent sets, as they are never chosen.
If $A$ is $\mathcal{F}$-independent, we define the *closure* of $A$ as the inclusion-wise biggest set

$X$ such that $\mathcal{F}(X) = A$. We denote it by $Cl(A) = X$. To show that the closure exists for an $\mathcal{F}$-independent set $A$, consider $Y = \bigcup\{X : \mathcal{F}(X) = A\}$. From Lemma 1.1.10, $\mathcal{F}$ is substitutable, thus $\mathcal{F}(Y) \subseteq A$. Since $A$ was one of the sets used in the union, $\mathcal{F}(Y) \subseteq A \subseteq Y$ and from the IRC$_1$ property this implies $\mathcal{F}(Y) = A$. Therefore $Y = Cl(A)$.

The $\subseteq$ relation defines a partial order over $2^E$. Every partial order can be extended to a linear ordering, let us call it $\subseteq'$. Let $A \preceq B$ if $Cl(A) \subseteq' Cl(B)$. This way we defined a linear order over all $\mathcal{F}$-independent sets.

We need to show that $\mathcal{F}(A) = \max_{\prec}\{X : X \subseteq A\}$. Suppose $B = \max_{\prec}\{X : X \subseteq A\}$ and $D = \mathcal{F}(A)$, $B \neq D$. Since $\mathcal{F}(A) \subseteq B \cup D \subseteq A$, from IRC$_1$, $\mathcal{F}(B \cup D) = \mathcal{F}(A) = D$. Since $\mathcal{F}$ is path-independent, $\mathcal{F}(Cl(D) \cup Cl(B)) = \mathcal{F}(\mathcal{F}(Cl(D)) \cup \mathcal{F}(Cl(B))) = \mathcal{F}(D \cup B) = D$ from the definition of $Cl(D)$, $Cl(D) \cup Cl(B) \subseteq Cl(D)$, so $Cl(B) \subseteq Cl(D)$ therefore in our ordering $B \prec D$, which contradicts the maximality of $B$. $\square$

Note that this proof works for infinite sets as well. However, there can be a linear order based choice function which is not path-independent.

**Example 1.1.15.** *Let E=$\{x,y,z\}$ and the preference order over the subsets is $\{x, y, z\} > \{x, z\} > \{x\} > \emptyset$. (All other subsets are worse than the empty set.) Then, for $A = \{x, y\}$ and $B = \{z\}$ we see that $\{x, y, z\} = \mathcal{F}(\{x, y, z\}) \neq \mathcal{F}(\mathcal{F}\{x, y\} + z) = \mathcal{F}(\{xz\}) = \{xz\}$ So $\mathcal{F}$ is not path-independent. In this example, $\mathcal{F}$ is not substitutable either.*

Theorem 1.1.14 and Lemma 1.1.10 together imply the following:

**Corollary 1.1.16.** *If $\mathcal{F}$ is substitutable and IRC$_1$ over a contract-set $E$, then $\mathcal{F}$ is linear order based.*

If $\mathcal{F}$ is a path-independent choice function, it would be very nice to find an opposite well-ordering $\prec$ of the sets (i.e., every set of the sets has a $\prec$-maximal element) which would define the choice function $\mathcal{F}$. However, this does not always exists.

**Example 1.1.17.** *Let $E$ be the $[0, 1]$ closed interval, and for every $A \subseteq E$, let $\mathcal{F}(A) = A \cap \left[\frac{\sup(A)}{2}, 1\right]$. This choice function is substitutable because if $A \subseteq B$ then $\sup(A) \leq \sup(B)$. We can easily check that $\mathcal{F}$ is IRC$_1$. If $x > y$ then $\mathcal{F}(\left[\frac{x}{2}, x\right] \cup \left[\frac{y}{2}, y\right]) = \left[\frac{x}{2}, x\right]$ therefore in the corresponding ordering $\left[\frac{y}{2}, y\right] \prec \left[\frac{x}{2}, x\right]$. Let $x_i = 1 - \frac{1}{i}$. There is an infinite chain $\left[\frac{x_1}{2}, x_1\right] \prec \left[\frac{x_2}{2}, x_2\right] \prec \ldots$ so there is no maximal choice among these sets. Therefore we cannot create a well-ordering.*

So we can summarize the connections between $IRC_1$, path-independence and the existence of an ordering as follows:

- A choice function $\mathcal{F}$ is path-independent if and only if $\mathcal{F}$ is IRC$_1$ and substitutable.

- If a choice function is path-independent, then $\mathcal{F}$ is linear order based. If $\mathcal{F}$ is linear order based, then it is $\mathrm{IRC}_1$.
- If a choice function is substitutable, then $IRC_1$, path-independence and linear order basedness are all equivalent to each other.

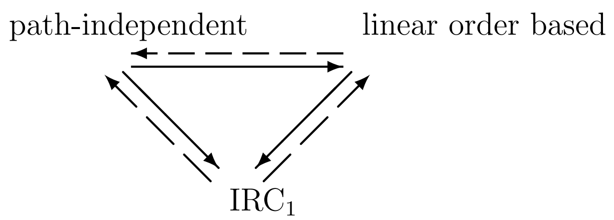
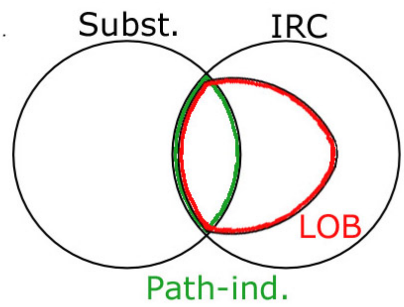



Figure 1.1: Connections.




On Figure 1.1, the dashed lines denote the implications where substitutability is needed.

On the other hand, if we consider the $\mathrm{IRC}_2$ property, it also does not match with a preference order over all subsets.

**Example 1.1.18.** *If $\mathcal{F}$ is substitutable and $IRC_2$ over an infinite contract-set $E$, then there may not exist an ordering $\prec$ over all subsets of $E$ for which $\mathcal{F}(A) = \max_{\prec}\{X : X \subseteq A\}$. Consider the same choice function as in Example 1.1.9. This choice function is substitutable. Let $A = \mathbb{Z}_+$ and $B = \{1, 2, 3\}$, $\mathcal{F}(A) = \{1\}$ so if there were a good preference ordering over all subsets of $\mathbb{Z}_+$, then $\{1\} > \{1, 2, 3\}$ so we not should have chosen $\{1, 2, 3\}$ from $B$.*

We will see, however, that typical scoring choice functions are not $\mathrm{IRC}_1$. Therefore, we shall study functions that are not necessarily $\mathrm{IRC}_1$ more generally.

### 1.1.1 Examples of Choice Functions

We list some typical choice functions, some of them come from practical applications, while some others are mostly theoretical, illustrating the flexibility of substitutable choice functions. Let $v$ be an agent (*i.e.*, a vertex of $G = (V, E)$), and let $E_v$ be the set of possible contracts involving $v$ ( *i.e.*, the edges from $v$).

1. The agent has a strict preference order, and always chooses exactly one contract, the best one available. (Except from the empty set.)

2. The agent's preferences are strict, and we allow polygamy ( *i.e.*, a college can have more than one student). The choice function picks the best $k$ contracts for some fixed $k$: $\mathcal{F}(X) =$ the best $k$ members of $X$. If $|X| \leq k$, then $\mathcal{F}(X) = X$.

3. We allow ties in the preference list. Here, $v$ chooses the best partner if it is unique and chooses the empty set if there is more than one best partner.

4. We allow ties in the preference list. Agent $v$ chooses the best partner if it is unique, and it chooses the set of best partners if there are two or more.

5. *Hungarian (H-scoring) choice function*: Every contract has a certain integral score: in the college admissions model, this is the number of points reached by the corresponding student at the particular college's entrance exam and the college has a quota $q$. If $X \subseteq E_v$ is a given set of contracts, then $v$ picks the score $t$, such that there are exactly $k$ contracts in $X$ having a score of at least $t$ and $k \leq q$, and there are more than $q$ contracts receiving a score of at least $t-1$. If no such $t$ exists, then $v$ picks $t = 0$. The choice function selects the contracts from $X$ having a score of at least $t$. For example, if $v$ is offered four contracts with scores of 3, 2, 2, and 1 and the quota of $v$ is $q = 2$, then $v$ chooses only the best contract with a score of 3 (*i.e.*, the score limit $t = 3$).

6. Let $\mathcal{H}_q$ be the following choice function on the set of contracts available for agent $v$: for any $A \subseteq E_v$, if $|A| \leq q$, then $\mathcal{H}_q(A) = A$, and if $|A| > q$, then $\mathcal{H}_q(A) = \emptyset$. Then $\mathcal{H}_q$ is a special case of H-scoring choice function, we can interpret it as there are more than $q$ students interested in the same college, with equal scores and the quota is $q$. If at most $q$ students apply, they are all accepted, but if more apply, the college rejects all of them.

   Later on $\mathcal{H}_1$ will mean that there are two students interested in the same college, with equal scores, and the quota is 1.

7. *Permissive (L-scoring) choice function*: Agent $v$ has a quota $q$ but she may choose more than $q$ contracts. Namely, $v$ chooses the best $k_2 \geq q$ contracts in a way that she chooses the best $k \leq q$ using the previous H-scoring method (with score limit $t$), and if $k < q$, then $v$ adds the next group of applicants with score $t - 1$. If the H-scoring function chooses exactly $k = q$ applicants, then $v$ keeps them and does not add new students. In the previous example, where there are four contracts with scores of 3, 2, 2, and 1 and the quota is $q = 2$, agent $v$ would set the score limit at 2 and pick three applicants with scores of 3, 2, and 2. (Here, H-scoring stands for *high* (or Hungarian) score limits, and L-scoring stands for *low* score limits.)

8. The *weighted scoring choice function* is similar to the H-scoring choice function, except that every contract also has a cost. Agent $v$ has a budget $m$ as a generalization of a quota. For a given set $X$ of contracts, $v$ determines $t$ in such a way that the total cost of contracts having a score of at least $t$ is not more than $m$, but the total cost of contracts having a score of at least $t-1$ is more than $m$. If no such $t$ exists, then $t = 0$. Now, $v$ chooses those contracts from $X$ that have a score of at least $t$. For example, if $v$ has four applicants according to the table below:

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| scores | 4 | 3 | 2 | 1 |
| cost | 9 | 5 | 5 | 1 |

   and the budget is 10, then $v$ chooses $a_1$ only. (Agent $v$ cannot skip any applicants and cannot choose a cost of $9 + 1$.)

9. *Strict hierarchical choice function*: Agent $v$ has a linear preference list over the contracts, and there is a downward closed set system $\mathcal{I}$ of subsets of $E_v$ (that is, if $A \in \mathcal{I}, B \subseteq A$, then $B \in \mathcal{I}$). Let $k$ be the greatest number, such that the set of $k$ best contracts of set $X$ belongs to $\mathcal{I}$. Let $\mathcal{F}(X)$ be the set of these $k$ best applicants.

10. *Weak hierarchical choice function*: Agent $v$ has a weak preference order (ties are allowed) over the contracts, and there is a downward closed set system $\mathcal{I}$ of subsets of $E_v$. Let $k$ be the greatest number such that the set of $k$ best contracts of set $A$ is in $\mathcal{I}$, and among equally good contracts, we choose all or none. Let $\mathcal{F}(A)$ be the set of these $k$ best applicants.

Assume that each contract $c$ has some score $s(c)$. A choice function $\mathcal{F}$ is *simple-loser-free* if any rejected contract has a lower score than any accepted contract. That is, $s(c') < s(c)$ holds whenever $c \in \mathcal{F}(X)$ and $c' \in X \setminus \mathcal{F}(X)$. A choice function is *loser-free*, if it is the direct sum of simple-loser-free choice functions.

Note that the Examples 1, 2, 4 and 7 above are IRC and all of the examples above are substitutable and loser-free.

**Remark 1.1.19.** *Any weighted scoring choice function is weak hierarchical, and any weak hierarchical choice function is loser-free. However, not every loser-free choice function is weak hierarchical:*

**Example 1.1.20.** *If $a > b > c$, $\mathcal{F}(A) = A$ if $|A| \leq 2$ and $\mathcal{F}(\{a, b, c\}) = \{a\}$, then $\mathcal{F}$ is loser-free and substitutable. However, $F(\{a, b\}) = \{a, b\}$, so $\{a, b\}$ is supposed to be a set in $\mathcal{I}$ but $\mathcal{F}(\{a, b, c\}) \neq \{a, b\}$, so this function is not hierarchical.*

*Not every weak hierarchical function is a weighted scoring choice function:*

**Example 1.1.21.** *The set of contracts is* $E = \{a, b, c, d\}$*, the preference order is* $a > b > c > d$*, and* $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}\}$*. Therefore, for example,* $\mathcal{F}(\{a, b, c\}) = \{a, b\}$*. However, if we want to describe it with weights: the weight limit is* $Q$*. The sets* $\{a, b\}$ *and* $\{c, d\}$ *have weight less than* $Q$*, but* $\{a, c\}$ *and* $\{b, d\}$ *have more weight than* $Q$*. Therefore, the weight of* $\{a, b, c, d\}$ *is simultaneously both less than and more than 2Q. This is a contradiction.*

## 1.2 On lattices

Let $\mathcal{L} = (X, \preceq)$ be a *partially ordered set*, where $X$ is a set, and $\preceq$ is a partial order over $X$, i.e., $\preceq$ is reflexive, antisymmetric, and transitive, but not all $x$ and $y$ has to be comparable. Partially ordered sets are often called *poset* for short. This $\mathcal{L} = (X, \preceq)$ is a *lattice* if any two elements $x$ and $y$ of $X$ have a least common upper bound $x \vee y$ (the *join* of $x$ and $y$) and a greatest common lower bound $x \wedge y$ (the *meet* of $x$ and $y$). A lattice $\mathcal{L}$ is *complete* if every subset $Y$ of $X$ has a least common upper bound $\bigvee Y$ and a greatest common lower bound $\bigwedge Y$. Clearly, every complete lattice $\mathcal{L}$ has a unique maximal element $1 := \bigvee X$ and a unique minimal element $0 := \bigwedge X$. A lattice $\mathcal{L} = (X, \preceq)$ is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1.2}$$

holds for all elements $x, y, z$ of $X$. Note that Condition (1.2) is equivalent to its dual, that is, for all $x, y, z \in X$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

holds. If a lattice $\mathcal{L} = (X, \preceq)$ is complete then $\mathcal{L}$ is called *infinitely distributive* if

$$x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}$$

holds for every element $x$ and every subset $Y$ of $X$. Note that, unlike distributivity, infinite distributivity does not imply its dual.

A most common example of a distributive complete lattice is the lattice $\mathcal{L} = (2^E, \subseteq)$ of subsets of a ground set $E$, and here the lattice operations $\vee$ and $\wedge$ are simply $\cap$ and $\cup$. This way we get back the model we discussed in the previous sections.

We can define properties of functions on a lattice similarly to the way we did with traditional choice functions.

If $\mathcal{L} = (X, \preceq)$ is a lattice, then a function $\mathcal{F} : X \to X$ is a *choice function* if $\mathcal{F}(x) \preceq x$ holds for every element $x$ of $X$. (This corresponds to choosing a subset of the offers in the original model.)

For choice function $\mathcal{F}$, an element $x \in X$ is $\mathcal{F}$*-independent* if $\mathcal{F}(x) = x$.

A mapping $\mathcal{F} : X \to X$ is *monotone* if $x \preceq y$ implies $\mathcal{F}(x) \preceq \mathcal{F}(y)$ and $\mathcal{F}$ is *antitone* if $x \preceq y$ implies $\mathcal{F}(y) \preceq \mathcal{F}(x)$.

A choice function $\mathcal{F} : X \to X$ is *path-independent* if $\mathcal{F}(x \vee y) = \mathcal{F}(x \vee \mathcal{F}(y))$ holds for all elements $x, y$ of $X$.

We say that $\mathcal{F}$ is IRC if $\mathcal{F}(x) \preceq y \preceq x$ implies $\mathcal{F}(x) = \mathcal{F}(y)$. (It would make no sense to define a condition similar to $\text{IRC}_2$ here.)

Observe that in the case of subset lattices, the defined notions above of a choice function correspond to the well-known notions of set-mappings under the same name.

Similarly to set-mapping, there is a connection between path-independence and IRC. The generalization of Lemma 1.1.10 is that every path-independent choice function has the IRC property and if $\mathcal{F}$ is substitutable and IRC then it is also path-independent. We will show this in Section 1.3.

For a function $f : \mathcal{L} \to \mathcal{L}$, an element $x$ of $\mathcal{L}$ is called a *fixed point* if $f(x) = x$.

**Theorem 1.2.1** (Tarski's fixed point theorem). *[47] Let $\mathcal{L}$ be a complete lattice and $f : \mathcal{L} \to \mathcal{L}$ be a monotone function on $\mathcal{L}$. Then the set $\mathcal{L}_f = \{x \in \mathcal{L} : f(x) = x\}$ of fixed points of $f$ is a nonempty, complete lattice on the restricted partial order.*

If a lattice $\mathcal{L}$ is finite in Theorem 1.2.1, there is a straightforward algorithm to find the least and greatest fixed points. Let 0 be the smallest element in lattice $\mathcal{L}$. Therefore, $0 \leq f(0)$ and from monotonicity $0 \leq f(0) \leq f(f(0)) \leq f(f(f(0))) \leq \cdots$. Since the lattice is finite, there exists an $i$, where $f^i(0) = f^{i+1}(0)$. Therefore, $f^i(0)$ is a fixed point.

**Statement 1.2.2.** *The fixed point $a = f^i(0)$ above is the least of all fixed points of $f$.*

*Proof.* Let $x$ be an arbitrary fixed point of $f$. Since $f$ is monotone, $0 \leq x \Rightarrow f(0) \leq f(x) = x$ and $f^j(0) \leq f^j(x) = x$ for every $j \geq 1$. We get that $a = f^i(0) \leq x$. □

Similarly, we can start with the greatest element 1. From $1 \geq f(1) \geq f(f(1)) \geq f(f(f(1))) \geq \cdots$, we see that there is a $j$, such that $f^j(1) = f^{j+1}(1)$. This $f^j(1)$ is the greatest of all fixed points of $f$.

## 1.3 Determinants

It this section we give a definition of substitutability for lattices with the help of determinants. For choice functions over subsets of a ground set $E$, this sophisticated definition was not necessary, but when we switch to defining choice functions over lattices, there is no good equivalent to $\overline{\mathcal{F}}(A) = A \setminus \mathcal{F}(A)$, so instead we use antitone functions to capture substitutability.

If $\mathcal{F} : 2^E \to 2^E$ is a choice function, then its determinant $\mathcal{D} : 2^E \to 2^E$ is also a set function on the same ground set. We say $\mathcal{D}$ is a *determinant* of the choice function $\mathcal{F}$ if $\mathcal{F}(Y) = Y \cap \mathcal{D}(Y)$ for every $Y \subseteq E$.

If $\mathcal{F}$ is a choice function over a lattice $\mathcal{L} = (X, \preceq)$, then $\mathcal{D} : X \to X$ is defined on the same lattice, and $\mathcal{D}$ is a *determinant* of the choice function $\mathcal{F}$ if $\mathcal{F}(x) = x \wedge \mathcal{D}(x)$ for every $x \in X$.

Note that this determinant is not related at all to the determinant of a matrix.

As every choice function is a determinant of itself, we see that a mapping is a choice function if and only if it has a determinant. Note that the determinant of a choice function might not be unique.

The following lemma eliminates the difference (or complementation) operation from the definition of substitutability, hence it allows us to extend the notion to lattices that have no such operation in general. That is, we can regard the following lemma as an equivalent definition of a substitutable choice function.

**Lemma 1.3.1.** *[20] A choice function $\mathcal{F} : 2^E \to 2^E$ is substitutable if and only if there exists an antitone determinant $\mathcal{D}$ of $\mathcal{F}$.*

*Proof.* Suppose that $\mathcal{F}$ is substitutable. Then $Y \mapsto Y \setminus \mathcal{F}(Y)$ is monotone by definition, hence $\mathcal{D}(Y) := E \setminus (Y \setminus \mathcal{F}(Y))$ is an antitone determinant of $\mathcal{F}$. Now assume that $\mathcal{D}$ is an antitone determinant of $\mathcal{F}$. Then $Y \subseteq Z$ implies that $Y \setminus \mathcal{F}(Y) = Y \setminus (Y \cap \mathcal{D}(Y)) = Y \setminus \mathcal{D}(Y) \subseteq Z \setminus \mathcal{D}(Y) \subseteq Z \setminus \mathcal{D}(Z) = Z \setminus (Z \cap \mathcal{D}(Z)) = Z \setminus \mathcal{F}(Z)$ and this is exactly what we had to prove. □

Note that a substitutable choice function $\mathcal{F}$ might have several antitone determinants $\mathcal{D}$. The next lemma states that there is a special one that has some useful extra property. First we state it for set-functions, then for any substitutable choice function over a lattice.

**Lemma 1.3.2.** *Suppose that $\mathcal{F} : 2^E \to 2^E$ is a substitutable choice function and define*

$$\mathcal{D}_{\mathcal{F}}(A) := \{e : e \in \mathcal{F}(A + e)\} \tag{1.3}$$

*as the set of all $A$-rational contracts. Then $\mathcal{D}_{\mathcal{F}}$ is an antitone determinant of $\mathcal{F}$. Moreover, if $\mathcal{D}$ is an antitone determinant of $\mathcal{F}$ then $\mathcal{D}_{\mathcal{F}}(A) \subseteq \mathcal{D}(A)$ holds for every $A \subseteq E$.*

*Proof of Lemma 1.3.2.* Let $A \subseteq B$. If $e \in \mathcal{D}_{\mathcal{F}}(B)$ then $e \in \mathcal{F}(B + e)$ Since $\mathcal{F}$ is substitutable, $e \in F(A + e)$ so $e \in \mathcal{D}_{\mathcal{F}}(A)$. Therefore $\mathcal{D}_{\mathcal{F}}(B) \subseteq \mathcal{D}_{\mathcal{F}}(A)$ i.e., $\mathcal{D}_F$ is antitone.

To prove the minimality property of determinant $\mathcal{D}_{\mathcal{F}}$, assume that $\mathcal{D}$ is an antitone determinant of $\mathcal{F}$. Now if $y \in \mathcal{F}(A + y)$ then

$$y \in \mathcal{F}(A + y) = (A + y) \cap \mathcal{D}(A + y) \subseteq \mathcal{D}(A + y) \subseteq \mathcal{D}(A)$$

hence $\mathcal{D}_{\mathcal{F}}(A) = \bigcup\{y \in A : y \in \mathcal{F}(A + y)\} \subseteq \mathcal{D}(A)$. □

We call this $\mathcal{D}_\mathcal{F}$ the *canonical determinant* of $\mathcal{F}$.

Another extremal determinant for choice function $\mathcal{F}$ is $\mathcal{D}'_\mathcal{F}(A) := E \setminus \overline{\mathcal{F}}(A)$ for every $A \subseteq E$.

**Lemma 1.3.3.** *$\mathcal{D}'_\mathcal{F}$ is an antitone determinant of $\mathcal{F}$ and it is the maximal among all antitone determinants i.e., if $\mathcal{D}$ is another antitone determinant of $\mathcal{F}$ then $\mathcal{D}'_\mathcal{F}(A) \supseteq \mathcal{D}(A)$.*

*Proof.* Since $\overline{\mathcal{F}}$ is monotone, clearly $E \setminus \overline{\mathcal{F}}$ is antitone, and $A \cap \mathcal{D}'_\mathcal{F}(A) = A \cap (E \setminus \overline{\mathcal{F}}(A)) = \mathcal{F}(A)$ so it is a determinant.

For every other determinant, $\mathcal{D}(A) \cap A = \mathcal{F}(A) = \mathcal{D}'_\mathcal{F}(A) \cap A$ and $\mathcal{D}(A) \cap (E \setminus A) \subseteq E \setminus A = \mathcal{D}'_\mathcal{F}(A) \cap (E \setminus A)$ so we can see that $\mathcal{D}(A) \subseteq \mathcal{D}'_\mathcal{F}(A)$. □

These determinants have some other useful properties.

**Lemma 1.3.4.** *Suppose that choice function $\mathcal{F} : 2^E \to 2^E$ has an antitone determinant $\mathcal{D}$ with the property that $\mathcal{D}(A) = \mathcal{D}(\mathcal{F}(A))$ holds for every $A \subseteq E$. Then $\mathcal{F}$ is path-independent.*

*For the other way around, if $\mathcal{F}$ is substitutable and path-independent, then for the canonical determinant $\mathcal{D}_\mathcal{F}$ the equality $\mathcal{D}_\mathcal{F}(\mathcal{F}(A)) = \mathcal{D}_\mathcal{F}(A)$ holds for every $A \subseteq E$.*

*Proof.* $\mathcal{F}(A \cup B) = (A \cup B) \cap \mathcal{D}(A \cup B) \subseteq (A \cup B) \cap \mathcal{D}(B) = (A \cap \mathcal{D}(B)) \cup (B \cap \mathcal{D}(B)) \subseteq A \cup \mathcal{F}(B) \subseteq A \cup B$. Consequently, $\mathcal{F}(A \cup B) \subseteq A \cup \mathcal{F}(B) \subseteq A \cup B$. Therefore

$$\mathcal{D}(A \cup B) \subseteq \mathcal{D}(A \cup \mathcal{F}(B)) \subseteq \mathcal{D}(\mathcal{F}(A \cup B)) = \mathcal{D}(A \cup B) \tag{1.4}$$

by the property that $\mathcal{D}(A) = \mathcal{D}(\mathcal{F}(A))$ and the antitonicity of $\mathcal{D}$. Thus we have equality throughout (1.4), in particular $\mathcal{D}(A \cup B) = \mathcal{D}(A \cup \mathcal{F}(B))$ holds. Now

$$\mathcal{F}(A \cup \mathcal{F}(B)) = (A \cup \mathcal{F}(B)) \cap \mathcal{D}(A \cup \mathcal{F}(B)) \subseteq (A \cup B) \cap \mathcal{D}(A \cup B) = \mathcal{F}(A \cup B)$$

and

$$\begin{aligned} \mathcal{F}(A \cup B) = (A \cup B) \cap \mathcal{D}(A \cup B) = (A \cup B) \cap \mathcal{D}(A \cup B) \cap \mathcal{D}(A \cup B) = \\ \mathcal{F}(A \cup B) \cap \mathcal{D}(A \cup \mathcal{F}(B)) \subseteq A \cup \mathcal{F}(B) \cap \mathcal{D}(A \cup \mathcal{F}(B)) = \mathcal{F}(A \cup \mathcal{F}(B)), \end{aligned}$$

so $\mathcal{F}(A \cup B) = \mathcal{F}(A \cup \mathcal{F}(B))$, that is, $\mathcal{F}$ is indeed path-independent.

Finally, path-independence of $\mathcal{F}$ directly implies that

$$\mathcal{D}_\mathcal{F}(A) = \bigcup\{y : y \in \mathcal{F}(A + y)\} = \bigcup\{y : y \in \mathcal{F}(\mathcal{F}(A) + y)\} = \mathcal{D}_\mathcal{F}(\mathcal{F}(A)).$$

□

When $\mathcal{F}$ is a choice function over a lattice, we can generalize the above results.

We observe some useful properties of substitutable and path-independent substitutable choice functions. Recall that in Lemma 1.1.10 we have shown the connection between path-independence and the property IRC. Now we present the same, but with choice functions over lattices.

**Lemma 1.3.5.** *Suppose that $\mathcal{L} = (X, \preceq)$ is a lattice. If a choice function $\mathcal{F} : X \to X$ is path-independent then $\mathcal{F}$ is IRC. On the other hand, if $\mathcal{L}$ is distributive and a choice function $\mathcal{F} : X \to X$ is substitutable and IRC then $\mathcal{F}$ is path-independent.*

*Proof.* If $\mathcal{F}(x) \preceq y \preceq x$ and $\mathcal{F}$ is path-independent, then $\mathcal{F}(y) = \mathcal{F}(y \vee \mathcal{F}(x)) = \mathcal{F}(y \vee x) = \mathcal{F}(x)$ and the first part follows. To see the second part, let $\mathcal{D}$ be an antitone determinant of $\mathcal{F}$. Then

$$\mathcal{F}(x \vee y) = (x \vee y) \wedge \mathcal{D}(x \vee y) \preceq (x \vee y) \wedge \mathcal{D}(y) = \\ (x \wedge \mathcal{D}(y)) \vee (y \wedge \mathcal{D}(y)) \preceq x \vee \mathcal{F}(y) \preceq x \vee y \quad (1.5)$$

Consequently, $\mathcal{F}(x \vee y) \preceq x \vee \mathcal{F}(y) \preceq x \vee y$ and $\mathcal{F}(x \vee \mathcal{F}(y)) = \mathcal{F}(x \vee y)$ by IRC. So $\mathcal{F}$ is indeed path-independent. □

The next lemma provides a sufficient condition on the determinant for path-independence of the corresponding choice function, and generalizes the first part of Lemma 1.3.4.

**Lemma 1.3.6** (Fleiner, Jankó)**.** *[20] Suppose that $\mathcal{L} = (X, \preceq)$ is a distributive complete lattice and a choice function $\mathcal{F} : X \to X$ has an antitone determinant $\mathcal{D}$ with the property that*

$$\mathcal{D}(x) = \mathcal{D}(\mathcal{F}(x)) \text{ holds for each element } x \text{ of } X. \quad (1.6)$$

*Then $\mathcal{F}$ is path-independent.*

*Proof.* As the calculation in (1.5) is valid in this case, we have $\mathcal{F}(x \vee y) \preceq x \vee \mathcal{F}(y) \preceq x \vee y$, hence

$$\mathcal{D}(x \vee y) \preceq \mathcal{D}(x \vee \mathcal{F}(y)) \preceq \mathcal{D}(\mathcal{F}(x \vee y)) = \mathcal{D}(x \vee y) \quad (1.7)$$

by Property (1.6) and the antitonicity of $\mathcal{D}$. Thus we have equality throughout (1.7), in particular $\mathcal{D}(x \vee y) = \mathcal{D}(x \vee \mathcal{F}(y))$ holds. Now

$$\mathcal{F}(x \vee \mathcal{F}(y)) = (x \vee \mathcal{F}(y)) \wedge \mathcal{D}(x \vee \mathcal{F}(y)) \preceq (x \vee y) \wedge \mathcal{D}(x \vee y) = \mathcal{F}(x \vee y)$$

and

$$\mathcal{F}(x \vee y) = (x \vee y) \wedge \mathcal{D}(x \vee y) = (x \vee y) \wedge \mathcal{D}(x \vee y) \wedge \mathcal{D}(x \vee y) = \\ \mathcal{F}(x \vee y) \wedge \mathcal{D}(x \vee \mathcal{F}(y)) \preceq x \vee \mathcal{F}(y) \wedge \mathcal{D}(x \vee \mathcal{F}(y)) = \mathcal{F}(x \vee \mathcal{F}(y)),$$

so $\mathcal{F}(x \vee y) = \mathcal{F}(x \vee \mathcal{F}(y))$, that is, $\mathcal{F}$ is indeed path-independent. □

As we have seen, if a choice function $\mathcal{F} : X \to X$ is substitutable then there might be several antitone determinants of $\mathcal{F}$. The following lemma states that, also for lattices, there is a determinant that is the minimal one. This is the lattice analogue of Lemma 1.3.2.

**Lemma 1.3.7** (Fleiner, Jankó). *[20] Suppose that $\mathcal{L} = (X, \preceq)$ is a complete lattice and $\mathcal{F} : X \to X$ is a substitutable choice function of it. Then for every $x, y \in \mathcal{L}$*

$$\mathcal{F}(x \vee y) \wedge x \preceq \mathcal{F}(x) \ . \tag{1.8}$$

*Furthermore, if $\mathcal{L}$ is infinitely distributive then*

$$\mathcal{D}_{\mathcal{F}}(x) := \bigvee\{y \in X : y \preceq \mathcal{F}(x \vee y)\} \text{ is an antitone determinant of } \mathcal{F}. \tag{1.9}$$

*If $\mathcal{D}$ is an antitone determinant of $\mathcal{F}$ then $\mathcal{D}_{\mathcal{F}}(x) \preceq \mathcal{D}(x)$ holds for every element $x$ of $X$. Finally, if $\mathcal{F}$ is path-independent then $\mathcal{D}_{\mathcal{F}}(x) = \mathcal{D}_{\mathcal{F}}(\mathcal{F}(x))$ holds for each element $x$ of $X$.*

*Proof.* Let $\mathcal{D}$ be an antitone determinant of a substitutable choice function $\mathcal{F}$. Then

$$\mathcal{F}(x \vee y) \wedge x = (x \vee y) \wedge \mathcal{D}(x \vee y) \wedge x = x \wedge \mathcal{D}(x \vee y) \preceq x \wedge \mathcal{D}(x) = \mathcal{F}(x)$$

proving (1.8). From (1.8) and the infinite distributivity of $\mathcal{F}$, we get

$$\begin{aligned} x \wedge \mathcal{D}_{\mathcal{F}}(x) = x \wedge \bigvee\{y \in X : y \preceq \mathcal{F}(x \vee y)\} = \bigvee\{x \wedge y : y \preceq \mathcal{F}(x \vee y)\} \preceq \\ \bigvee\{x \wedge \mathcal{F}(x \vee y) : y \in X\} \preceq \bigvee\{\mathcal{F}(x) : y \in X\} = \mathcal{F}(x) \ . \end{aligned} \tag{1.10}$$

Moreover, $\mathcal{F}(x) = \mathcal{F}(x \vee \mathcal{F}(x))$ as $\mathcal{F}(x) \preceq x$, so $\mathcal{F}(x) \preceq \mathcal{D}_{\mathcal{F}}(x)$, hence $\mathcal{F}(x) \preceq x \wedge \mathcal{D}_{\mathcal{F}}(x)$. Together with (1.10), this proves that $\mathcal{D}_{\mathcal{F}}$ is indeed a determinant of $\mathcal{F}$.

To show the antitone property of $\mathcal{D}_{\mathcal{F}}$, let $x_1 \preceq x_2$. Our goal is to prove that $\mathcal{D}_{\mathcal{F}}(x_2) \preceq \mathcal{D}_F(x_1)$, so assume that $y \preceq \mathcal{F}(x_2 \vee y)$. Again, (1.8) shows that

$$y \preceq \mathcal{F}(x_2 \vee y) \wedge (x_1 \vee y) = \mathcal{F}(x_1 \vee y \vee x_2) \wedge (x_1 \vee y) \preceq \mathcal{F}(x_1 \vee y),$$

and due to (1.9), this is exactly what we need.

To prove the minimal property of determinant $\mathcal{D}_{\mathcal{F}}$, assume that $\mathcal{D}$ is an antitone determinant of $\mathcal{F}$. Now if $y \preceq \mathcal{F}(x \vee y)$ then

$$y \preceq \mathcal{F}(x \vee y) = (x \vee y) \wedge \mathcal{D}(x \vee y) \preceq \mathcal{D}(x \vee y) \preceq \mathcal{D}(x)$$

hence $\mathcal{D}_{\mathcal{F}}(x) = \bigvee\{y \in X : y \preceq \mathcal{F}(x \vee y)\} \preceq \bigvee\{\mathcal{D}(x) : y \preceq \mathcal{F}(x \vee y)\} = \mathcal{D}(x)$.

Finally, path-independence of $\mathcal{F}$ directly implies (1.6):

$$\mathcal{D}_{\mathcal{F}}(x) = \bigvee\{y : y \preceq \mathcal{F}(x \vee y)\} = \bigvee\{y : y \preceq \mathcal{F}(\mathcal{F}(x) \vee y)\} = \mathcal{D}_{\mathcal{F}}(\mathcal{F}(x)),$$

and this finishes the proof. $\square$

# Chapter 2

# Two-Sided Markets

## 2.1 Stability Concepts

In this section, we formulate different stability concepts, which we shall study later. Dominating stability is the most straightforward generalization of the nonexistence of blocking edges in the marriage model. Three-stability is also of similar kind, here we partition the set of all contracts into three parts: chosen, dominated by men, and dominated by women. If we continue this line of thought, to get four-stability, we define four contract sets: chosen, dominated only by men, dominated only by women, and dominated by both.

### 2.1.1 Dominating Stability

In the original stable marriage model, a matching is stable if it dominates every other contract, so for every $e = (m, w) \notin S$, either $\mu(m) >_m e$ or $\mu(w) >_w e$. A natural generalization of this notion is dominating stability.

Consider a two-sided market with choice functions $\mathcal{F}$ and $\mathcal{G}$ on the two sides, as we have seen in the Preliminaries. We say that a contract set $X$ is $\mathcal{F}$-*dominated* by the contract set $S$ if $(\mathcal{F}(S \cup X)) \cap X = \emptyset$. This means that if the agent can choose from the union of sets $S$ and $X$, she is not going to choose anything from $X$.

Recall that for a substitutable choice function $\mathcal{F}$, $\mathcal{D}_\mathcal{F}$ is the canonical antitone determinant and $\mathcal{D}_\mathcal{F}(S)$ consists of all the edges that are not $\mathcal{F}$-dominated by $S$. A subset $S$ of $E$ is called *dominating stable*, if $\mathcal{D}_\mathcal{F}(S) \cap \mathcal{D}_\mathcal{G}(S) = S$. Therefore, every contract which is not part of the scheme $S$ is either $\mathcal{F}$-dominated or $\mathcal{G}$-dominated by $S$.

Equivalently, we can say that $S \subseteq E$ is dominating stable, if $\mathcal{F}(S) = \mathcal{G}(S) = S$ and for every $e \notin S$, $e \notin \mathcal{F}(S + e)$ or $e \notin \mathcal{G}(S + e)$, so $S$ is acceptable but for any $e \notin S$ one side of the market would not like to add contract $e$.

**Remark 2.1.1.** *If $S$ is dominating stable, then $\mathcal{F}(S) = S = \mathcal{G}(S)$, so the set $S$ is acceptable.*

*Proof.* Since $\mathcal{D}_{\mathcal{F}}(S) \cap \mathcal{D}_{\mathcal{G}}(S) = S$, the determinant $\mathcal{D}_{\mathcal{F}}(S) \supseteq S$, so $\mathcal{F}(S) = S \cap \mathcal{D}_{\mathcal{F}}(S) = S$. A similar proof applies for $\mathcal{G}$. □

**Example 2.1.2.** *Think of the marriage model again, where men and women have strict preferences. The common choice function of men is $\mathcal{F}$, an the choice function of women is $\mathcal{G}$. These functions choose the single best option for every player. If $S$ is dominating stable, then since $\mathcal{F}(S) = S = \mathcal{G}(S)$, the set $S$ is a matching, and for every $e = mw \notin S$, the contract $e \notin \mathcal{F}(S+e)$ or $e \notin \mathcal{G}(S+e)$, so that one of $m$ or $w$ does not want to choose $mw$ instead of their current marriage. Therefore, in this case dominating stability is equivalent to the original stable marriage definition of Gale and Shapley.*

Unfortunately, it turns out that for non-IRC choice functions, even for substitutable $\mathcal{F}$ and $\mathcal{G}$, a dominating stable solution might not exist. Although this is a direct generalization of the original stability notion of Gale and Shapley, it seems that in practical applications, this notion is not very helpful.

**Example 2.1.3.** *Let $\mathcal{F}$ and $\mathcal{G}$ be the following functions, defined on a set of three contracts: $\{a,b,c\}$: $\mathcal{F}$ accepts everything, and $\mathcal{G}$ prefers $a$ to $b$, $b$ to $c$ and $c$ to $a$. Now, $\mathcal{F}$ is substitutable and IRC, and $\mathcal{G}$ is substitutable, but not IRC.*

| | $\emptyset$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{a,b\}$ | $\{b,c\}$ | $\{a,c\}$ | $\{a,b,c\}$ |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{F}$ | $\emptyset$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{a,b\}$ | $\{b,c\}$ | $\{a,c\}$ | $\{a,b,c\}$ |
| $\mathcal{G}$ | $\emptyset$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\emptyset$ |
| $\mathcal{D}_{\mathcal{F}}$ | $\{a,b,c\}$ | $\{a,b,c\}$ | $\{a,b,c\}$ | $\{a,b,c\}$ | $\{a,b,c\}$ | $\{a,b,c\}$ | $\{a,b,c\}$ | $\{a,b,c\}$ |
| $\mathcal{D}_{\mathcal{G}}$ | $\{a,b,c\}$ | $\{a,c\}$ | $\{a,b\}$ | $\{b,c\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\emptyset$ |

*Suppose that $S$ is dominating stable. Since $\mathcal{G}(S) = S$, the cardinality of $S$ is at most 1. However, then, $\mathcal{D}_{\mathcal{F}}(S) \cap \mathcal{D}_{\mathcal{G}}(S) = \mathcal{D}_{\mathcal{G}}(S) \neq S$, because every contract has a determinant with cardinality at least 2.*

A similar example appeared in the paper of Aygün and Sönmez [6], they also showed that a dominating stable set does not need to exist.

Consider a classical example: the job market between doctors and hospitals. This is a many-to-one matching market, which means the hospitals are allowed to hire multiple doctors up to a given quota, but doctors can work at only one hospital at a time. Both sides have a strict preference ordering over an acceptable subset of members of the other side. An *allocation* is a set of doctor-hospital agreements. Here, a *blocking* pair means a doctor $d$ and a hospital $h$ such that the $d$ is unemployed (analogous to single in the marriage model) or $d$ prefers hospital $h$ to the institution to which he is assigned. The hospital $h$ either has not filled its quota or prefers $d$ to some doctor who is assigned it. An assignment is stable if there is no blocking pair.

It is easy to see that the stability criterion here is exactly dominating stability.

We also mention the Rural Hospital Theorem, since it will be needed at the end of this Thesis. In an allocation, a hospital is *undersubscribed* if it has more places available than realized applications.

**Theorem 2.1.4** (Rural Hospitals). *[Roth] [43] For a given doctor-hospital instance, the following properties hold:*

1. *the same doctors are assigned in all stable matchings;*

2. *each hospital is assigned the same number of doctors in all stable matchings;*

3. *any hospital that is undersubscribed in one stable matching is assigned exactly the same set of doctors in all stable matchings.*

In [28], Hatfield and Milgrom defined stable allocations similar to our dominating stability definition, namely a doctor-hospital allocation is stable if there is no blocking contract set. Hatfield and Milgrom [28] introduced stability in the following way: Given a two-sided many-to-one market, namely doctors and hospitals. Let $\mathcal{F}$ be the choice function of doctors, and $\mathcal{G}$ the choice function of hospitals. For an individual hospital $h$, the choice function is $\mathcal{G}_h$.

A set of contracts $S \subseteq E$ is a stable allocation (in the following we are going to call it *group-stable*) if

1. $\mathcal{F}(S) = \mathcal{G}(S) = S$ and

2. there exists no hospital $h$ and set of contracts $X' \neq \mathcal{G}_h(S)$ such that $X' = \mathcal{G}_h(S \cup X') \subseteq \mathcal{F}(S \cup X')$.

If the first condition fails, $S$ is not individually rational, if the second fails, there is a blocking set for the outcome $S$.

**Statement 2.1.5.** *If $\mathcal{F}$ and $\mathcal{G}$ are substitutable, and there are no parallel contracts, group-stability and dominating stability are equivalent.*

By parallel contracts, we mean there is more than one possible contract between a given hospital and a given doctor. Let us denote the contracts out of $S$ incident by hospital $h$ with $S_h$.

*Proof.* It is easy to see that $\mathcal{F}(S) = \mathcal{G}(S) = S$ is fundamentally part of both definitions, so we only have to pay attention to (2).

Suppose there is a blocking set $X'$ for outcome $S$, so $X' = \mathcal{G}_h(S \cup X') \subseteq \mathcal{F}(S \cup X')$. Notice that $X' \subset S$ is impossible, since in this case $X' = \mathcal{G}_h(S \cup X') = \mathcal{G}_h(S) = S_h$. Pick a contract $e \in X' \setminus \mathcal{G}_h(S)$. From substitutability, $e \in \mathcal{G}_h(S + e)$ and $e \in \mathcal{F}(S + e)$, therefore $S$ is not dominating stable.

On the other hand, suppose there exists a contract $e = hd$ such that $e \in \mathcal{G}(S+e)$ and $e \in \mathcal{F}(S + e)$. The hospital corresponding to this contract is $h$, and let $X' = \mathcal{G}_h(S + e)$.

Obviously $e \in X'$ and for doctor $d$, $e \in \mathcal{F}_d(S \cup X')$. Other doctors do not receive any new contract, therefore from individual rationality $X' \subseteq \mathcal{F}(S \cup X')$. □

**Example 2.1.6.** *If there are parallel contracts in the system, every dominating stable outcome is group-stable but there might be a group-stable outcome which is not dominating stable. Consider one hospital $h$, two doctors $d_1, d_2$ and three contracts: $a = d_1h$, $b = d_1h$, $c = d_2h$. Hospital $h$ has the preference order $a > b > c$ and its quota is 2, so it always chooses the best two out of the offered contracts. Doctor $d_1$ has preference $a > b$ and can choose only one, doctor $d_2$ accepts $c$ if offered.*

*Let $S = \{b, c\}$. It is easy to see that $a \in \mathcal{G}(S + a)$ and $a \in \mathcal{F}(S + a)$, so $S$ is not dominating stable. To find a blocking set $X' \neq \mathcal{G}_h(S)$, contract $a$ must be in $X'$, and $\mathcal{G}_h(S + a) = \{a, b\}$ therefore $X' = \{a, b\}$. But $\{a, b\} \nsubseteq \mathcal{F}(\{a, b, c\}) = \{a, c\}$ therefore it is not a proper blocking set.*

### 2.1.2 Three-Stability

The following stability concept was defined by Fleiner [17]. Given a two-sided market, where the choice functions of each side over the contracts are $\mathcal{F}$ and $\mathcal{G}$. A subset $S$ of $E$ is *three-stable*, if there exist subsets $A$ and $B$ of $E$, such that $\mathcal{F}(A) = S = \mathcal{G}(B)$ and $A \cup B = E$, $A \cap B = S$. A pair $(A, B)$ with this property is called a *three-stable pair*, and $S$ is a *three-stable set*.

The explanation of the name, three-stable, is that we partition the set $E$ of contracts into three parts, as shown in Figure 2.1, $S$, $A \setminus S$ and $B \setminus S$, where $S$ is stable, $A \setminus S$ is $\mathcal{F}$-dominated by $S$ and $B \setminus S$ is $\mathcal{G}$-dominated by $S$.

**Example 2.1.7.** *In the original marriage model, $\mathcal{F}$ and $\mathcal{G}$ select the single best partner for the men and women. It is easy to see that every three-stable set $S$ is a matching, and it is stable, since men prefer contracts in $S$ to $A \setminus S$ and women prefer $S$ to $B \setminus S$.*

*On the other hand, if $S$ is a stable matching, then it is also a three-stable set with the pair $(A, B)$, where we define $A \setminus S$ as the set of contracts that the men prefer less than the contracts of $S$ and $B := S \cup (E \setminus A)$. So we conclude that three-stable sets yield exactly the stable marriages in this case.*

**Example 2.1.8.** *Figure 2.2 illustrates a small example for a possible market. There are two possible contracts, $a$ and $b$, and $\mathcal{F} = \mathcal{H}_1$, $\mathcal{G} = \mathcal{H}_1$, that is, $\mathcal{F}(\{a\}) = \mathcal{G}(\{a\}) = \{a\}$, $\mathcal{F}(\{b\}) = \mathcal{G}(\{b\}) = \{b\}$, $\mathcal{F}(\{a, b\}) = \mathcal{G}(\{a, b\}) = \emptyset$. These choice functions are substitutable but not IRC. Then, only $S = \emptyset$ is three-stable, and it can be achieved with two distinct three-stable pairs, where $A = \{a, b\}$, $B = \emptyset$ or $A = \emptyset$, $B = \{a, b\}$*

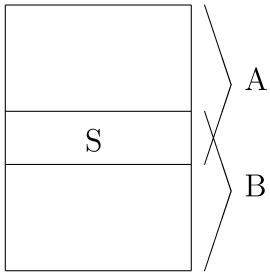



Figure 2.1: A three-partition of the edge-set.

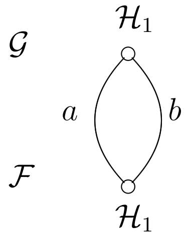



Figure 2.2: Example graph.

### 2.1.3 Four-Stability

We introduce the notion of four-stability. It is similar to three-stability, but when there is a double-dominated contract $e$ in the three-stable partition, we can choose whether $e$ should belong to $A$ or $B$. Now, in four-stability, we put $e$ in a fourth contract-set. Therefore, while three-stability is a more natural definition, four-stability has nicer properties and is more useful; as we will show, it is closely related to score-stability. Moreover, for IRC choice functions, for any four-stable set, the corresponding $(A, B)$ pair is unique, which cannot be said of three-stable pairs.

Again, we have a two-sided market, the choice functions of the two sides are $\mathcal{F}$ and $\mathcal{G}$. A subset $S$ of $E$ is *four-stable*, if there exists subsets $A$ and $B$ of $E$, such that $A \cap B = S$ and $\mathcal{D}_{\mathcal{F}}(A) = B, \mathcal{D}_{\mathcal{G}}(B) = A$. We call a pair $(A, B)$ fulfilling this property a *four-stable pair.*

**Remark 2.1.9.** *If $S$ is a four-stable set, then $\mathcal{F}(S) = S = \mathcal{G}(S)$, that is, the set $S$ is acceptable.*

*Proof.* Since $\mathcal{F}(A) = \mathcal{D}_{\mathcal{F}}(A) \cap A = B \cap A = S$, and $\mathcal{F}$ is substitutable this implies $\mathcal{F}(S) = S$. A similar proof applies for $\mathcal{G}$. □

This concept is called four-stable because we partition the set $E$ of contracts into four parts, as seen in Figure 2.3: $S$ is stable, $A \setminus S$ is $\mathcal{F}$-dominated by S, $B \setminus S$ is $\mathcal{G}$-dominated by $S$, and the contracts in $E \setminus (A \cup B)$ are both $\mathcal{F}$- and $\mathcal{G}$-dominated.

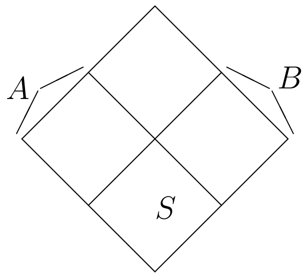



Figure 2.3: A four-partition of the edge-set.

**Example 2.1.10.** *We consider the same example as we did for three-stability in Figure 2.2: There are two possible contracts, $a$ and $b$, and $\mathcal{F} = \mathcal{H}_1$, $\mathcal{G} = \mathcal{H}_1$. Now, there are three four-stable solutions:*

- $S = \emptyset$, *where* $A = \{a, b\}$, $B = \emptyset$ *or* $A = \emptyset$, $B = \{a, b\}$
- $S = \{a\}$, $A = \{a\}, B = \{a\}$
- $S = \{b\}, A = \{b\}, B = \{b\}$.

**Statement 2.1.11.** *[Fleiner, Jankó] [19] If $\mathcal{F}$ and $\mathcal{G}$ are substitutable and $\mathcal{F}$ is IRC, then for a four-stable set $S$ there exists a unique four-stable pair $(A, B)$.*

*Proof.* Suppose that there are two different four-stable pairs for $S$: $(A, B)$ and $(A', B')$. We can assume that there exists a contract $b$ for which $b \in B$, but $b \notin B'$. Since $S \subseteq B'$, it follows that $b \notin S$. Moreover $b \in \mathcal{F}(A + b)$, but $b \notin \mathcal{F}(A' + b)$, because $\mathcal{D}_{\mathcal{F}}(A) = B$ and $\mathcal{D}_{\mathcal{F}}(A') = B'$. By $A' \setminus S = \overline{\mathcal{F}}(A') \subseteq \overline{\mathcal{F}}(A' + b)$, we get $\mathcal{F}(A' + b) \subseteq S \subseteq (A' + b)$, hence $\mathcal{F}(A' + b) = S$.
We know that $(A' + b) \setminus S = \overline{\mathcal{F}}(A' + b)$ and $A \setminus S = \overline{\mathcal{F}}(A)$.
Since $\mathcal{F}$ is substitutable, $\overline{\mathcal{F}}(A' \cup A + b)$ is a superset of both, $(A' \cup A + b) \setminus S \subseteq \overline{\mathcal{F}}(A' \cup A + b)$, so $\mathcal{F}(A' \cup A + b) \subseteq S$.
From $\mathcal{F}(A' + b) = S$, we get $\mathcal{F}(A + b) \subseteq \mathcal{F}(A + b) \cup \mathcal{F}(A' + b) \subseteq (A + b)$, so $\mathcal{F}(A + b) = \mathcal{F}(\mathcal{F}(A + b) \cup \mathcal{F}(A' + b))$.
Using that $\mathcal{F}$ is path-independent from Lemma 1.1.10, $b \in \mathcal{F}(A + b) = \mathcal{F}(\mathcal{F}(A + b) \cup \mathcal{F}(A' + b)) = \mathcal{F}((A + b) \cup (A' + b)) = \mathcal{F}(A' \cup A + b) \subseteq S$. Therefore, $b \in S$, a contradiction. $\square$

### 2.1.4 Score-Stability

In this subsection, we describe the stability notion used in the Hungarian college admission scheme. For this reason, we shall call the agents colleges and applicants, and application is a synonym for contract. The mathematical model of the Hungarian college admissions system is close to stable matchings. Our model is a simplified version of the one that is used in practice. Biró and Kiselgof [10], Azevedo and Leshno [8] also examined the mathematics behind stable score limits.

Assume that we have $n$ applicants $A_1, A_2, \ldots, A_n$ and $m$ colleges $C_1, C_2, \ldots, C_m$. Let $E$ be the set of all contracts. It is convenient to think that $E$ is the set of edges of the bipartite graph with color classes $\{A_1, \ldots, A_n\}$ and $\{C_1, \ldots, C_m\}$, where each edge, $A_iC_j$, of the graph corresponds to a contract between applicant $A_i$ and college $C_j$. Every applicant has a strict preference order over the colleges she applies to, and each college assigns some score $s(A_iC_j)$ (an integer between one and $M$) to each of its applicants. Moreover, each college $C$ has a quota $q(C)$ on admissible applicants. According to the law, no college can accept more applicants than its quota; moreover if an applicant $A_i$ with a certain score $s(A_iC_j)$ is not acceptable to some college $C_j$, then any applicant with the same or lower score has to be unacceptable for $C_j$.

Let us the denote the preference order of applicant $A_i$ with $>_i$. To determine the admissions after all information is known, each college has to declare a score limit. Let the score limits for colleges $C_1, C_2, \ldots, C_m$ be $t_1, t_2, \ldots, t_m$, respectively. Each applicant will study at her most preferred college where she has a high enough score. More precisely, applicant $A_i$ is assigned to college $C_j$ if $s(A_iC_j) \geq t_j$ ( i.e., score $s(A_iC_j)$ of $A_i$ at $C_j$ is not less than threshold $t_j$ for $C_j$) and $s(A_iC_{j'}) < t_{j'}$ for $j' >_i j$ ( i.e., score $s(A_iC_{j'})$ of $A_i$ at $C_{j'}$ is less than the score limit $t_{j'}$, if $A_i$ likes $C_{j'}$ more than $C_j$). The vector of declared score limits $(t_1, t_2, \ldots, t_m)$ is called a *score vector*. The stability notion below is defined according to the requirements of Hungarian law.

A score vector $(t_1, t_2, \ldots, t_m)$ is *valid* if no college exceeds its quota with these score limits.

A score vector $(t_1, t_2, \ldots, t_m)$ is *violable* if for every college $C_j$ either $t_j = 0$ or the score vector $(t_1, t_2, \ldots, t_{j-1}, t_j - 1, t_{j+1}, \ldots, t_m)$ would assign more than $q(C_j)$ students to $C_j$, that is, no single college can decrease its score limit without exceeding its quota.

A score vector, $\underline{s}$, is *stable* if $\underline{s}$ is valid and violable.

The college admissions model above determines a natural choice function for applicants and another one for the colleges. Therefore, for subset $X \subseteq E$ of contracts, $\mathcal{F}_i(X)$ denotes the most preferred contract from $X$ of applicant $A_i$, and $\mathcal{F}(X)$ is the common choice function of all applicants, $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \cdots + \mathcal{F}_n$. Similarly, $\mathcal{G}_j(X)$ denotes the set of contracts that college $C_j$ would choose if it could select freely. More precisely, let $X_j$ denote the set of contracts with $C_j$ in $X$, and let $C_j$ declare a score limit $t_j$ such that no more than $q(C_j)$ contracts from $X_j$ have score of at least $t_j$, but either $t_j = 0$ or more than $q(C_j)$ contracts have a score of at least $t_j - 1$. Let $\mathcal{G}_j(X)$ be the set of all contracts in $X_j$ exceeding the score limit $t_j$. Define choice function $\mathcal{G} : 2^E \to 2^E$ as the common choice function of all colleges, $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 + \cdots + \mathcal{G}_m$.

*It is easy to see that choice function $\mathcal{F}$ of the applicants is IRC, but choice function $\mathcal{G}$ of the colleges is not.*

For example, $\mathcal{G} = \mathcal{H}_1$ is a typical scoring choice function, there are two equally good contracts $a$ and $b$, and the quota is 1. However, $\mathcal{G}(\{a, b\}) = \emptyset \subseteq \{a\} \subseteq \{a, b\}$ and $\mathcal{G}(\{a\}) \neq \mathcal{G}(\{a, b\})$, so it is not IRC.

### 2.1.5 Generalized Score-Stability

We can generalize the above framework, keeping the main property needed to ensure the existence of a stable solution, namely the loser-free property that allows us to extend the model in a way that is fairly generalized and has economically interesting choice functions. Assume that each contract $c$ has some score $s(c)$.

Recall that choice function $\mathcal{G}$ is *simple-loser-free* if any rejected contract has a lower score than any accepted contract. That is, $s(c') < s(c)$ holds whenever $c \in \mathcal{G}(X)$ and $c' \in X \setminus \mathcal{G}(X)$. A choice function is *loser-free*, if it is the direct sum of simple-loser-free choice functions.

Let $\mathcal{F}$ be direct sum of substitutable choice functions of the applicants, and $\mathcal{G}$ is a direct sum of simple-loser-free, substitutable choice functions $G_j$ of the colleges.

We say a set $\{A_{i_1}, A_{i_2}, \dots, A_{i_k}\}$ of applicants is *feasible* for college $C_j$ if $\mathcal{G}_j(X) = X$ for contract set $X = \{A_{i_1}C_j, A_{i_2}C_j, \dots A_{i_k}C_j\}$, otherwise, it is *infeasible*. Each contract $c$ has a score $s(c)$.

Let $P : \mathbb{N}^E \to 2^E$ be a function, that codes the scores of the applicants: $P(\underline{t})$ is the set of contracts above the score limit given by score vector $\underline{t}$. $P(\underline{t}) = \{AC \in E : s(AC) \geq t(C)\}$. Therefore, $P(\underline{0}) = E$. There exists a score vector $\underline{T}$ where $P(\underline{T}) = \emptyset$ (assign 1+the highest possible score to every college). The function $P$ is antitone on the scores: if $\underline{t}_1 \leq \underline{t}_2$ then $P(\underline{t}_1) \supseteq P(\underline{t}_2)$.

**Lemma 2.1.12** (Fleiner, Jankó). *[19] A choice function $\mathcal{G} : 2^E \to 2^E$ is loser-free if and only if there exists a function $P_{\mathcal{G}} : 2^E \to \mathbb{N}^E$, such that for every contract-set $A \subseteq E$, $P_{\mathcal{G}}$ declares a score-limit where the accepted contracts above the score limit are exactly the set accepted by $\mathcal{G}(A)$, i.e., $P(P_{\mathcal{G}}(A)) \cap A = \mathcal{G}(A)$.*

*Proof.* If $\mathcal{G}$ is loser-free, the set of accepted contracts from $A$ are all above a score vector, let the maximal score vector they reach be $P_{\mathcal{G}}(A)$. On the other hand, if we are given a score limit by $P_{\mathcal{G}}$, there can be no student that is missed out while others with the same score get in, so $\mathcal{G}$ must be loser-free. □

We can generalize the validity and stability of score vectors to this generalized setting the following way: If contracts above score limit $\underline{t}$ are offered, students choose $\mathcal{F}(P(\underline{t}))$, and from this, the contract set $\mathcal{G}(\mathcal{F}(P(\underline{t})))$ is acceptable for the colleges. Therefore, the score vector $\underline{t}$ is *valid* if and only if $\mathcal{G}(\mathcal{F}(P(\underline{t}))) = \mathcal{F}(P(\underline{t}))$.

Score vector $\underline{t}$ is *violable* if for any $1 \leq j \leq m$, the score vector $(t_1, t_2, \dots, t_{j-1}, t_j - 1, t_{j+1}, \dots, t_m)$ is not valid for college $C_j$, or $t_j = 0$. We call $\underline{t}$ *stable* if it is both valid and violable.

A contract-set $S$ is *score-stable* is $S = \mathcal{F}(P(\underline{t}))$ for some stable score vector $\underline{t}$.

**Example 2.1.13.** *If the choice function $\mathcal{F}$ of the applicants corresponds to a linear preference order, and the choice function $\mathcal{G}$ of the colleges is the Hungarian scoring choice function, we get the score stability defined in the previous subsection.*

*If the choice function $\mathcal{G}$ of the colleges is the permissive (L-scoring) choice function, defined as in Subsection 1.1.1 then we call the score-stable lets* L-stable. *This stability concept appeared in [10] by Biró and Kiselgof.*

**Lemma 2.1.14.** *If $\underline{t}$ is valid, but $\underline{t}' = (t_1, t_2, \ldots, t_{j-1}, t_j - 1, t_{j+1}, \ldots, t_m)$ is not valid, then the only college that can get an infeasible set of students at score vector $\underline{t}'$ is college $C_j$.*

*Proof.* The set of offered places increases at college $C_j$ and stays unchanged at other colleges. For applicant $A_i$, if she rejected a college, $C_k$, earlier (where $C_k \neq C_j$), she will also reject $C_k$ when she has more choices, so colleges other than $C_j$ cannot have too many students.

With score limit $\underline{t}$, the set of students going to college $C_k$ is $\mathcal{F}(P(\underline{t})) \cap E(C_k)$, denote it by $Z$. Similarly, for score limit $\underline{t}'$, denote it by $Z'$. As we have seen, $Z' \subseteq Z$, and from the substitutability, $\mathcal{G}_k(Z) = Z$ implies $\mathcal{G}_k(Z') = Z'$. □

We call a score vector $\underline{t}$ *$C_j$-valid* , if it is acceptable for college $C_j$, *i.e.*, $\mathcal{G}_j(\mathcal{F}(P(\underline{t}))) = \mathcal{F}(P(\underline{t})) \cap E(C_j)$.

The following two lemmas help to understand how the set of the valid score vectors look:

**Lemma 2.1.15.** *Let $\underline{t}$ and $\underline{t}'$ be two score vectors, such that $\underline{t}$ is $C_j$-valid. Suppose that for college $C_j$, score limit $t'_j \geq t_j$, but $t'_i \leq t_i$ for every college $C_i \neq C_j$. Then, $\underline{t}'$ is also $C_j$-valid.*

*Proof.* If $A_i$ is a student of $C_j$ when the score vector is $\underline{t}$ (*i.e.*, $A_iC_j \in \mathcal{F}(P(\underline{t}))$), then $A_i$ can leave $C_j$ at $\underline{t}'$ if she does not reach $t'_j$ or gets a better opportunity at another college.
If student $A_l$ does not go to college $C_j$ at the score vector $\underline{t}$ ($A_lC_j \notin \mathcal{F}(P(\underline{t}))$), then she will not go to $C_j$ under score vector $\underline{t}'$, because if $A_l$ does not reach $t_j$, then she does not reach the higher limit $t'_j$. If $A_l$ reaches $t_j$, but chooses a better college $C_k$ instead, since $t'_k \leq t_k$, she will stay in college $C_k$. Therefore, the set of students assigned to $C_j$ with $\underline{t}'$ is the subset of the set of students going to $C_j$ under $\underline{t}$, $(\mathcal{F}(P(\underline{t}')) \cap E(C_j)) \subseteq (\mathcal{F}(P(\underline{t})) \cap E(C_j))$. The choice function $\mathcal{G}_j$ of college $C_j$ is substitutable. Since $\mathcal{F}(P(\underline{t})) \cap E(C_j)$ is valid, a subset of it is also valid, so $\underline{t}'$ is $C_j$-valid. □

**Lemma 2.1.16.** *Let $\underline{t}^1$ and $\underline{t}^2$ be two valid score vectors, and let $\underline{t}^{min}$ be their pointwise minimum ($t_j^{min} = \min(t_j^1, t_j^2)$ for every $1 \leq j \leq m$). Then, $\underline{t}^{min}$ is also valid.*

*Proof.* Let the set of contracts above with score vectors $\underline{t}^1$ and $\underline{t}^2$ be $P(\underline{t}^1) = A$ and $P(\underline{t}^2) = B$. Then, $P(t_{min}) = A \cup B$. Suppose that $t_j^1 = t_j^{min}$ for college $C_j$. Since $A \subseteq A \cup B$, from substitutability $\overline{\mathcal{F}}(A) \subseteq \overline{\mathcal{F}}(A \cup B)$. Considering the set of contracts of college $C_j$, $E(C_j) \cap A = E(C_j) \cap (A \cup B)$, *i.e.*, $C_j$ accepts the same set of contracts with score vector $\underline{t}^1$ as with $\underline{t}^{min}$. Therefore, $E(C_j) \cap \mathcal{F}(A) \supseteq E(C_j) \cap \mathcal{F}(A \cup B)$. Since

$\mathcal{G}$ is substitutable, if a set is valid, then its subset is also valid. $\mathcal{G}(E(C_j) \cap \mathcal{F}(A)) = E(C_j) \cap \mathcal{F}(A)$, so $\overline{\mathcal{G}}(E(C_j) \cap \mathcal{F}(A)) = \emptyset$. From substitutability, $\overline{\mathcal{G}}(E(C_j) \cap \mathcal{F}(A \cup B)) = \emptyset$, so $(A \cup B)$ is also valid for college $C_j$.
In other words, if we change the score limit from $\underline{t}^1$ to $\underline{t}^{min}$, college $C_j$ keeps its score limit, while other colleges may decrease it. Applicants may leave $C_j$, but no new students will arrive at $C_j$, so the score limit $\underline{t}^{min}$ is $C_j$-valid. We can use the same argument for every college, therefore $\underline{t}^{min}$ is valid. □

A graph $G$ is simple if it has neither parallel edges nor loops. Therefore, between a given student and college, only one contract is permitted. In some applications, for example in the college enrollment system, the underlying graph is simple: one cannot apply to the same department, in the same year, twice. For the sake of generalizations, the underlying graph in our model may not be simple.

Assume that $\mathcal{F}$ and $\mathcal{G}$ are substitutable choice functions and $\mathcal{G}$ is also loser-free. Define the following function $f : \mathbb{N}^E \to \mathbb{N}^E$:

$$f(\underline{t}) = P_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(P(\underline{t}))).$$

Therefore, we take all contracts above score limit $\underline{t}$ (this is $P(\underline{t})$) and add those contracts that are not dominated by $P(\underline{t})$. Then, $f(\underline{t})$ is the score limit that the colleges choose for this set.

**Lemma 2.1.17.** *The function $f$ is monotone, i.e., if $\underline{t}_1 \leq \underline{t}_2$ then $f(\underline{t}_1) \leq f(\underline{t}_2)$*

*Proof.* If $\underline{t}_1 \leq \underline{t}_2$, then $P(\underline{t}_1) \supseteq P(\underline{t}_2)$. Since $\mathcal{D}_{\mathcal{F}}$ is antitone, $\mathcal{D}_{\mathcal{F}}(P(\underline{t}_1)) \subseteq \mathcal{D}_{\mathcal{F}}(P(\underline{t}_2)))$. For a larger contract set, $P_{\mathcal{G}}$ assigns a higher score limit, so $P_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(P(\underline{t}_1)) \leq P_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(P(\underline{t}_2)))$. Therefore, $f$ is indeed a monotone function. □

**Statement 2.1.18.** *[Fleiner, Jankó][19] If the underlying graph $G$ is simple, and the choice functions $\mathcal{F}$ and $\mathcal{G}$ are substitutable and $\mathcal{G}$ is loser-free, then score vector $\underline{t}$ is stable if and only if $\underline{t} = P_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(P(\underline{t})))$.*

*Proof.* Let $J = \{e \notin P(\underline{t}) : e \in \mathcal{F}(P(\underline{t}) + e)\}$ be the set of contracts that $\mathcal{F}$ prefers to $\mathcal{F}(P(\underline{t}))$. In other words, $J = (\mathcal{D}_{\mathcal{F}}(P(\underline{t}))) \setminus P(\underline{t})$, therefore, $\mathcal{D}_{\mathcal{F}}(P(\underline{t})) = \mathcal{F}(P(\underline{t})) \cup J$. (See Figure 2.4.)

Suppose $\underline{t}$ is a fixed point. Let $B = \mathcal{D}_{\mathcal{F}}(P(\underline{t}))$, and we use that: $P(P_{\mathcal{G}}(B)) \cap B = \mathcal{G}(B)$. From this, $P(\underline{t}) \cap (\mathcal{D}_{\mathcal{F}}(P(\underline{t}))) = P(P_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(P(\underline{t})))) \cap (\mathcal{D}_{\mathcal{F}}(P(\underline{t}))) = \mathcal{G}(\mathcal{D}_{\mathcal{F}}(P(\underline{t})))$. Since $P(\underline{t}) \cap (\mathcal{D}_{\mathcal{F}}(P(\underline{t}))) = \mathcal{F}(P(t))$, we get that $\mathcal{G}(\mathcal{F}(P(\underline{t})) \cup J) = \mathcal{F}(P(\underline{t}))$, and since $\mathcal{G}$ is substitutable, $\overline{\mathcal{G}}(\mathcal{F}(P(\underline{t}))) \subseteq \overline{\mathcal{G}}(\mathcal{F}(P(\underline{t})) \cup J) = J$. Therefore, $\mathcal{G}(\mathcal{F}(P(\underline{t}))) = \mathcal{F}(P(\underline{t}))$, so $\underline{t}$ is valid.

To prove that $\underline{t}$ is violable, assume that college $C_j$ lowers its score limit by 1. Let $\underline{t}' = (t_1, t_2, \ldots, t_{j-1}, t_j - 1, t_{j+1}, \ldots, t_m)$. Then, at college $C_j$, the accepted $P(\underline{t})$ increases with some contracts.

Now, we use that the graph is simple. If $A_iC_j \in P(\underline{t})$, then applicant $A_i$ will also be accepted under $\underline{t}'$. If $A_i$ is not accepted at college $C_j$ with score vector $\underline{t}$, but she has a score of at least $t_j - 1$, then she will go to $C_j$ if and only if $C_jA_i \in J$, because she got only one new chance. Therefore, $\mathcal{F}(P(\underline{t}')) \cap E(C_j) = (\mathcal{F}(P(\underline{t})) \cup J) \cap E(C_j)$.

Other colleges cannot have new students, so they stay valid.

From $\mathcal{F}(P(\underline{t})) \cup J$, the scoring function $P_{\mathcal{G}_j}$ for college $C_j$ chooses score limit $t_j$. Therefore it also chooses $t_j$ from $\mathcal{F}(P(\underline{t}')$, so $\underline{t}'$ is not valid.

Now, assume that $\underline{t}$ is valid and violable. Therefore, $\mathcal{G}(\mathcal{F}(P(\underline{t}))) = \mathcal{F}(P(\underline{t}))$, so $\mathcal{G}(\mathcal{F}(P(\underline{t})) \cup J)$ accepts contracts in $\mathcal{F}(P(\underline{t}))$ (because contracts in $J$ do not reach score limit $\underline{t}$). Therefore, $P_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(P(\underline{t}))) \leq \underline{t}$.

As before, $\underline{t}' = (t_1, t_2, \ldots, t_{j-1}, t_j - 1, t_{j+1}, \ldots, t_m)$. Function $P$ is antitone, so $P(\underline{t}) \subseteq P(\underline{t}')$. Since $\mathcal{D}_{\mathcal{F}}$ is antitone, $\mathcal{D}_{\mathcal{F}}(P(\underline{t})) \supseteq \mathcal{D}_{\mathcal{F}}(P(\underline{t}')) \supseteq \mathcal{F}(P(\underline{t}'))$. As $\underline{t}$ is violable, $\mathcal{F}(P(t'))$ is infeasible for college $C_j$, so $\mathcal{D}_{\mathcal{F}}(P(\underline{t}))$ is too, and we get $P_{\mathcal{G}_j}(\mathcal{D}_{\mathcal{F}}(P(\underline{t}))) \geq t_j$. This is valid for every college. Therefore $P_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(P(\underline{t}))) = \underline{t}$. We did not use that $G$ is simple in the second direction and in the "valid" part of the first direction, so these parts remain true for general bipartite graphs. $\square$

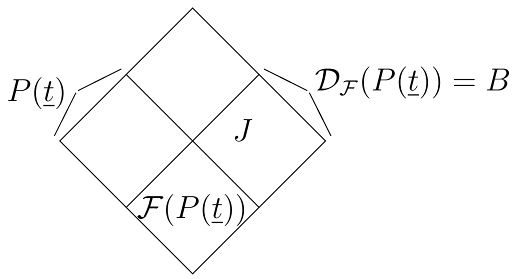



Figure 2.4: A four-partition.

Statement 2.1.18 and Tarski's fixed point theorem implies the following corollary:

**Theorem 2.1.19.** *[Fleiner, Jankó][19] If graph $G$ is simple, choice functions $\mathcal{F}$ and $\mathcal{G}$ are substitutable and $\mathcal{G}$ is loser-free, then the score-stable sets form a non-empty lattice.*

Moreover, we can show a link with four-stability for every bipartite graph.

**Statement 2.1.20.** *[Fleiner, Jankó][19] If choice functions $\mathcal{F}$ and $\mathcal{G}$ are substitutable, $\mathcal{G}$ is loser-free and $\mathcal{F}$ is IRC, then the following two statements are equivalent: (i) $S = \mathcal{F}(P(\underline{t}))$ for some score vector $\underline{t}$, such that $f(\underline{t}) = \underline{t}$. (ii) The contract set $S$ is four-stable.*

*Proof.* (i)$\Rightarrow$(ii) If $\underline{t}$ is a fixed point, $\underline{t} = P_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(P(\underline{t})))$, then let $B = \mathcal{D}_{\mathcal{F}}(P(\underline{t}))$. As we have seen in the proof of Statement 2.1.18, $\mathcal{F}(P(\underline{t})) = P(\underline{t}) \cap (\mathcal{D}_{\mathcal{F}}(P(\underline{t})) = \mathcal{G}(\mathcal{D}_{\mathcal{F}}(P(\underline{t}))) = \mathcal{G}(\mathcal{F}(P(\underline{t})) \cup J)$, so $S = \mathcal{G}(B)$.

This gives $(E \setminus \mathcal{D}_{\mathcal{G}}(B)) \cap B = \overline{\mathcal{G}}(B) = J$. From the contracts outside $B$, the set $B$ must dominate contracts under the score limit $\underline{t}$, since if colleges do not accept contracts from $J$, then they will not accept other contracts with the same or lower scores. (It cannot happen that for some college $C_j$, all contracts in $J$ have a score of $t_j - 2$ or less and some $e \notin B$ has a score of $t_j - 1$, because in that case $P_{\mathcal{G}_j}$ would have chosen $t_j - 1$ and $\underline{t}$ would not be stable.) Therefore, $A = \mathcal{D}_{\mathcal{G}}(B) \subseteq P(\underline{t})$.

Since $\mathcal{F}$ is IRC and $S = \mathcal{F}(P(\underline{t})) \subseteq A \subseteq P(\underline{t})$, we find that $\mathcal{F}(A) = S$.

From Lemma 1.3.4, $\mathcal{D}_{\mathcal{F}}(A) = \mathcal{D}_{\mathcal{F}}(P(t)) = B$, so $S$ is indeed four-stable.

(i)$\Leftarrow$(ii)

If $S$ is four-stable, then there exist $A$ and $B$, such that $\mathcal{F}(A) = S, \mathcal{G}(B) = S$. Let the score limit be $\underline{t} = P_{\mathcal{G}}(B)$. We want to know what $P(\underline{t})$ is. It is sure that $P(\underline{t}) \cap B = S$, because $P(P_{\mathcal{G}}(B)) \cap B = \mathcal{G}(B)$, and $A = \mathcal{D}_{\mathcal{G}}(B) \subseteq P(\underline{t})$.

Since $\mathcal{D}_{\mathcal{F}}(A) = B$, substitutability implies that dominated contracts will not be chosen from $P(\underline{t})$, since $A \subseteq P(\underline{t}) \subseteq A \cup (E \setminus B)$. Therefore, $\overline{\mathcal{F}}(P(\underline{t})) \supseteq P(\underline{t}) \setminus A$. Then, $\mathcal{F}(P(\underline{t})) \subseteq A \subseteq P(\underline{t})$. From the IRC property, $S = \mathcal{F}(A) = \mathcal{F}(P(\underline{t}))$.
Using Lemma 1.3.4 again, $\mathcal{D}_{\mathcal{F}}(P(\underline{t})) = \mathcal{D}_{\mathcal{F}}(A) = B, \mathcal{D}_{\mathcal{F}}(P(\underline{t})) = B$ and $P_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(P(\underline{t}))) = P_{\mathcal{G}}(B) = \underline{t}$. Therefore $\underline{t}$ is a fixed point. $\square$

As a corollary of Statements 2.1.18 and 2.1.20, we get the following theorem:

**Theorem 2.1.21.** *[Fleiner, Jankó][19] If choice functions $\mathcal{F}$ and $\mathcal{G}$ are substitutable, $\mathcal{G}$ is loser-free and the applicants' choice function $\mathcal{F}$ is path independent, then every score-stable set is also four-stable. Furthermore, if we require that the graph $G$ is simple, then score-stability is equivalent to four-stability.*

**Example 2.1.22.** *Figure 2.5 illustrates a counterexample for Theorem 2.1.21 if the underlying graph is not simple.*

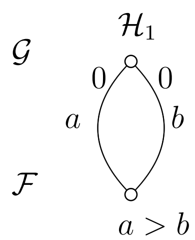



Figure 2.5: A counterexample.

*There are one college and one student, and the student applies both for mathematics and physics, but prefers mathematics. She achieved a zero score on both. The college has a common quota of 1 for these two faculties. If the score limit is 1, the college accepts nobody.*

*If the score limit is 0, the college accepts both contracts a and b, and the applicant prefers a, so only a is realized. This is valid and stable. Therefore, the only score-stable solution is* $\underline{t} = 0$ *and* $S = \mathcal{F}(P(\underline{t})) = \{a\}$. *However, there are two four-stable sets:*

- $S = \emptyset$ *with* $A = \emptyset, B = \{a, b\}$
- $S = \{a\}$ *with* $A = \{a\}, B = \{a\}$.

*The fixed points of* $f$ *are the same as the four-stable sets:* $\{a\}$ *and* $\emptyset$.

### 2.1.6 Connections between Different Stability Concepts

For a given two-sided market with choice functions $\mathcal{F}$ and $\mathcal{G}$ for each side, we have defined four kinds of stability:

1. A subset $S$ of $E$ is *three-stable*, if there exist subsets $A$ and $B$ of $E$, such that $\mathcal{F}(A) = S = \mathcal{G}(B)$ and $A \cup B = E$, $A \cap B = S$.
2. A subset $S$ of $E$ is *four-stable*, if there exist subsets $A$ and $B$ of $E$, such that $A \cap B = S$ and $\mathcal{D}_{\mathcal{F}}(A) = B, \mathcal{D}_{\mathcal{G}}(B) = A$.
3. A subset $S$ of $E$ is *dominating stable*, if $\mathcal{D}_{\mathcal{F}}(S) \cap \mathcal{D}_{\mathcal{G}}(S) = S$.
4. If $\mathcal{F}$ is substitutable, $\mathcal{G}$ is substitutable and loser-free we defined *generalized score-stability* in Subsection 2.1.5.

We will examine the connections between these definitions with regard to the IRC property. Aygün and Sönmez [6] showed that if $\mathcal{F}$ and $\mathcal{G}$ are substitutable and IRC, three-stable and dominating stable are equivalent, but without IRC, they are not. We extend this by considering all four definitions of stability.

**Theorem 2.1.23.** *[Fleiner, Jankó][19] If* $\mathcal{F}$ *and* $\mathcal{G}$ *are substitutable and IRC, then three-stability, four-stability and dominating stability are equivalent.*

*Proof. Three-stable* $\Rightarrow$ *dominating*
There are $A$ and $B$, such that $\mathcal{F}(A) = S = \mathcal{G}(B)$. From Lemma 1.3.4, $\mathcal{D}_{\mathcal{F}}(S) = \mathcal{D}_{\mathcal{F}}(A) \subseteq (E \setminus A) \cup S$. The same goes for $\mathcal{G}$, so $\mathcal{D}_{\mathcal{G}}(S) \subseteq (E \setminus A) \cup S$. Since $A \cup B = E$, their intersection is $\mathcal{D}_{\mathcal{F}}(S) \cup \mathcal{D}_{\mathcal{G}}(S) \subseteq S$. Additionally, $S$ is acceptable, so $\mathcal{D}_{\mathcal{F}}(S) \cap \mathcal{D}_{\mathcal{G}}(S) = S$.

*Dominating* $\Rightarrow$ *four-stable*
We know that $\mathcal{D}_{\mathcal{F}}(S) \cap \mathcal{D}_{\mathcal{G}}(S) = S$. Let $A = \mathcal{D}_{\mathcal{G}}(S)$ and $B = \mathcal{D}_{\mathcal{F}}(S)$. $A \subseteq S \cup (E \setminus \mathcal{D}_{\mathcal{F}}(S))$, so $\mathcal{F}(A) = S$. From Lemma 1.3.4, $\mathcal{D}_{\mathcal{F}}(A) = \mathcal{D}_{\mathcal{F}}(S) = B$. Similarly, $\mathcal{D}_{\mathcal{G}}(B) = \mathcal{D}_{\mathcal{G}}(S) = A$. With this $(A, B)$ pair, $S$ is four-stable.

*Four-stable $\Rightarrow$ three-stable*
There exist subsets $A$, $B$ of $E$, such that $\mathcal{F}(A) = S = \mathcal{G}(B)$, $A \cap B = S$, and $\mathcal{D}_{\mathcal{F}}(A) = B, \mathcal{D}_{\mathcal{G}}(B) = A$. Let $D = E \setminus (A \cup B)$ and $A' = A \cup D$. Now, $A' \cup B = E$, $A' \cap B = S$, and from Lemma 1.3.4, $\mathcal{D}_F(S) = \mathcal{D}_F(A) = B = A' \setminus S$. Therefore $\mathcal{F}(A') = S$, $\mathcal{G}(B) = S$, so with the pair $(A', B)$, $S$ is three-stable. □

**Theorem 2.1.24.** *[Fleiner, Jankó][19] If $\mathcal{F}$ and $\mathcal{G}$ are substitutable choice functions, $\mathcal{F}$ is IRC, (but $\mathcal{G}$ may not be) then, every four-stable set is three-stable.*

*Proof.* Notice that the third part of the proof of Theorem 2.1.23 did not use that $G$ was IRC. □

Compared to other stability concepts, three-stability, four-stability and dominating stability are defined on every substitutable choice function, $\mathcal{F}$ and $\mathcal{G}$, but for score stability, we need substitutability on one side and a substitutable, loser-free function on the other side.

As we showed in Theorem 2.1.21, if a choice function $\mathcal{F}$ is substitutable and IRC and $\mathcal{G}$ is substitutable loser-free, then every score-stable set is also four-stable.

**Theorem 2.1.25** (Fleiner, Jankó). *[19] If $\mathcal{F}$ is substitutable and $\mathcal{G}$ is substitutable and loser-free, then every score-stable solution is three-stable.*

*Proof.* Let $S \subseteq E$ be the enrollment realized from a stable score vector $\underline{t}$. Define $A$ as the set of contracts above score vector $\underline{t}$, i.e. $A = P(\underline{t})$. Let $B$ be the union of $S$ and the set of contracts under score limit $\underline{t}$, i.e. $B = S \cup (E \setminus P(\underline{t}))$. From all contracts above score vector $\underline{t}$, the applicants choose contract set $S$, so $\mathcal{F}(A) = S$. If colleges choose from contract set $B$, just like from all contracts, they would set the score limit to $\underline{t}$, so $\mathcal{G}(B) = S$. If colleges would like to accept one more contract, all the contracts with the same score are in $B$, and accepting all would contradict the stability of $\underline{t}$. It is easy to see that $A \cup B = E, A \cap B = S$. Therefore $S$ is three-stable. □

Theorems 2.1.21, 2.1.23, 2.1.24, and 2.1.25 are summarized in Figure 2.6 below. In the notation, 3 stands for three-stable, 4 for four-stable, $d$ for dominating stable and $s$ for score-stable sets.

The solid lines denote implications that are true even if the underlying graph is not simple. The dashed lines denote the extra implications when the graph of possible contracts is simple.

**Statement 2.1.26.** *For all the implications that are not shown in Theorems 2.1.21, 2.1.23, 2.1.24 and 2.1.25, and therefore not present in the above picture, we can show a counterexample.*

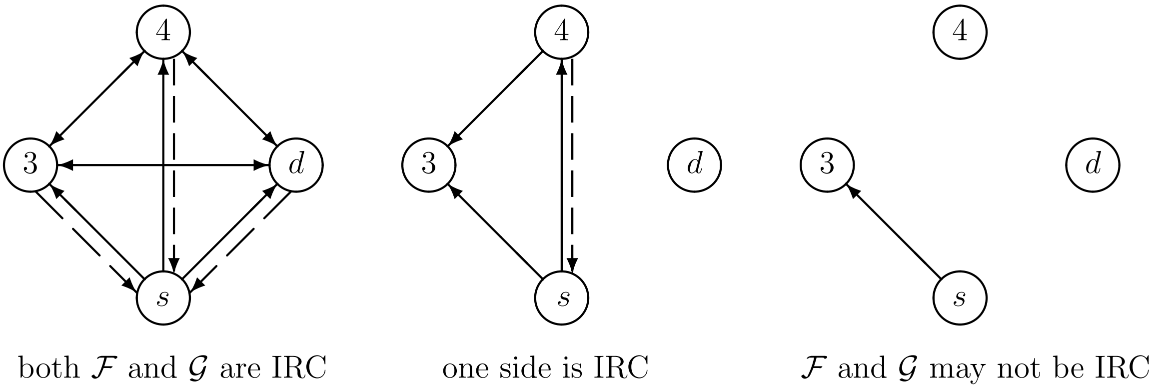



Figure 2.6: Graphs of the connections.

*Proof.* Table 2.1 describes the choice functions, usually as a direct sum of the choice functions of individual colleges or applicants. The notation $\mathcal{H}_1(a, b)$ means a college chooses from equally good contracts $a$ and $b$ and its quota is 1.

Table 2.2 shows the stable sets for each notion for the seven examples.

| **Number** | **Simple graph** | **F** | **IRC** | **G** | **IRC** |
|---|---|---|---|---|---|
| I | no | $\mathcal{H}_1$ | no | $\mathcal{H}_1$ | no |
| II | no | $a > b$ | yes | $\mathcal{H}_1$ | no |
| III | no | $a > b$ | yes | $b > a$ | yes |
| IV | yes | $a + \mathcal{H}_1(b, c)$ | no | $\mathcal{H}_1(a, b) + c$ | no |
| V | yes | $a + \mathcal{H}_1(b, c, d)$ | no | $\mathcal{H}_1(a, b) + c + d$ | no |
| VI | yes | $(a > c) + (d > b)$ | yes | $\mathcal{H}_1(a, b) + \mathcal{H}_1(c, d)$ | no |
| VII | yes | $(a > c) + (d > b)$ | yes | $\mathcal{H}_1(a, b) + \mathcal{H}_2(c, d)$ | no |

Table 2.1: The choice functions for the seven examples.

$\square$

| **Number** | **Three-Stable** | **Four-Stable** | **Dominating Stable** | **Score-Stable** | $\underline{t}$ **Fixed** |
|---|---|---|---|---|---|
| I | $\emptyset$ | $\emptyset, \{a\}, \{b\}$ | $\{a\}, \{b\}$ | $\emptyset$ | $\emptyset$ |
| II | $\emptyset, \{a\}$ | $\emptyset, \{a\}$ | $\{a\}, \{b\}$ | $\{a\}$ | $\emptyset, \{a\}$ |
| III | $\{a\}, \{b\}$ | $\{a\}, \{b\}$ | $\{a\}, \{b\}$ | $\{a\}$ | $\{a\}, \{b\}$ |
| IV | $\{a\}, \{c\}$ | $\{a, c\}$ | $\{a, c\}, \{b\}$ | $\{a\}$ | $\{a\},$ |
| V | $\emptyset, \{a\}$ | $\{a\}$ | $\{b\}, \{a, c\}, \{a, d\}$ | $\{a\}$ | $\{a\}$ |
| VI | $\emptyset, \{a, d\}$ | $\emptyset, \{a, d\}$ | $\{a, d\}, \{b, c\}$ | $\emptyset, \{a, d\}$ | $\emptyset, \{a, d\}$ |
| VII | $\{a, d\}, \{c, d\}$ | $\{a, d\}$ | $\{a, d\}$ | $\{a, d\}$ | $\{a, d\}$ |

Table 2.2: Stable sets in case of different stability notions.

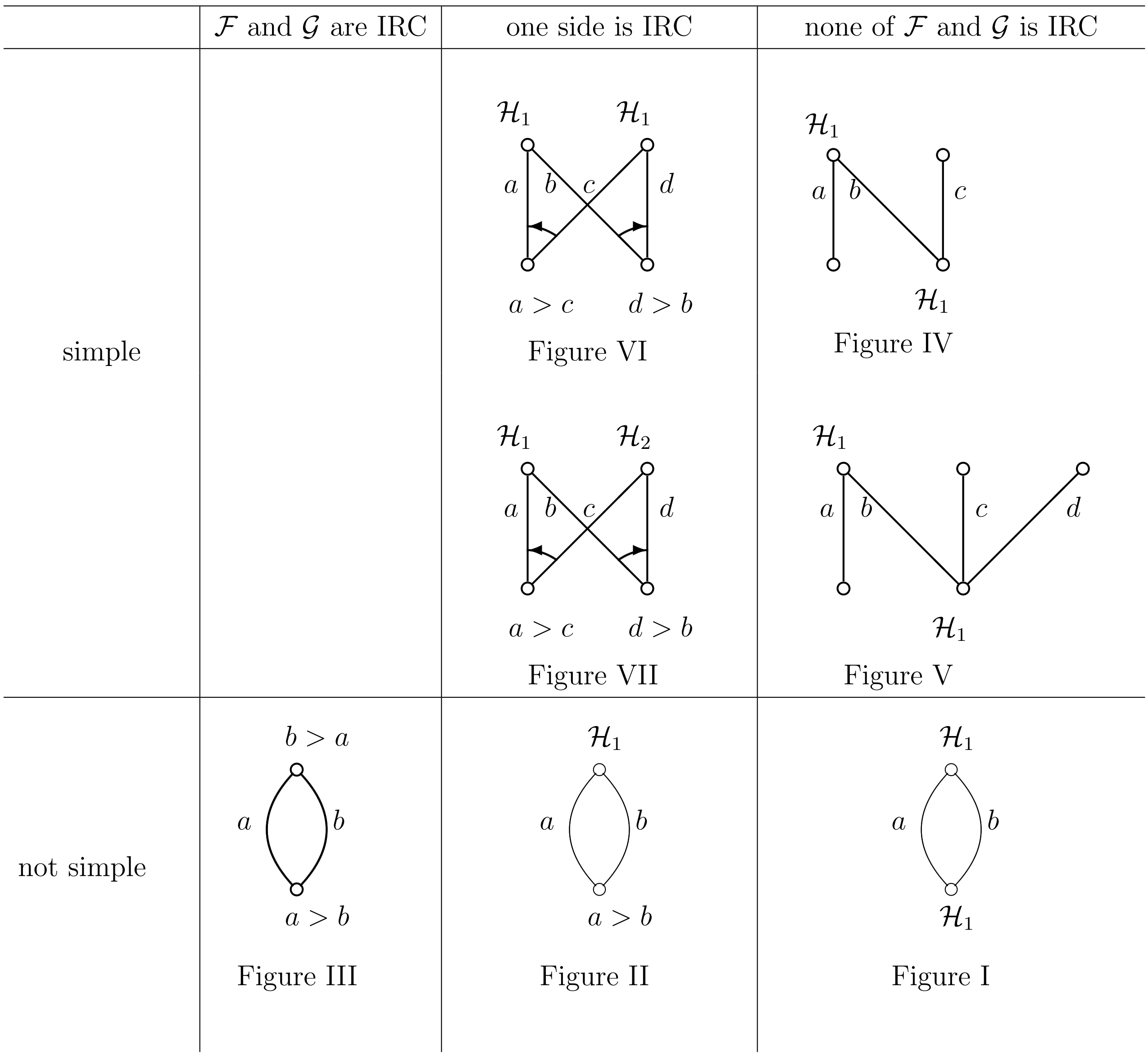



Figure 2.7: Graphs of seven examples.

### 2.1.7 Stability on lattices

Let $\mathcal{L} = (X, \preceq)$ be a lattice. Dominating stability and four-stability can be generalized nicely to choice functions over lattices.

We can generalize some of the stability definitions we used in the previous subsections to lattices in the following way: Assume that $\mathcal{L} = (X, \preceq)$ is a lattice and $\mathcal{F}$ and $\mathcal{G}$ are substitutable choice functions.

- An element $s$ of $X$ is *$\mathcal{FG}$-stable* if

$$\mathcal{F}(s) = \mathcal{G}(s) = s \qquad \text{and} \tag{2.1}$$
$$\mathcal{F}(s \vee x) \wedge \mathcal{G}(s \vee x) \preceq s \qquad \text{holds for each element } x \text{ of } X. \tag{2.2}$$

  Intuitively, this means that $s$ is both $\mathcal{F}$-independent and $\mathcal{G}$-independent, moreover, if some other option $x$ is offered together with $s$, then it is impossible that both these choice functions select $x$. We denote the set of $\mathcal{FG}$-stable elements by $\mathcal{S}(\mathcal{FG})$.

- An element $s$ of $X$ is *dominating stable* if

$$\mathcal{D}_{\mathcal{F}}(s) \wedge \mathcal{D}_{\mathcal{G}}(s) = s. \tag{2.3}$$

- An element $s$ of $X$ is called *four-stable* if there exist elements $a$ and $b$ of $X$ such that

$$s = a \wedge b,\ \mathcal{D}_{\mathcal{F}}(a) = b \text{ and } \mathcal{D}_{\mathcal{G}}(b) = a \tag{2.4}$$

  hold.

Note that $\mathcal{FG}$-stability is one possible generalization of dominating stability, since every $x \not\preceq s$ is either $\mathcal{F}$-dominated or $\mathcal{G}$-dominated.

However, we also defined the exact generalization of dominating stability, with canonical determinants. We show that this definition is equivalent to $\mathcal{FG}$-stability.

**Lemma 2.1.27** (Fleiner, Jankó). *[20] Assume that $\mathcal{F}$ and $\mathcal{G}$ are substitutable choice functions on an infinitely distributive complete lattice $\mathcal{L} = (X, \preceq)$. An element $s$ of $X$ is $\mathcal{FG}$-stable if and only if it is dominating stable.*

*Moreover, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are antitone determinants of $\mathcal{F}$ and $\mathcal{G}$ and $\mathcal{D}_1(s) \wedge \mathcal{D}_2(s) = s$ holds for some element $s$ of $X$ then $s$ is $\mathcal{FG}$-stable.*

*Proof.* Assume first that $s$ is $\mathcal{FG}$-stable. From Lemma 1.3.7 and the infinite distributivity of $L$, we get

$$
\begin{gathered}
s=\mathcal{F}(s)\wedge\mathcal{G}(s)=s\wedge\mathcal{D}_{\mathcal{F}}(s)\wedge\mathcal{D}_{\mathcal{G}}(s)\preceq\mathcal{D}_{\mathcal{F}}(s)\wedge\mathcal{D}_{\mathcal{G}}(s)=\\
\bigvee\{y:y\preceq\mathcal{F}(s\vee y)\}\wedge\bigvee\{z:z\preceq\mathcal{G}(s\vee z)\}=\bigvee\{y\wedge z:y\preceq\mathcal{F}(s\vee y),z\preceq\mathcal{G}(s\vee z)\}\preceq\\
\bigvee\{y\wedge z:y\wedge z\preceq\mathcal{F}(s\vee y),y\wedge z\preceq\mathcal{G}(s\vee z)\}\preceq\\
\bigvee\{y\wedge z:y\wedge z\preceq\mathcal{F}(s\vee(y\wedge z)),y\wedge z\preceq\mathcal{G}(s\vee(y\wedge z))\}=\\
\bigvee\{x:x\preceq\mathcal{F}(s\vee x),x\preceq\mathcal{G}(s\vee x)\}=\\
\bigvee\{x:x\preceq\mathcal{F}(s\vee x)\wedge\mathcal{G}(s\vee x)\}\preceq\bigvee\{x:x\preceq s\}=s\ .
\end{gathered}
$$

So we have equality throughout, in particular (2.3) holds.

Now assume that $s$ is dominating stable. Then

$$s=s\wedge s\succeq\mathcal{F}(s)\wedge\mathcal{G}(s)=s\wedge\mathcal{D}_{\mathcal{F}}(s)\wedge s\wedge\mathcal{D}_{\mathcal{G}}(s)=s\wedge(\mathcal{D}_{\mathcal{F}}(s)\wedge\mathcal{D}_{\mathcal{G}}(s))=s\wedge s=s,$$

hence we have equality throughout, in particular $\mathcal{F}(s)\wedge\mathcal{G}(s)=s$. As $\mathcal{F}$ and $\mathcal{G}$ are choice functions, $\mathcal{F}(s)=\mathcal{G}(s)=s$ follows. To see that $\mathcal{F}(s\vee x)\wedge\mathcal{G}(s\vee x)\preceq s$ holds let $x$ be an arbitrary element of $X$. By Lemma 1.3.7 and the antitone property of the canonical determinants, we see that

$$\mathcal{F}(s\vee x)\wedge\mathcal{G}(s\vee x)=(s\vee x)\wedge\mathcal{D}_{\mathcal{F}}(s\vee x)\wedge\mathcal{D}_{\mathcal{G}}(s\vee x)\preceq\mathcal{D}_{\mathcal{F}}(s)\wedge\mathcal{D}_{\mathcal{G}}(s)=s$$

and this finishes the proof of the first part of the Lemma.

To prove the second part, observe that

$$s\succeq\mathcal{F}(s)=s\wedge\mathcal{D}_1(s)\succeq s\wedge\mathcal{D}_1(s)\wedge\mathcal{D}_2(s)=s\wedge s=s$$

and a similar argument shows that $\mathcal{G}(s)=s$ as well. By Lemma 1.3.7 we get that

$$s\preceq\mathcal{D}_{\mathcal{F}}(s)\wedge\mathcal{D}_{\mathcal{G}}(s)\preceq\mathcal{D}_1(s)\wedge\mathcal{D}_2(s)=s,$$

hence we have equality throughout, in particular $\mathcal{D}_{\mathcal{F}}(s)\wedge\mathcal{D}_{\mathcal{G}}(s)=s$, proving the $\mathcal{F}\mathcal{G}$-stability of $s$. □

The following consequence of Lemma 2.1.27 is an important characterization of $\mathcal{F}\mathcal{G}$-stable sets in the case where the substitutable choice functions $\mathcal{F}$ and $\mathcal{G}$ are also path-independent.

Recall that a choice function $\mathcal{F}:X\to X$ on a lattice $L$ is path-independent if $\mathcal{F}(x\vee y)=\mathcal{F}(x\vee\mathcal{F}(y))$ holds for all elements $x,y$ of $X$. $\mathcal{F}$ is IRC if $\mathcal{F}(x)\preceq y\preceq x$ implies $\mathcal{F}(x)=\mathcal{F}(y)$.

**Lemma 2.1.28** (Fleiner, Jankó)**.** *[20] Assume that $\mathcal{F}$ and $\mathcal{G}$ are path-independent, substitutable choice functions on an infinitely distributive complete lattice $\mathcal{L}=(X,\preceq)$. Then $s\in X$ is $\mathcal{F}\mathcal{G}$-stable if and only if $s$ is four-stable.*

*Furthermore,* (2.4) *implies $\mathcal{F}(a)=\mathcal{G}(b)=s$ and $a=\mathcal{D}_{\mathcal{G}}(s)$, $b=\mathcal{D}_{\mathcal{F}}(s)$.*

Notice that property $s = a \wedge b$ and $\mathcal{D}_{\mathcal{F}}(a) = b$ and $\mathcal{D}_{\mathcal{G}}(b) = a$ is exactly four-stability. So this Lemma proves the equivalence between dominating stability and four-stability for substitutable, path-independent lattice choice functions thus this is a generalization of a part of Theorem 2.1.23

*Proof.* For sufficiency, assume that $s = a \wedge b$ and $\mathcal{D}_{\mathcal{F}}(a) = b$ and $\mathcal{D}_{\mathcal{G}}(b) = a$. By Lemma 1.3.7, $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}_{\mathcal{G}}$ have property $\mathcal{D}_{\mathcal{F}}(x) = \mathcal{D}_{\mathcal{F}}(\mathcal{F}(x))$ and $\mathcal{D}_{\mathcal{G}}(x) = \mathcal{D}_{\mathcal{G}}(\mathcal{G}(x))$ thus

$$b = \mathcal{D}_{\mathcal{F}}(a) = \mathcal{D}_{\mathcal{F}}(\mathcal{F}(a)) = \mathcal{D}_{\mathcal{F}}(a \wedge \mathcal{D}_{\mathcal{F}}(a)) = \mathcal{D}_{\mathcal{F}}(a \wedge b) = \mathcal{D}_{\mathcal{F}}(s) \tag{2.5}$$

and similarly

$$a = \mathcal{D}_{\mathcal{G}}(b) = \mathcal{D}_{\mathcal{G}}(\mathcal{G}(b)) = \mathcal{D}_{\mathcal{G}}(b \wedge \mathcal{D}_{\mathcal{G}}(b)) = \mathcal{D}_{\mathcal{G}}(b \wedge a) = \mathcal{D}_{\mathcal{G}}(s)$$

hence $\mathcal{D}_{\mathcal{F}}(s) \wedge \mathcal{D}_{\mathcal{G}}(s) = a \wedge b = s$. So $s$ is $\mathcal{F}\mathcal{G}$-stable by Lemma 2.1.27.

To see necessity, assume that $s$ is $\mathcal{F}\mathcal{G}$-stable, that is, $\mathcal{D}_{\mathcal{F}}(s) \wedge \mathcal{D}_{\mathcal{G}}(s) = s$ by Lemma 2.1.27. Define $a := \mathcal{D}_{\mathcal{G}}(s)$ and $b := \mathcal{D}_{\mathcal{F}}(s)$ and observe that

$$\mathcal{F}(a) = a \wedge \mathcal{D}_{\mathcal{F}}(a) \preceq a \wedge \mathcal{D}_{\mathcal{F}}(s) = a \wedge b = s \preceq a$$

so $\mathcal{F}(a) \preceq s \preceq a$, hence $\mathcal{F}(a) = \mathcal{F}(s) = s \wedge \mathcal{D}_{\mathcal{F}}(s) = s \wedge b = s$. Now $\mathcal{D}_{\mathcal{F}}(a) = \mathcal{D}_{\mathcal{F}}(\mathcal{F}(a)) = \mathcal{D}_{\mathcal{F}}(s) = b$ and a similar argument shows that $\mathcal{D}_{\mathcal{G}}(b) = a$.

To see the second part, observe first that

$$\mathcal{G}(b) = b \wedge \mathcal{D}_{\mathcal{G}}(b) = b \wedge a = s \text{ and } \mathcal{F}(a) = a \wedge \mathcal{D}_{\mathcal{F}}(a) = a \wedge b = s\ ,$$

hence we get

$$a = \mathcal{D}_{\mathcal{G}}(b) = \mathcal{D}_{\mathcal{G}}(\mathcal{G}(b)) = \mathcal{D}_{\mathcal{G}}(s) \text{ and } b = \mathcal{D}_{\mathcal{F}}(a) = \mathcal{D}_{\mathcal{F}}(\mathcal{F}(a)) = \mathcal{D}_{\mathcal{F}}(s)$$

and this finishes the proof. □

The following lemma is a generalization of the stable marriage theorem.

**Lemma 2.1.29** (Fleiner, Jankó)**.** *[20] If $\mathcal{L} = (X, \preceq)$ is an infinitely distributive complete lattice and $\mathcal{F}$ and $\mathcal{G}$ are substitutable path-independent choice functions then there exists an $\mathcal{F}\mathcal{G}$-stable element $s$ of $X$.*

*Proof.* Define a mapping

$$\mathcal{M}(x) := \mathcal{D}_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(x)). \tag{2.6}$$

As both $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}_{\mathcal{G}}$ are antitone, $\mathcal{M}$ is monotone, and by Tarski's fixed point theorem (Theorem 1.2.1) there exists a fixed point $a$ of $\mathcal{M}$. Define $b := \mathcal{D}_{\mathcal{F}}(a)$. Now $a = \mathcal{M}(a) = \mathcal{D}_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(a)) = \mathcal{D}_{\mathcal{G}}(b)$, hence $s = a \wedge b$ is four-stable, so by Lemma 2.1.28, it is also an $\mathcal{F}\mathcal{G}$-stable set. □

Note that there is a generalization of the deferred acceptance algorithm of Gale and Shapley that finds an $\mathcal{FG}$-stable element in the case where the lattice $L$ is finite. This generalized algorithm finds a fixed point $a$ of a monotone function $\mathcal{M}$ in the proof of Lemma 2.1.29. This is done according to the remark after Theorem 1.2.1. Namely, if 0 is the least element of the lattice $L$ then $0 \preceq \mathcal{M}(0)$ implies $\mathcal{M}(0) \preceq \mathcal{M}(\mathcal{M}(0))$, and this yields $\mathcal{M}(\mathcal{M}(0)) \preceq \mathcal{M}(\mathcal{M}(\mathcal{M}(0)))$. So if $L$ is a finite lattice then the chain $0 \preceq \mathcal{D}_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(0)) \preceq \mathcal{D}_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(\mathcal{D}_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(0)))) \preceq \cdots$ must converge to a fixed point $a$. Then $s := a \wedge \mathcal{D}_{\mathcal{F}}(a)$ is an $\mathcal{FG}$-stable element of $L$.

## 2.2 Algorithms

In this section, we show algorithms to find three-stable, four-stable and score-stable allocations. These stable solutions always exist, if $\mathcal{F}$ and $\mathcal{G}$ are substitutable (and $\mathcal{G}$ is also loser-free in the case of score-stability). Moreover, these algorithms give us the men-optimal or women-optimal solutions. We show a close connection between Tarski's fixed point theorem and the Gale–Shapley algorithm.

### 2.2.1 A Generalized Gale–Shapley Algorithm for Three- and Four-Stability

For three-stable sets, we can generalize the Gale–Shapley algorithm to the case where both choice functions are substitutable, but they do not have to be IRC. It is a special case of the monotone function iteration that finds a fixed point of a monotone function. The following algorithm is now well-known; it appeared in [28]:

Let $\mathcal{F}$ be the choice function of men (or students), and let $\mathcal{G}$ be the choice function of women (or colleges). In the male-proposing version, in every step, let $X_i$ be the contract-set the men choose from, and let $Y_i$ be the contract-set women can select from. First, the let $X_1 = E$, so men choose from all contracts and propose to $Y_1 = \mathcal{F}(E) = E \setminus \overline{\mathcal{F}}(E)$. Women choose $\mathcal{G}(\mathcal{F}(E))$ and reject $\overline{\mathcal{G}}(\mathcal{F}(E))$. In the second step, men choose from all contracts, except for the previously rejected ones: $X_2 = E \setminus \overline{\mathcal{G}}(Y_1) = E \setminus \overline{\mathcal{G}}(F(X_1))$. The men choose $\mathcal{F}(X_2)$, the women take these contracts and the previously rejected contracts and choose from $Y_2 = \mathcal{F}(X_2) \cup \overline{\mathcal{G}}(\mathcal{F}(X_1)) = E \setminus \overline{\mathcal{F}}(X_2)$. Since $\mathcal{G}$ is substitutable, if a contract was rejected earlier, it will be rejected in this step, too.

Here, this algorithm differs from the original Gale–Shapley, since there, women choose only from their current proposals. However, if $\mathcal{G}$ is IRC, then $\mathcal{G}(Y_2) \subseteq \mathcal{F}(X_2) \subseteq Y_2$ implies $\mathcal{G}(Y_2) = \mathcal{G}(\mathcal{F}(X_2))$, so putting back already refused proposals to the choice set does not change the outcome.

The general step of the algorithm is as follows: for a given $X_i$, let $Y_i = E \setminus \overline{\mathcal{F}}(X_i)$, and let $X_{i+1} = E \setminus \overline{\mathcal{G}}(Y_i)$.

Define the following function, $f$:

$$f(X_i, Y_i) = (E \setminus \overline{\mathcal{G}}(Y_i), E \setminus \overline{\mathcal{F}}(E \setminus \overline{\mathcal{G}}(Y_i)))$$

We can define a partial order on pairs with $(A', B') \sqsubseteq (A, B)$ if $A' \subseteq A$ and $B' \supseteq B$.

Observe that $f$ is monotone for this ordering. The iteration of this monotone function gives us a fixed pair $(X_i, Y_i)$, which corresponds to a three-stable pair $(A, B)$. If we start our iteration from the pair $(X_1, Y_1) = (E, \mathcal{F}(E))$, we get the male-optimal matching, if we start from $(X_1, Y_1) = (\emptyset, \emptyset)$, we get the female-optimal one.

There is an alternative algorithm similar to the previous one. Define a function $f' : 2^E \times 2^E \to 2^E \times 2^E$ by:

$$f'(A, B) := (E \setminus (\overline{\mathcal{G}}(B)), E \setminus (\overline{\mathcal{F}}(A))).$$

If $\mathcal{F}$ and $\mathcal{G}$ are substitutable, then $f'$ is monotone for the order $\sqsubseteq$, since, if $B$ decreases, then $\overline{\mathcal{G}}(B)$ decreases, so $E \setminus \overline{\mathcal{G}}(B)$ increases. Similarly, if $A$ increases, then $E \setminus \overline{\mathcal{F}}(A)$ decreases.

As before, three-stable pairs are exactly the fixed points of $f'$. We start the iteration from $(A_1, B_1) = (E, \emptyset)$ for the men-optimal solution or with $(A_1, B_1) = (\emptyset, E)$ for the women-optimal solution.

For four-stability, we define the monotone function $f''$ as follows:

$$f''(A, B) := (\mathcal{D}_{\mathcal{G}}(B), \mathcal{D}_{\mathcal{F}}(A))$$

If $\mathcal{F}$ and $\mathcal{G}$ are substitutable, then $\mathcal{D}_{\mathcal{F}}, \mathcal{D}_{\mathcal{G}}$ are antitone. Therefore, $f''$ is monotone for the order $\sqsubseteq$. Fixed points of $f''$ are four-stable pairs $(A, B)$, since $A = \mathcal{D}_{\mathcal{G}}(B)$, $B = \mathcal{D}_F(A)$.

If we start the iteration of $f''$ from $(A_1, B_1) = (E, \emptyset)$, we get a four-stable pair with the largest possible $A$ and smallest possible $B$, so it is men-optimal. Starting with the pair $(A_1, B_1) = (\emptyset, E)$ leads to the women-optimal solution.

In these algorithms we had a $\sqsubseteq$-monotone function over $2^E \times 2^E$ and we stopped when this function came to a fixed point. Therefore the runtime of the algorithms is at most the *height* (i.e., the maximum cardinality of a chain) of the lattice $(2^E \times 2^E, \sqsubseteq)$, which is $2|E|$. So the algorithms terminate in at most $2|E|$ steps, the same as the Gale–Shapley algorithm.

### 2.2.2 Algorithms for Score-Stability

In this subsection, we describe algorithms for generalized score-stability, hence also for score-stability. These algorithms are well-known by now, see e.g. [9]

1. **The score-decreasing algorithm:** colleges start from a valid score vector $\underline{t_0}$ (e.g., $\underline{t_C} := (M+1, \ldots, M+1)$). First, if there is a college $C_i$ that can lower its

score limit without getting too many students, then $C_i$ will decrease its score limit to the lowest score, such that $C_i$ still gets a feasible set of students. Here, $C_i$ chooses from free students and students who prefer $C_i$ to their college, so it chooses score limit $P_{\mathcal{G}_i}(\mathcal{D}_{\mathcal{F}}(P(\underline{t}))$. Then, we choose another college, and iterate this score-decreasing step. (It is convenient to check $C_1$ first, then $C_2$, then all colleges one-by-one. After $C_m$, we return to $C_1$ again.) The algorithm terminates if no college wants to lower its score limit any more. As soon as no college can decrease its score limit, the score vector is stable. Let $\underline{s_C}$ denote the stable score vector that we get by running the score-decreasing algorithm on $\underline{t_C}$.

**Theorem 2.2.1** (Fleiner, Jankó). *[19] If a stable score vector $\underline{t}$ is the output of the score-decreasing algorithm with input $\underline{t_0}$, where $\underline{t_0}$ is valid, then $\underline{t}$ is stable and $\underline{t}$ is the maximum of all the stable score vectors that are not greater than $\underline{t_0}$. Consequently, $\underline{s_C}$ is the maximum of all stable score vectors. Furthermore, $\underline{s_C}$ is applicant-pessimal.*

2. **The score-increasing algorithm:** colleges start with some violable score vector $\underline{t_0}$ (e.g., $\underline{t_A} = (0, \dots, 0)$), and keep on raising their score limits. If there is a college $C_i$ that has an infeasible set of students, then it raises the score limit to the lowest score where it becomes feasible. Therefore, it chooses $P_{\mathcal{G}_i}(\mathcal{F}(P(\underline{t}))$. Then, another college $C_j$ increases the score limit and all colleges one-by-one. The algorithm stops if no college wants to raise its score limit. Let $\underline{s_A}$ be the stable score vector the score-increasing algorithm outputs from input $\underline{t_A}$.

**Theorem 2.2.2** (Fleiner, Jankó). *[19] If a score vector $\underline{t}$ is the output of the score-increasing algorithm with input $\underline{t_0}$, where $\underline{t_0}$ is violable, then $\underline{t}$ is stable, and it is the minimum of all the stable score vectors that are not less than $\underline{t_0}$. Consequently, $\underline{s_A}$ is the minimum of all stable score vectors. Moreover, $\underline{s_A}$ is applicant-optimal.*

In [**?**] Biró showed that in the score-decreasing algorithm, if we start form a reasonably high score-vector $\underline{t_0}$ where every university is under-subscribed, and in the the score-increasing algorithm if we start from a zero, then the outputs $\underline{s_C}$ and $\underline{s_A}$ are the maximal and minimal stable score vectors. Here we show a bit more general statement since we allow any valid or violable starting score-vector.

**Theorem 2.2.3** (Fleiner, Jankó). *[19] The score-decreasing algorithm and the score-increasing algorithm terminates in $O(m^2n)$ and $O(mn)$ time, respectively.*

In the proofs of Theorems 2.2.1 and 2.2.2, we use the alternative versions of the score-decreasing/increasing algorithms, where in every step, a college decreases or increases its score limit only by 1. These modified algorithms also find stable solutions, as one step of the score decreasing or score increasing algorithm can be regarded as several steps of this modified algorithm. From Lemma 2.1.15, if a score vector $\underline{t}$ is valid and $\underline{t}' = (t_1, \dots, t_{j-1}, t_j - k, \dots, t_m)$ is also valid, then for every $1 \leq k' \leq k$,

$\underline{t}'' = (t_1, \ldots, t_{j-1}, t_j - k', \ldots, t_m)$ is valid (it is $C_j$-valid, because $\underline{t}'$ is valid, and valid for other colleges, because $\underline{t}$ is valid).

From the maximal/minimal property of the output solution, we see that the output of the algorithm does not depend on the order in which the colleges modify their score limits. Note that these algorithms may use $m(M+1)$ steps, for example, if we start decreasing from $\underline{t_C} := (M+1, \ldots, M+1))$, but only $(0, \ldots, 0)$ is a stable score vector.

*Proof of Theorem 2.2.1.* From the algorithm description, the fact that no college can decrease its score limit implies that $\underline{t}$ is a stable vector. Suppose that there exists a stable score vector $\underline{t}^1 \leq \underline{t}_0$, where $\underline{t}^1 \not\leq \underline{t}$. Therefore, $\underline{t} = (t_1, \ldots, t_m)$ and $\underline{t}^1 = (t_1^1, \ldots, t_m^1)$, and $t_i^1 > t_i$ for some $i$. Define the set:

$$T = \{\underline{x} \in \mathbb{N}^m : x_j \geq t_j^1 \quad \forall j \in \{1, \ldots, m\}\}$$

The algorithm starts from $\underline{t}_0 \in T$ and ends with $\underline{t} \notin T$, so there is a step when the score vector leaves $T$: from score vector $\underline{w}^1 = (w_1, w_2, \ldots, t_k^1, \ldots, w_m) \in T$, we move to $\underline{w}^2 = (w_1, w_2, \ldots, t_k^1 - 1, \ldots, w_m) \notin T$. Since this step is possible, both $\underline{w}^1$ and $\underline{w}^2$ are valid score vectors. We know that $\underline{w}^1 \geq \underline{t}^1$. Using Lemma 2.1.16, both $\underline{t}^1$ and $\underline{w}^2$ are valid, so their minimum $\underline{w}^3 = (t_1^1, t_2^1, \ldots, t_k^1 - 1, \ldots, t_m^1)$ is also valid. Therefore, $\underline{w}^3$ is stable, and it can be reached from $\underline{t}^1$ by lowering the score limit of $C_k$. Therefore, $\underline{t}^1$ is not violable, hence it cannot be stable, a contradiction.

Since every stable score vector is less than or equal to $\underline{t_0} = (M+1, \ldots, M+1)$, the biggest of all stable score vectors is $\underline{s_C}$. Every student is accepted by fewer colleges than in any other stable admissions, so $\underline{s_C}$ is applicant-pessimal. Figure 2.8 shows a possible layout. □

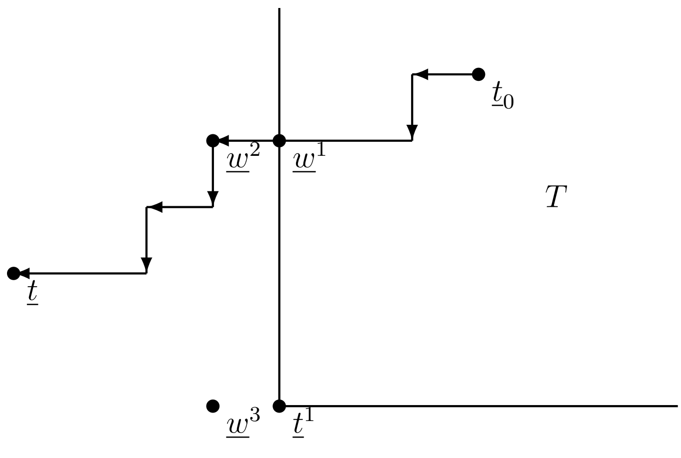



Figure 2.8: The score-decreasing algorithm.

*Proof of Theorem 2.2.2.* It follows from the algorithm that $\underline{t}$ is valid. Suppose that it is not stable, *i.e.*, there is a college $C_j$ such that $\underline{t}' = (t_1, t_2, \ldots, t_{j-1}, t_j - 1, t_{j+1}, \ldots, t_m)$

is still valid. If $t_j^0 \leq t_j - 1$, look at the step where college $C_j$ raises its score from $t_j - 1$ to $t_j$, moving from score vector $\underline{v}^1$ to $\underline{v}^2$.

Since the score limits in the algorithm always increase, $\underline{v}^1 \leq \underline{t}$ and $v_j^1 = t_j - 1$, therefore $\underline{v}^1 \leq \underline{t}'$. We use Lemma 2.1.15: the score limit $\underline{t}'$ is valid, so $\underline{v}^1$ is also $C_j$-valid. However, then the algorithm would not have increased $\underline{v}^1$ to $\underline{v}^2$, a contradiction. Therefore, $\underline{t}$ is stable.

If $t_j^0 = t_j$, since $\underline{t}_0$ is violable, the score vector $\underline{t}_0' = (t_1^0, t_2^0, \dots, t_{j-1}^0, t_j^0 - 1, t_{j+1}^0, \dots, t_m^0)$ is not valid for $C_j$. From $\underline{t}_0 \leq \underline{t}$, we get that $\underline{t}_0' \leq \underline{t}'$. Using Lemma 2.1.15 again, if $\underline{t}'$ were valid, then $\underline{t}_0'$ would be $C_j$-valid. Therefore, $\underline{t}'$ is not valid. Therefore, $\underline{t}$ is indeed stable.

To show that $\underline{t}$ is minimal, suppose that there is a stable score limit $\underline{t}^1$ such that $\underline{t}_0 \leq \underline{t}^1$, but $\underline{t} \not\leq \underline{t}^1$, *i.e.*, $t_j^1 < t_j$ for some $j$. Let:

$$T' = \{\underline{x} \in \mathbb{N}^n : x_i \leq t_i^1 \quad \forall i \in \{1, \dots, m\}\}.$$

Since $\underline{t} \notin T'$, but $\underline{t}_0 \in T'$, there is a step such that when we leave $T'$, we move from $\underline{w}^1$ to $\underline{w}^2$. There is a college $C_i$, where $\underline{w}_i^1 = \underline{t}_i^1$. For other colleges, $\underline{w}_k^1 \leq \underline{t}_k^1$, so by Lemma 2.1.15, $\underline{w}^1$ was $C_i$-stable. Therefore, $C_i$ does not want to increase its score limit.
Therefore, $\underline{s}_A$ is the smallest of all stable score vectors, so every student gets accepted at as many colleges as possible, and they choose what is best for them. Therefore, $\underline{s}_A$ is applicant-optimal. Figure 2.9 shows a possible layout. □

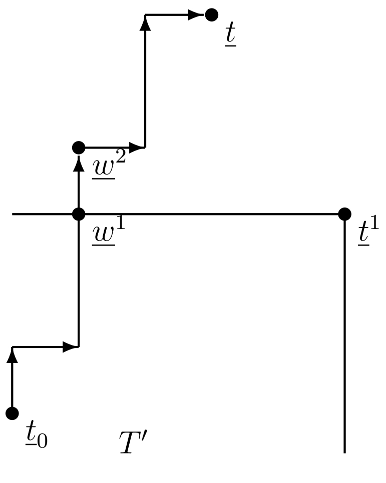



Figure 2.9: The score-increasing algorithm.

*Proof of Theorem 2.2.3.* In this proof, we return to the algorithm versions where colleges increase/lower their score limits as much as they can. We call the set of realized contracts at some score vector an *enrollment*. Each of the $n$ students can go to one of the $m$ colleges or remain unmatched. Therefore, there are at most $n^{m+1}$ possible enrollments. In the score-decreasing algorithm, the applicants always change to preferred assignments. In the score-increasing algorithm, the students' positions get worse. Therefore, we cannot return to an earlier enrollment in these algorithms.

If we order all enrollments according to the applicants' preference order, the longest chain contains $n(m+1)$ enrollments. It goes from "everyone gets the best college" to "everyone gets the worst college". In the score-decreasing algorithm, college $C_i$ may lower its score limit without changing the enrollment, taking the same students as before. If all $m$ colleges do this, we get the minimal score vector for that given enrollment, next time we come to college $C_i$, it has to change to a different enrollment or stop. Therefore, in the algorithm, there can be at most $m$ consecutive steps without changing the enrollment. Therefore, the number of steps is $O(m^2n)$.

In the score-increasing algorithm, every step will change the enrollment. Hence, if $C_i$ increase its score limit, the set of students going to $C_i$ was infeasible before this step and feasible after the step. Thus, the number of steps is $O(mn)$. □

## 2.3 The Lattice Property

Tarski's Theorem implies the following corollary for three-stability.

**Theorem 2.3.1** (Fleiner)**.** *[17] If $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ are substitutable choice functions, then three-stable pairs form a nonempty complete lattice for the partial order $\sqsubseteq$.*

Define the function $f : 2^E \times 2^E \to 2^E \times 2^E$ by:

$$f(A, B) := (E \setminus (\overline{\mathcal{G}}(B)), E \setminus (\overline{\mathcal{F}}(A))) = (E \setminus (B \setminus \mathcal{G}(B)), E \setminus (A \setminus \mathcal{F}(A)))$$

It is straightforward to see that three-stable pairs are exactly the fixed points of $f$. Therefore, since $f$ is monotone, three-stable pairs form a lattice.

A similar theorem can be proven for the four-stable $(A, B)$ pairs:

**Theorem 2.3.2** (Fleiner, Jankó)**.** *[19] If $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ are substitutable choice functions, then the four-stable pairs form a nonempty complete lattice for the partial order $\sqsubseteq$.*

The function

$$f''(A, B) := (\mathcal{D}_{\mathcal{G}}(B), \mathcal{D}_{\mathcal{F}}(A))$$

is monotone, and its fixed points are exactly the four-stable pairs, so we can use Tarski's theorem again.

As in Theorem 2.1.19, if a graph $G$ is simple, choice functions $\mathcal{F}$ and $\mathcal{G}$ are substitutable and $\mathcal{G}$ is loser-free, then the (generalized) score-stable sets form a non-empty lattice.

### 2.3.1 A Generalization of Blair's Theorem

Recall that in the traditional stable marriage model with strict preference ordering, from Conways's observation, if $S_1$ and $S_2$ are two stable marriage schemes and every man chooses the better of his partners in $S_1$ and $S_2$, we get a stable matching that we denote by $S_1 \vee S_2$. When the women pick the better partner the same way, we get $S_1 \wedge S_2$. The notations $\vee$ and $\wedge$ are not mere coincidence. It is easy to see, that if we take the partial ordering defined by the common preference of the men $\leq_M$, $S_1 \vee S_2 \geq_M S_1$ and $S_1 \vee S_2 \geq_M S_2$, and it is the smallest upper bound for these two marriage schemes. Since women's preferences over the stable matchings are the opposite of men's, $S_1 \wedge S_2$ is the greatest common lower bound of $S_1$ and $S_2$, according to $\leq_M$. Therefore, the set of all stable marriage schemes form a lattice over $\leq_M$, and also over $\leq_W$.

Blair's theorem [11] generalizes this to a situation where the men's and women's preferences are described with substitutable, IRC choice functions, $\mathcal{F}$ and $\mathcal{G}$. Here the stable sets will form a lattice according to a partial order defined by the choice function $F$, namely, if $S \subseteq E$ and $S' \subseteq E$ are two $\mathcal{F}$-rational sets, let $S' \leq_{\mathcal{F}} S$ if $\mathcal{F}(S \cup S') = S$.

**Observation 2.3.3.** *[11] If $\mathcal{F}$ is substitutable and IRC, then $\leq_{\mathcal{F}}$ is indeed a partial order over the $\mathcal{F}$-rational sets. In particular, $A \leq_{\mathcal{F}} B \leq_{\mathcal{F}} C$ implies $A \leq_{\mathcal{F}} C$.*

Blair proved the lattice property of dominating stable sets assuming the IRC property of the choice functions [11]. As we saw in Theorem 2.1.23, if $\mathcal{F}$ and $\mathcal{G}$ are both IRC, dominating stability, three-stability and four-stability are equivalent, so Blair's theorem holds for each of these notions.

**Theorem 2.3.4** (Blair)**.** *[11] If $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ are substitutable, IRC choice functions, then the dominating stable sets form a lattice for the partial order $\leq_{\mathcal{F}}$.*

We generalize the above lattice property for four-stability, and there is a close connection between score-stability and four-stability, so this generalization can be used for student-college admission as well. Usually, when colleges define score limits, their choice function is not IRC. To find a theorem fitting this scenario, we require IRC on only one side.

**Theorem 2.3.5** (Generalization of Blair's theorem)**.** *[Fleiner, Jankó] [19] If $\mathcal{F}$ and $\mathcal{G}$ are substitutable choice functions and $\mathcal{F}$ is IRC, then the four-stable sets form a lattice for the partial order $\leq_{\mathcal{F}}$.*

*Proof.* It was proven in Statement 2.1.11 that for any given stable set $S$ there is a unique four-stable pair $(A, B)$. In the following, we will show that $S \leq_{\mathcal{F}} S'$ if and only if $(A, B) \sqsubseteq (A', B')$ for their corresponding pairs.

$(A, B) \sqsubseteq (A', B') \Rightarrow S \leq_F S'$.
From the ordering of the four-stable pairs, $S \subseteq A \subseteq A'$ and $S' \subseteq A'$, so $S \cup S' \subseteq A'$. Since $\mathcal{F}$ is IRC, $S' = \mathcal{F}(A') \subseteq S \cup S' \subseteq A'$ implies $\mathcal{F}(S \cup S') = S'$.

$S \leq_{\mathcal{F}} S' \Rightarrow (A, B) \sqsubseteq (A', B')$ .
Suppose that $B' \nsubseteq B$. Consequently, $\exists\, b$, such that $b \notin B$, but $b \in B'$. From Lemma 1.3.4, $\mathcal{D}_{\mathcal{F}}(A) = \mathcal{D}_{\mathcal{F}}(S)$, so:
$b \notin B \Rightarrow b \notin \mathcal{D}_{\mathcal{F}}(A) = \mathcal{D}_{\mathcal{F}}(S) \Rightarrow b \in \mathcal{F}(S + b)$
$b \in B' \Rightarrow b \in \mathcal{D}_{\mathcal{F}}(A') = \mathcal{D}_{\mathcal{F}}(S') \Rightarrow b \notin \mathcal{F}(S' + b)$
We know that $\mathcal{F}(S \cup S') = S'$. Therefore $\mathcal{F}(S \cup S' + b) \subseteq (S' + b) \subseteq (S \cup S' + b)$. Since $\mathcal{F}$ is IRC, $\mathcal{F}(S \cup S' + b) = \mathcal{F}(S' + b) \ni b$, hence $b \in \mathcal{F}(S + b)$, a contradiction.
Similarly, from $B' \subseteq B$, we get $\mathcal{D}_{\mathcal{F}}(B') \supseteq \mathcal{D}_{\mathcal{F}}(B)$ by the antitonicity of $\mathcal{D}_{\mathcal{F}}$ and hence $A' \supseteq A$.

*The stable sets form a lattice.*
We have seen that there is an order preserving bijection between the stable sets and stable pairs. As stable pairs form a lattice, stable sets do as well. □

If only one of $\mathcal{F}$ and $\mathcal{G}$ is IRC, dominating stable sets may not form a lattice. Moreover, dominating stable sets do not necessarily exist, as we have seen in Example 2.1.3.

**Example 2.3.6** (Dominating stable sets do not form a lattice)**.** *Given one college $C_1$, two applicants $A_1, A_2$, and two contracts $a = A_1C_1$, $b = A_2C_1$. The college has a quota of 1, and both applicants want to go to $C_1$, so $\mathcal{G} = \mathcal{H}_1, \mathcal{F} = \mathcal{H}_2$. The dominating stable solutions are $\{a\}$ and $\{b\}$, however, $a$ and $b$ are incomparable, since $\mathcal{F}(\{a, b\}) = \{a, b\}$. Therefore, the dominating stable sets do not form a lattice.*

**Remark 2.3.7.** *If $\mathcal{F}$ and $\mathcal{G}$ are substitutable choice functions, but neither of them is IRC, then the lattice property does not always hold for four-stability. Moreover, a four-stable set $S$ may have more than one corresponding $(A, B)$ pair.*

**Example 2.3.8.** *We have two contracts, $a$ and $b$. The choice function is $\mathcal{H}_1$ for both sides: See Figure 2.2. In this situation, we have four four-stable pairs:*

| | | |
|---|---|---|
| $A = \emptyset$ | $B = \{a, b\}$ | $S = \emptyset$ |
| $A = \{a\}$ | $B = \{a\}$ | $S = \{a\}$ |
| $A = \{b\}$ | $B = \{b\}$ | $S = \{b\}$ |
| $A = \{a, b\}$ | $B = \emptyset$ | $S = \emptyset$ |

*Now, $\emptyset \leq_{\mathcal{F}} \{a\}$ and $\emptyset \leq_{\mathcal{F}} \{b\}$, but $\{a\}$ and $\{b\}$ are incomparable. Therefore, these two sets do not have a supremum. However, if $\mathcal{F}$ is not IRC, then $\leq_{\mathcal{F}}$ does not define a partial order, see choice function $\mathcal{G}$ in Example 2.1.3, therefore it is not so surprising that there is no lattice.*

We can state a version of Blair's theorem also for lattice choice functions.

Assume that $\mathcal{L} = (X, \preceq)$ is a lattice and $\mathcal{F}$ is a substitutable path-independent choice function. We say that $x$ is *$\mathcal{F}$-superior* to $y$ (denoted by $y \leq_{\mathcal{F}} x$) if $\mathcal{F}(x \vee y) = x$ holds. Relation $\leq_{\mathcal{F}}$ is a partial order on $\mathcal{F}$-independent elements according to the following lemma.

**Lemma 2.3.9.** *If $\mathcal{L} = (X, \preceq)$ is a lattice and $\mathcal{F}$ is a substitutable path-independent choice function then $\leq_{\mathcal{F}}$ is a partial order on $\mathcal{F}$-independent elements of $X$.*

*Proof.* We need to prove that $\leq_{\mathcal{F}}$ is reflexive, antisymmetric and transitive. If $x$ is $\mathcal{F}$-independent then $x = \mathcal{F}(x) = \mathcal{F}(x \vee x)$, that is, $x \leq_{\mathcal{F}} x$, proving reflexivity. Now if $x \leq_{\mathcal{F}} y \leq_{\mathcal{F}} x$, then $y = \mathcal{F}(x \vee y) = x$, hence $\leq_{\mathcal{F}}$ is indeed antisymmetric. At last, if $x \leq_{\mathcal{F}} y \leq_{\mathcal{F}} z$ holds then by path-independence of $\mathcal{F}$ we have

$$\mathcal{F}(x \vee z) = \mathcal{F}(x \vee \mathcal{F}(y \vee z)) = \mathcal{F}(x \vee y \vee z) = \mathcal{F}(\mathcal{F}(x \vee y) \vee z) = \mathcal{F}(y \vee z) = z,$$

hence $x \leq_{\mathcal{F}} z$, proving the transitivity of $\leq_{\mathcal{F}}$. □

One can generalize Blair's theorem [11] on the lattice structure of stable matchings to our setting as follows. First, for four-stable elements, we can generalize 2.3.5.

**Theorem 2.3.10** (Fleiner, Jankó)**.** *Assume that $\mathcal{L} = (X, \preceq)$ is an infinitely distributive complete lattice and $\mathcal{F}$ and $\mathcal{G}$ are substitutable choice functions, $\mathcal{F}$ is also path-independent. Then the partial order $\leq_{\mathcal{F}}$ defines a lattice on the set of four-stable elements of $X$.*

*Proof.* It is easy to see that $(a, b)$ is an four-stable pair if and only if $a$ is a fixed point of $\mathcal{M}(x) = \mathcal{D}_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(x))$ and $b = \mathcal{D}_{\mathcal{F}}(a)$.

As $\mathcal{M}$ is monotone, $(X_{\mathcal{M}}, \preceq)$ is a lattice according to Theorem 1.2.1. So to prove Theorem 2.3.10, we should show that there is a bijection between four-stable $(a, b)$ pairs and four-stable elements and this bijection order-preserving.

For a given $(a, b)$ let $s = a \wedge b$ and for a given four-stable element $s$, let $b = \mathcal{D}_{\mathcal{F}}(s)$ and $a = \mathcal{D}_{\mathcal{G}}(b)$. Since $\mathcal{F}$ is path-independent, we use equation 2.5 from the proof of Lemma 2.1.28 to obtain $b = \mathcal{D}_{\mathcal{F}}(s)$. As we know, $a = \mathcal{D}_{\mathcal{G}}(b)$ so $s$ unambiguously defines $(a, b)$.

Consider two four-stable sets $s$ and $s'$ with their corresponding four-stable pairs $(a, b)$ and $(a', b')$. Assume first that $a \preceq a'$. So $\mathcal{F}(a) = s$, and $\mathcal{F}(a') = s'$, and path-independence of $\mathcal{F}$ shows

$$\mathcal{F}(s \vee s') = \mathcal{F}(\mathcal{F}(a) \vee \mathcal{F}(a')) = \mathcal{F}(a \vee a') = \mathcal{F}(a') = s' \ .$$

Hence $a \preceq a'$ implies $s \leq_{\mathcal{F}} s'$.

If $s \leq_{\mathcal{F}} s'$ then $s' = \mathcal{F}(s \vee s')$ and $b = \mathcal{D}_{\mathcal{F}}(s)$, $b' = \mathcal{D}_{\mathcal{F}}(s')$ by Lemma 2.1.28. By (1.6) and path-independence of $\mathcal{F}$, we get

$$b = \mathcal{D}_{\mathcal{F}}(s) \succeq \mathcal{D}_{\mathcal{F}}(s \vee s') = \mathcal{D}_{\mathcal{F}}(\mathcal{F}(s \vee s')) = \mathcal{D}_{\mathcal{F}}(s') = b' \ ,$$

thus $a' = \mathcal{D}_{\mathcal{G}}(b') \succeq \mathcal{D}_{\mathcal{G}}(b) = a$ by the antitone property of $\mathcal{D}_{\mathcal{G}}$. □

Now for dominating stable elements, as known as $\mathcal{FG}$-stable elements:

**Theorem 2.3.11** (Fleiner, Jankó). *[20] Assume that $\mathcal{L} = (X, \preceq)$ is an infinitely distributive complete lattice and $\mathcal{F}$ and $\mathcal{G}$ are substitutable path-independent choice functions. Then partial order $\leq_{\mathcal{F}}$ defines a lattice on the set $S(\mathcal{FG})$ of $\mathcal{FG}$-stable elements of $X$. Moreover, $\leq_{\mathcal{F}} |_{S(\mathcal{FG})} \equiv \geq_{\mathcal{G}} |_{S(\mathcal{FG})}$, that is, if $s$ and $s'$ are $\mathcal{FG}$-stable elements then $s \leq_{\mathcal{F}} s'$ and $s' \leq_{\mathcal{G}} s$ are equivalent.*

*Proof.* It follows from Lemma 2.1.28 that $s \mapsto \mathcal{D}_{\mathcal{G}}(s)$ is a bijection between $S(\mathcal{FG})$ and set $X_{\mathcal{M}}$ of fixed points of mapping $\mathcal{M}(x) = \mathcal{D}_{\mathcal{G}}(\mathcal{D}_{\mathcal{F}}(x))$. As $\mathcal{M}$ is monotone, $(X_{\mathcal{M}}, \preceq)$ is a lattice according to Theorem 1.2.1. So to prove the first part of Theorem 2.3.11, it is enough to show that the bijection between $\mathcal{FG}$-stable elements and fixed points of $\mathcal{M}$ is order-preserving. We showed this in Theorem 2.3.10, therefore $\mathcal{FG}$-stable elements form a lattice for partial order $\leq_{\mathcal{F}}$.

To show the second part, observe that if $s \leq_{\mathcal{F}} s'$ then $a \preceq a'$. Moreover, $b' := \mathcal{D}_{\mathcal{F}}(a') \preceq \mathcal{D}_{\mathcal{F}}(a) = b$ by the antitone property of $\mathcal{D}_{\mathcal{F}}$. So path-independence of $\mathcal{G}$ shows that

$$\mathcal{G}(s \vee s') = \mathcal{G}(\mathcal{G}(b) \vee \mathcal{G}(b')) = \mathcal{G}(b \vee b') = \mathcal{G}(b) = s \ ,$$

i.e. $s' \leq_{\mathcal{G}} s$. □

## 2.3.2 The Lattice of Stable Score Vectors

Stable score vectors form a lattice, even if the graph is not simple, so a stronger version of Theorem 2.1.19 is also true:

**Theorem 2.3.12.** *[Fleiner, Jankó][19] If choice functions $\mathcal{F}$ and $\mathcal{G}$ are substitutable and $\mathcal{G}$ is loser-free, then the score-stable sets form a non-empty lattice.*

**Remark 2.3.13.** *If we consider L-stable score vectors, they also form a lattice, since the permissive scoring choice function used in L-stability is also loser-free.*

We prove Theorem 2.3.12 by taking a the pointwise minimum of $\underline{t}^1$ and $\underline{t}^2$ and starting from there. The score-decreasing algorithm terminates at the stable score vector $\underline{t}^1 \wedge \underline{t}^2$.

*Proof of Theorem 2.3.12.* We know from Theorems 2.2.1 and 2.2.2 that there exist a greatest and a least stable score vector. Let $\underline{t}^1$ and $\underline{t}^2$ be two arbitrary stable score vectors. We want to show that they have a join and a meet. Using Lemma 2.1.16, $\underline{t}^{min} = \min(\underline{t}^1, \underline{t}^2)$ is valid. Let us start the score-decreasing algorithm from $\underline{t}^{min}$. From the algorithm, we get a stable score vector, $\underline{t}$. From Theorem 2.2.1, $\underline{t}$ is the biggest among all the stable score vectors smaller than or equal to $\underline{t}^{min}$. Therefore, $\underline{t} = \underline{t}^1 \wedge \underline{t}^2$, because for every stable vector, such that $\underline{t}' \leq \underline{t}^1, \underline{t}^2$, $t' \leq \underline{t}^{min}$, therefore $\underline{t}' \leq t$.

We finish by showing the existence of $\underline{t}^1 \vee \underline{t}^2$. There exists a common upper bound of $\underline{t}^1$ and $\underline{t}^2$, for example $\underline{s}_C$. Since the lattice is finite, there has to be at least one least common upper bound. Suppose there exist two least common upper bounds: $\underline{a}$ and $\underline{b}$. Since $\underline{t}^1$ is a lower bound of $\underline{a}$ and $\underline{b}$, $\underline{t}^1 \leq \underline{a} \wedge \underline{b}$. Similarly, $\underline{t}^2 \leq \underline{a} \wedge \underline{b}$. Therefore, we found a common upper bound of $\underline{t}^1$ and $\underline{t}^2$ smaller than $\underline{a}$, a contradiction. $\square$

## 2.4 Weighted Kernels on Two Posets

Sands, Sauer, and Woodrow proved an interesting generalization of the stable marriage theorem [24] by Gale and Shapley in [45]. Namely, if digraph $D$ is the union of two acyclic digraphs, say $D_1$ and $D_2$ then there is a subset $K$ of the vertices of $D$ such that neither $D_1$ nor $D_2$ contains a directed path between two vertices of $K$, but from any vertex of $D$ outside $K$ there is a directed path of $D_1$ or of $D_2$ to some vertex of $K$. The same result can also be formulated in terms of partially ordered sets as follows. If $\preceq_1$ and $\preceq_2$ are two partial orders on the same ground set $V$ then there is a common antichain $K$ of these posets such that for any element $v \in V \setminus K$ of the ground set there exists a vertex $k \in K$ such that $v \preceq_1 k$ or $v \preceq_2 k$ holds. This latter formulation comes from [17] by Fleiner (see also [15]) and the proof is based on a choice function framework and Tarski's well-known fixed point theorem [47]. Fleiner also described a generalization of the deferred acceptance algorithm by Gale and Shapley that finds such an antichain. Moreover, as an application of Blair's theorem in [11], it turned out that these antichains form a lattice under a natural partial order.

Recently, Aharoni Berger and Gorelik generalized the Sands Sauer Woodrow result to a weighted setting in [2]. They follow the terminology by Sands Sauer and Woodrow and call the above common antichain a kernel. They describe a generalized model, define weighted kernels for weighted posets and prove that for any integral weighted pair of posets, there exists an integral weighted kernel that has a so-called tame property.

As we have seen in the Preliminaries, a *partially ordered set* or *poset* is a pair $P = (V, \preceq)$ of a ground set $V$ and a partial order (i.e. a reflexive, antisymmetric and transitive binary relation) $\preceq$ on $V$. Elements $u$ and $v$ of poset $P$ are *comparable* if $u \preceq v$ or $v \preceq u$ holds, otherwise $u$ and $v$ are *incomparable.* A *chain* of the above poset $P$ is a subset $C$ of $V$ such that its elements are pairwise comparable. Subset $A$ of $V$ is an *antichain* if no two different elements of $A$ are comparable. The following result of Sands Sauer and Woodrow is a generalization of the stable marriage theorem of Gale and Shapley [24].

**Theorem 2.4.1** (Sands, Sauer, and Woodrow [45]). *If $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ are finite posets on the same ground set $V$ then there exists a subset $A$ of $V$ that is a common antichain of $P_1$ and $P_2$ and for any element $v$ of $V$ there is some element $a$ of $A$ such that $v \preceq_1 a$ or $v \preceq_2 a$.*

Note that the original result of Sands, Sauer, and Woodrow in [45] was formulated in terms of arc 2-colored digraphs and oriented paths. The antichain $A$ in Theorem

2.4.1 is called a kernel in their terminology. In fact the main result in [45] is somewhat more general than Theorem 2.4.1 above and corresponds to a certain acyclic case in the digraph terminology. Still, it is not difficult to deduce the Sands, Sauer, and Woodrow result from Theorem 2.4.1. Note also that the marriage model of Gale and Shapley in [24] can also be translated to the language of Theorem 2.4.1. Namely, the common ground set of the two posets consists of all possible marriages and $\preceq_1$ is given by the men's and $\preceq_2$ by the women's preferences. This construction provides a bijection between stable marriage schemes and kernels $A$ in Theorem 2.4.1.

Let us return to our model. Fix a demand function $w : V \to \mathbb{R}_+$. A weight function $f : V \to \mathbb{R}_+$ is *$\preceq$-independent (with respect to $w$)* if $\widetilde{f}(C) := \sum\{f(c) : c \in C\} \leq max\{w(c) : c \in C\}$ holds for any chain $C$ of $P$, that is, if the total weight of no chain exceeds the maximum demand of its elements. Clearly, if demand function $w \equiv \mathbb{1}$ then $A$ is an antichain if and only if its characteristic function $\chi_A$ is independent (here $\mathbb{1}$ denotes the constant 1 function on $V$). A weight function $f$ is *$\preceq$-tame (with respect to $w$)* if for every chain $C = \{c_1 \preceq c_2 \preceq \cdots \preceq c_k\}$ with $f(c_1) > 0$ we have $\widetilde{f}(C) \leq w(c_1)$, that is, if the total weight of no chain exceeds the demand of its minimal element unless this minimal element has weight zero. It is easy to see that if weight function $f$ is $\preceq$-tame then $f$ is $\preceq$-independent. We say that element $v$ of $V$ is *$\preceq$-dominated* by $f$ if there is a chain $C = \{v = c_1 \preceq c_2 \preceq \cdots \preceq c_k\}$ such that $\widetilde{f}(C) \geq w(v)$, or in other words, if there is a chain starting at $v$ of total weight not less than the demand of $v$.

Let $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ be posets on the same ground set $V$. A common antichain $K$ of $P_1$ and $P_2$ is a *kernel* if each element $v$ of $V \setminus K$ is *dominated* by $K$, that is, if there is an element $k$ of $K$ such that $v \preceq_1 k$ or $v \preceq_2 k$ holds. If $w : V \to \mathbb{R}_+$ is a demand function then weight function $f : V \to \mathbb{R}_+$ is a *weighted kernel* if $f$ is both $\preceq_1$-independent and $\preceq_2$-independent and moreover each element $v$ of $V$ is $\preceq_1$-dominated or $\preceq_2$-dominated (or both). The above weight function $f$ is called *integral* if $f : V \to \mathbb{Z}_+$. It is easy to see that for $w \equiv \mathbb{1}$ an integral weighted kernel is exactly the characteristic function of a kernel. The main result of Aharoni, Berger, and Gorelik states the following.

**Theorem 2.4.2** (Aharoni, Berger, and Gorelik [2])**.** *For any pair $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ of posets and for any demand function $w : V \to \mathbb{Z}_+$, there exists an integral weighted kernel $f : V \to \mathbb{Z}_+$ that is both $\preceq_1$-tame and $\preceq_2$-tame.*

Note that if the common ground set $V$ of the two posets is infinite then Theorem 2.4.2 might not hold, for example when $P_1 = P_2 = ([0, 1], \leq)$, $w(1) = 0$ and $w(x) = 1$ for each $0 \leq x < 1$. Although this condition is not stated in [2], the authors clearly require this assumption on finiteness throughout their paper.

### 2.4.1 Existence and Structure of Weighted Kernels

To prove the Aharoni, Berger, and Gorelik result in [2], we show that weighted kernels are exactly the $\mathcal{F}_1\mathcal{F}_2$-stable elements of an appropriate lattice for certain substitutable path-independent choice functions $\mathcal{F}_1$ and $\mathcal{F}_2$. First we define the lattice we work with.

Let $V$ be a finite set and let $w : V \to \mathbb{Z}_+$ be a demand function. Define the poset $\mathcal{L}^w := (\{f : V \to \mathbb{Z}_+, f \leq w\}, \leq)$ of weight functions on $V$.

**Observation 2.4.3.** *The lattice $\mathcal{L}^w$ is an infinitely distributive complete lattice with lattice operations* min *and* max*.*

Let $P = (V, \preceq)$ be a finite poset and let $w$ and $\mathcal{L}^w$ be as in the definition above. We define a choice function $\mathcal{F}^w_\preceq$ on $\mathcal{L}^w$ that always picks a $\preceq$-tame weight function which is maximal in some sense. Namely, let $V = \{v_1, v_2, \ldots, v_n\}$ be a linear extension of $\succeq$, that is, if $v_i \preceq v_j$ then $j \leq i$ holds. (So $v_1$ is $\preceq$-maximal element of $V$ and $v_{i+1}$ is a $\preceq$-maximal element of $V \setminus \{v_1, v_2, \ldots, v_i\}$ for $i = 1, 2, \ldots$.) For any weight function $f$ in $\mathcal{L}^w$, define $\mathcal{F}^w_\preceq(f)$ for each of the values of $v_1, v_2, \ldots, v_n$ in this order in a certain greedy manner. By this we mean that after we calculated the values of $\left[\mathcal{F}^W_\preceq(f)\right](v_1), \ldots, \left[\mathcal{F}^W_\preceq(f)\right](v_{i-1})$, we determine a value $\left[\mathcal{F}^W_\preceq(f)\right](v_i) = \alpha$ such that $\alpha \leq f(v_i)$ and $\alpha$ is maximal with the property that $\mathcal{F}^w_\preceq(f)$ is $\preceq$-tame on any chain $v_i \prec l_1 \prec l_2 \prec \cdots$ starting at $v_i$. More precisely,

$$\left[\mathcal{F}^w_\preceq(f)\right](v_i) = \min\left\{f(v_i), \max\left\{0, w(v_i) - \left[\widehat{\mathcal{F}^w_\preceq}(f)\right](v_i)\right\}\right\} \tag{2.7}$$

where $\left[\widehat{\mathcal{F}^w_\preceq}(f)\right](v) = 0$ if $v$ is a $\preceq$-maximal element of $V$, otherwise

$$\left[\widehat{\mathcal{F}^w_\preceq}(f)\right](v) = \max\left\{\left[\mathcal{F}^w_\preceq(f)\right](u_1) + \left[\mathcal{F}^w_\preceq(f)\right](u_2) + \cdots : v \prec u_1 \prec u_2 \prec \ldots\right\}. \tag{2.8}$$

By definition, $\left[\mathcal{F}^w_\preceq(f)\right](v_i) \leq f(v_i)$ holds for each element $v_i$ of $V$, hence mapping $\mathcal{F}^w_\preceq$ is a choice function on $\mathcal{L}^w$. Moreover, $\mathcal{F}^w_\preceq(f)$ is $\preceq$-tame for any weight $f \in \mathcal{L}^w$ as we have chosen each value $[\mathcal{F}^w_\preceq(f)](v)$ such that every chain $v \preceq u_1 \preceq u_2 \preceq \cdots$ satisfies the property that tameness requires. The following Lemma describes a determinant of $\mathcal{F}^w_\preceq(f)$.

**Lemma 2.4.4.** *Let $P = (V, \preceq)$ be a finite poset with a demand function $w$ and a lattice $\mathcal{L}^w = (\{f : V \to \mathbb{Z}_+, f \leq w\}, \leq)$. For any $f \in \mathcal{L}^w$ and $v \in V$ define $\left[\mathcal{M}^w_\preceq(f)\right](v) = 0$ if $v$ is $\preceq$-maximal otherwise let*

$$\begin{aligned}\left[\mathcal{M}^w_\preceq(f)\right](v) := \max\{&f'(c_1) + f'(c_2) + \cdots + f'(c_k) : \\ &f' \leq f \text{ and } f' \text{ is } \preceq\text{-tame and } v \prec c_1 \prec c_2 \prec \ldots \prec c_k\}\end{aligned} \tag{2.9}$$

*as the maximum total $f'$-weight of a chain above $v$ where $f'$ is a $\preceq$-tame lower bound of $f$. Then $\mathcal{D}^w_\preceq := \max\{0, w - \mathcal{M}^w_\preceq\}$ is a determinant of $\mathcal{F}^w_\preceq$, that is*

$$\left[\mathcal{F}^w_\preceq(f)\right](v) = \min\left\{f(v), \max\{0, w(v) - \left[\mathcal{M}^w_\preceq(f)\right](v)\}\right\}. \tag{2.10}$$

*Moreover,* $\mathcal{D}^w_\preceq = \mathcal{D}_{\mathcal{F}^w_\preceq}$*, i.e.* $\mathcal{D}^w_\preceq$ *is the canonical determinant of* $\mathcal{F}^w_\preceq$*.*

*Proof.* To show (2.10), according to (2.7), (2.8) and (2.9), it is enough to prove that for each element $v_j$ of $V$

$$\left[\mathcal{M}^w_\preceq(f)\right](v_j) = \left[\widehat{\mathcal{F}^w_\preceq}(f)\right](v_j) \tag{2.11}$$

holds. We apply induction on $j$. (Recall that $v_1, v_2, \ldots$ is a linear extension of the reverse order $\succeq$ of $\preceq$.) If $v_j$ is $\preceq$-maximal in $V$ (e.g. if $j = 1$) then both sides of (2.11) equal 0 by definition. Assume now that $v_j$ is not $\preceq$-maximal in $V$ and (2.11) holds for $1, 2, \ldots, j-1$, in particular for all elements of $V$ above $v_j$. We have seen that $\mathcal{F}^w_\preceq(f)$ is $\preceq$-tame, so the right hand side of (2.11) is a lower bound of the left hand side. To show the opposite inequality, pick a chain $v_j \prec c_1 \prec c_2 \prec \cdots \prec c_k$ and a weight function $f' \in \mathcal{L}^w$ that achieves the maximum in (2.9). We may assume that $f'(c_1) > 0$. We distinguish two cases. If $f(c_1) + \left[\widehat{\mathcal{F}^w_\preceq}(f)\right](c_1) \geq w(c_1)$ then by (2.7) and the $\preceq$-tame property of $f'$ we have

$$\left[\widehat{\mathcal{F}^w_\preceq}(f)\right](v_j) \geq \left[\mathcal{F}^w_\preceq(f)\right](c_1) + \left[\widehat{\mathcal{F}^w_\preceq}(f)\right](c_1) \geq w(c_1) \geq$$
$$f'(c_1) + f'(c_2) + \cdots + f'(c_k) = \left[\mathcal{M}^w_\preceq(f)\right](v_j)\ .$$

Otherwise, if $f(c_1) + \left[\widehat{\mathcal{F}^w_\preceq}(f)\right](c_1) < w(c_1)$ then $\left[\mathcal{F}^w_\preceq(f)\right](c_1) = f(c_1)$ again by (2.7). As (2.11) holds for $c_1$ by induction, we get

$$\left[\mathcal{M}^w_\preceq(f)\right](v_j) = f'(c_1) + f'(c_2) + \cdots + f'(c_k) \leq f(c_1) + f'(c_2) + \cdots + f'(c_k) \leq$$
$$f(c_1) + \left[\mathcal{M}^w_\preceq(f)\right](c_1) = f(c_1) + \left[\widehat{\mathcal{F}^w_\preceq}(f)\right](c_1) = \left[\mathcal{F}^w_\preceq(f)\right](c_1) + \left[\widehat{\mathcal{F}^w_\preceq}(f)\right](c_1) \leq \left[\widehat{\mathcal{F}^w_\preceq}(f)\right](v_j)$$

This proves the induction step and justifies (2.10), hence $\mathcal{D}^w_\preceq$ is indeed a determinant of $\mathcal{F}^w_\preceq$. The fact that $\mathcal{D}^w_\preceq$ is the canonical determinant of $\mathcal{F}^w_\preceq$ immediately follows from definition (1.9) and the observation that the value $\left[\mathcal{D}^w_\preceq(f)\right](v)$ does not depend on $f(v)$. This finishes the proof. □

**Lemma 2.4.5.** *The mapping* $\mathcal{M}^w_\preceq$ *in* (2.9) *is monotone and the choice function* $\mathcal{F}^w_\preceq$ *is substitutable and path-independent.*

*Proof.* Monotonicity of $\mathcal{M}^w_\preceq$ directly follows from its definition (2.9). Namely, if $f, g \in \mathcal{L}^w$, $f \leq g$, $v \in V$ and the weight function $f' \in \mathcal{L}^w$ defines the value of $\left[\mathcal{M}^w_\preceq(f)\right](v)$ then $f' \leq f$ implies $f' \leq g$, so $\left[\mathcal{M}^w_\preceq(f)\right](v) \leq \left[\mathcal{M}^w_\preceq(g)\right](v)$ holds. The monotone property of $\mathcal{M}^w_\preceq$ and (2.10) immediately implies that $\mathcal{D}^w_\preceq$ is antitone, hence it is a determinant of some substitutable choice function $\mathcal{F}$ and $\mathcal{F} = \mathcal{F}^w_\preceq$ by to Lemma 2.4.4.

As $\mathcal{F}^w_\preceq(f)$ is $\preceq$-tame for any weight function $f \in \mathcal{L}^w$, (2.11) implies that $\mathcal{M}^w_\preceq(f) = \widehat{\mathcal{F}^w_\preceq}(f) = \mathcal{M}^w_\preceq\left(\mathcal{F}^w_\preceq(f)\right)$ and consequently, $\mathcal{D}^w_\preceq(f) = \mathcal{D}^w_\preceq(\mathcal{F}^w_\preceq(f))$ holds for $\mathcal{D}^w_\preceq$ which is an antitone determinant of $\mathcal{F}^w_\preceq$ by Lemma 2.4.4. Path-independence of $\mathcal{F}^w_\preceq$ follows directly from Lemma 1.3.6. □

The following lemma together with Lemma 2.1.29 immediately imply Theorem 2.4.2 of Aharoni, Berger and Gorelik.

**Lemma 2.4.6.** *[Fleiner, Jankó][20] Assume that $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ are finite posets, $w : V \to \mathbb{Z}_+$ is a demand function and weight function $f$ is $\preceq_1$-tame and $\preceq_2$-tame. Then $f$ is a weighted kernel if and only if $f$ is $\mathcal{F}^w_{\preceq_1}\mathcal{F}^w_{\preceq_2}$-stable.*

*Proof.* Assume first that $f$ is a weighted kernel. To show that $f$ is $\mathcal{F}^w_{\preceq_1}\mathcal{F}^w_{\preceq_2}$-stable, it is enough to prove by Lemma 2.1.27 that

$$f = \min\left\{\mathcal{D}_{\mathcal{F}^w_{\preceq_1}}(f), \mathcal{D}_{\mathcal{F}^w_{\preceq_2}}(f)\right\}. \tag{2.12}$$

As $f$ is $\preceq_1$-tame and $\preceq_2$-tame, $f = \mathcal{F}^w_{\preceq_1}(f) = \mathcal{F}^w_{\preceq_2}(f)$, so $f \le \min\left\{\mathcal{D}_{\mathcal{F}^w_{\preceq_1}}(f), A_{\mathcal{F}^w_{\preceq_2}}(f)\right\}$ by the definition of the determinant. Now pick any $v \in V$. As $f$ is a weighted kernel, $v$ is either $\preceq_1$-dominated or $\preceq_2$-dominated by $f$ (or both). In the first case $\left[\mathcal{D}^w_{\preceq_1}(f)\right](v) = f(v)$ and in the second one $\left[\mathcal{D}^w_{\preceq_2}(f)\right](v) = f(v)$ holds, that is

$$f \ge \min\left\{\mathcal{D}^w_{\preceq_1}(f), \mathcal{D}^w_{\preceq_2}(f)\right\} = \min\left\{\mathcal{D}_{\mathcal{F}^w_{\preceq_1}}(f), \mathcal{D}_{\mathcal{F}^w_{\preceq_2}}(f)\right\},$$

by Lemma 2.4.4. This proves (2.12) hence the $\mathcal{F}^w_{\preceq_1}\mathcal{F}^w_{\preceq_2}$-stability of $f$.

Now suppose that $f$ is $\mathcal{F}^w_{\preceq_1}\mathcal{F}^w_{\preceq_2}$-stable. As $f = \mathcal{F}^w_{\preceq_1}(f) = \mathcal{F}^w_{\preceq_2}(f)$, $f$ is both $\preceq_1$-tame and $\preceq_2$-tame. Moreover, (2.12) holds by (2.3). Pick any $v \in V$. Now Lemma 2.4.4 implies

$$f(v) = \min\left\{\left[\mathcal{D}_{\mathcal{F}^w_{\preceq_1}}(f)\right](v), \left[\mathcal{D}_{\mathcal{F}^w_{\preceq_2}}(f)\right](v)\right\} = \min\left\{\left[\mathcal{D}^w_{\preceq_1}(f)\right](v), \left[\mathcal{D}^w_{\preceq_2}(f)\right](v)\right\}$$

So either $f(v) = \left[\mathcal{D}^w_{\preceq_1}(f)\right](v)$ or $f(v) = \left[\mathcal{D}^w_{\preceq_2}(f)\right](v)$ (or both). In the first case $v$ is $\preceq_1$-dominated by $f$ and in the second case $v$ is $\preceq_2$-dominated by $f$ according to the definition (1.9) of the canonical determinant and Lemma 1.3.7. This proves that $f$ is indeed a weighted kernel. □

Theorem 2.3.11 and Lemma 2.4.6 yields the following generalization of Theorem 2.4.2.

**Corollary 2.4.7.** *For any posets $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ and for any demand function $w : V \to \mathbb{Z}_+$, there exists a $\preceq_1$-tame and $\preceq_2$-tame weighted kernel. Moreover, $\preceq_1$-tame and $\preceq_2$-tame weighted kernels form a complete lattice under $\le_{\mathcal{F}^w_{\preceq_1}}$.*

Although $\mathcal{F}$-superiority in Corollary 2.4.7 is defined before Lemma 2.3.9 in a general setting, the following lemma provides a choice function free characterization of $\le_{\mathcal{F}^w_{\preceq_1}}$.

**Lemma 2.4.8.** *Assume that $P = (V, \preceq)$ is a poset, $w : V \to \mathbb{Z}_+$ is a demand function and $f$ and $g$ are $\preceq$-tame weights in $\mathcal{L}^w$. Then $f \le_{\mathcal{F}^w_{\preceq}} g$ if and only if*

$$f(v) \ge g(v) \text{ holds whenever } v \text{ is not dominated by } f. \tag{2.13}$$

*Proof.* Assume first that $f \leq_{\mathcal{F}^w_\preceq} g$, that is, $\mathcal{F}^w_\preceq(\max\{f,g\}) = f$. As $\mathcal{F}^w_\preceq$ is path-independent and substitutable by Lemma 2.4.5, we get

$$\mathcal{D}^w_\preceq(f) = \mathcal{D}^w_\preceq(\mathcal{F}(\max\{f,g\}) = \mathcal{D}^w_\preceq(\max\{f,g\})$$

by Lemma 1.3.7. To justify (2.13), suppose that $v$ is not dominated by $f$. Consequently, $\left[\mathcal{D}^w_\preceq(f)\right](v) > f(v)$ as $\mathcal{D}^w_\preceq$ is the canonical determinant of $\mathcal{F}^w_\preceq$ by Lemma 2.4.4. Now

$$f(v) = \left[\mathcal{F}^w_\preceq(\max\{f,g\})\right](v) = \min\left\{\left[\mathcal{D}^w_\preceq(\max\{f,g\})\right](v), \max\{f(v),g(v)\}\right\} = \\ \min\left\{\left[\mathcal{D}^w_\preceq(f)\right](v), \max\{f(v),g(v)\}\right\} = \max\{f(v),g(v)\},$$

where the last equality holds because $\left[\mathcal{D}^w_\preceq(f)\right](v)$ is greater than $f(v)$, which is the left hand side. Therefore, $f(v) \geq g(v)$ and (2.13) follows.

To show the opposite implication, assume that (2.13) holds for $\preceq$-tame weight functions $f$ and $g$ and suppose indirectly that $f \not\leq_{\mathcal{F}^w_\preceq} g$, that is, $\left[\mathcal{F}^w_\preceq(\max\{f,g\})\right](v) \neq f(v)$ for some element $v$ of $V$. Pick a $\preceq$-maximal $v$ with the above property, that is,

$$\left[\mathcal{F}^w_\preceq(\max\{f,g\})\right](u) = f(u) \text{ whenever } v \prec u. \tag{2.14}$$

Now $\max\{f(v),g(v)\} \geq \left[\mathcal{F}^w_\preceq(\max\{f,g\})\right](v) > f(v)$ by (2.14) and (2.10), so $g(v) > f(v)$. Hence $v$ is dominated by $f$ due to (2.13). Now (2.14) and (2.7) imply that $\left[\mathcal{F}^w_\preceq(\max\{f,g\})\right](v) = f(v)$, a contradiction. This concludes the proof. □

### 2.4.2 Further Generalizations

Our approach can be applied to prove other generalizations of the Sands, Sauer, and Woodrow result than the one by Aharoni, Berger, and Gorelik. To do so, we may define other lattices than the lattice $\mathcal{L}^w$ we used in Section 2.4.1. A natural extension is if we define a "continuous" version of $\mathcal{L}^w$ on functions $f : V \to \mathbb{R}_+$ and we allow the demand function $w : V \to \mathbb{R}_+$ to be nonintegral. Aharoni, Berger, and Gorelik remark in [2] that a nonintegral analogue of Theorem 2.4.2 holds by the well-known Scarf lemma [46]. Note that we get the same result by applying our framework. To do so, we only need to copy the argument word for word in Section 2.4.1, replacing lattice $\mathcal{L}^w$ by $(\mathcal{L}^w)' := \{f : V \to \mathbb{R}_+, f \leq w\}$, which is also an infinitely distributive complete lattice. As a side product of this approach, one can deduce the lattice property of weighted kernels, which does not seem to follow from the application of the Scarf lemma.

We may also use our framework to deduce the many-to-one and many-to-many generalizations of the stable marriage theorem of Gale and Shapley. There, we are given a bipartite graph $G$ with color classes $U$ and $V$ and a quota function $b : U \cup V \to \mathbb{Z}_+$ and each vertex $v$ of $G$ has a linear preference order $\preceq_v$ on the set $E(v)$ of edges that are incident with $v$. A subset $M$ of $E(G)$ is a *b-matching* if each vertex $v$ of $G$ is incident with at most $b(v)$ edges of $M$. A $b$-matching $M$ is *stable* if for any edge $e = uv$ of $E(G) \setminus M$ there exist either $b(u)$ edges of $M$ that are all preferred to $e$ by $u$ or there exist $b(v)$ edges of $M$ that are all preferred to $e$ by $v$ (or both conditions hold). It is easy to see that if $b \equiv \mathbb{1}$ then a stable $b$-matching is exactly a stable matching. The generalization of the stable marriage theorem of Gale and Shapley states that for any quota function $b$, there exists a stable $b$-matching. We can deduce this result from our framework by defining two partial orders on the ground set $E(G)$. The first order corresponds to preferences in $U$ and the second to preferences in $V$. Define two demand functions $w_1$ and $w_2$ on $E$ such that if an edge $e$ has end vertices $u$ and $v$ in $U$ and $V$, respectively then $w_1(e) = b(u)$ and $w_2(e) = b(v)$. We work on the lattice $\mathcal{L}^{\mathbb{1}}$ (hence all weight functions are characteristic functions of sets of edges) and define choice functions $\mathcal{F}_1$ and $\mathcal{F}_2$ by determinants $\mathcal{D}^{w_1}_{\preceq_1}$ and $\mathcal{D}^{w_2}_{\preceq_2}$, respectively. (As we work in $\mathcal{L}^{\mathbb{1}}$, these choice functions can be interpreted as ordinary set choice functions.) In this model, it is easy to see that characteristic vectors of stable $b$-matchings are exactly the $\mathcal{F}_1\mathcal{F}_2$-stable weight functions of $\mathcal{L}^{\mathbb{1}}$ that form a nonempty complete lattice by Tarski's fixed point Theorem (Theorem 1.2.1).

But playing with the underlying lattice is not the only option to find a generalization of Theorem 2.4.1. We may work on our well known lattice $\mathcal{L}^w$ with path-independent substitutable choice functions other than $\mathcal{F}^w_{\preceq}$. One possibility is to replace the sum operation with maximization in the definition of $\mathcal{F}^w_{\preceq}$. More precisely, if $P = (V, \preceq)$ is a poset such that $v_1, v_2, \ldots, v_n$ is a linear extension of $\succeq$, $w : V \to \mathbb{Z}_+$ is a demand function and $f \in \mathcal{L}^w$ is a weight function then observe that (2.7) can be rewritten as

$$\left[\mathcal{F}^w_{\preceq}(f)\right](v_i) = \begin{cases} 0 & \text{if } \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](v_j) \geq w(v_i) \\ \min\left\{f(v_i), w(v_i) - \left[\widehat{\mathcal{F}^w_{\preceq}}(f)\right](v_j)\right\} & \text{otherwise.} \end{cases} \tag{2.15}$$

Now consider the following modification of (2.15)

$$\left[\mathcal{G}^w_{\preceq}(f)\right](v_i) = \begin{cases} 0 & \text{if } \max\left\{\left[\mathcal{G}^w_{\preceq}(f)\right](v_j) : v_i \prec v_j\right\} \geq w(v_i) \\ \min\left\{f(v_i), w(v_i)\right\} & \text{otherwise} \end{cases} \tag{2.16}$$

Similarly as we did in Section 2.4.1, one can prove that $\mathcal{G}^w_{\preceq}$ is substitutable and path-independent. To motivate the choice function $\mathcal{G}^w_{\preceq}$, let us say that a weight function $f$ of $\mathcal{L}^w$ is $\preceq$*-reasonable* if $f(v) > 0$ implies that $f(u) \leq w(v)$ holds whenever $v \preceq u$. We say that a weight function $f$ $\preceq$*-covers* $v$ if there is an element $u$ of $V$ such that $v \preceq u$ and $f(u) \geq w(v)$. With these notions, for every $v_i$ the choice function $\mathcal{G}^w_{\preceq}$ picks the maximum value $\left[\mathcal{G}^w_{\preceq}(f)\right](v_i)$ such that the choice $\mathcal{G}^w_{\preceq}(f)$ is $\preceq$-reasonable. Theorem 2.3.11 yields the following generalization of Theorem 2.4.2.

**Theorem 2.4.9.** *For any pair $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ of posets and for any demand function $w : V \to \mathbb{Z}_+$, there exists a weight function $f$ such that $f$ is both $\preceq_1$-reasonable and $\preceq_2$-reasonable and any element $v$ of $V$ is $\preceq_1$-covered or $\preceq_2$-covered by $f$.*

We may also mix the two kinds of choice functions we have seen so far.

**Theorem 2.4.10.** *For any pair $P_1 = (V, \preceq_1)$ and $P_2 = (V, \preceq_2)$ of posets and for any demand function $w : V \to \mathbb{Z}_+$, there exists a weight function $f$ such that $f$ is $\preceq_1$-tame and $\preceq_2$-reasonable and any element $v$ of $V$ is $\preceq_1$-dominated or $\preceq_2$-covered by $f$.*

Clearly, Theorem 2.4.1 is a special case of Theorems 2.4.9, 2.4.10 and 2.4.7 for $w \equiv \mathbb{1}$. Note that one can define further interesting path-independent substitutable choice functions that provide nontrivial results when Theorem 2.3.11 is applied. For example, if $0 < \alpha \leq 1$ then we can modify the definition (2.7) as

$$\left[\mathcal{F}^{w,\alpha}_{\preceq}(f)\right](v_i) = \min\left\{f(v_i), \max\left\{0, w(v_i) - \alpha \cdot \left[\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}(f)\right](v_i)\right\}\right\}.$$

**Lemma 2.4.11.** *For any poset $P = (V, \preceq)$, any demand function $w : V \to \mathbb{Z}_+$ and for any $0 < \alpha \leq 1$, the choice function $\mathcal{F}^{w,\alpha}_{\preceq} : \mathcal{L}^w \to \mathcal{L}^w$ is substitutable and path-independent.*

*Sketch of the proof.* Observe that

$$\begin{aligned}
\left[\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}(f)\right](v) = \max_{v \prec u}\left\{\left[\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}(f)\right](u) + \left[\mathcal{F}^{w,\alpha}_{\preceq}(f)\right](u)\right\} = \\
\max_{v \prec u}\left\{\left[\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}(f)\right](u) + \min\left\{f(u), \max\left\{0, w(u) - \alpha \cdot \left[\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}(f)\right](u)\right\}\right\}\right\} = \\
\max_{v \prec u}\left\{\min\left\{\left[\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}(f)\right](u) + f(u), \max\left\{\left[\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}(f)\right](u), w(u) + (1-\alpha) \cdot \left[\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}(f)\right](u)\right\}\right\}\right\}.
\end{aligned}$$

From this formula, it is easy to prove by induction on $|V|$ that $\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}$ is monotone in $f$. Consequently, the determinant $\mathcal{D}^{w,\alpha}_{\preceq}$ of $\mathcal{F}^{w,\alpha}_{\preceq}$ is antitone in $f$ where $\left[\mathcal{D}^{w,\alpha}_{\preceq}\right](f) = \max\left\{0, w - \alpha \cdot \left[\widehat{\mathcal{F}^{w,\alpha}_{\preceq}}(f)\right]\right\}$. So $\mathcal{F}^{w,\alpha}_{\preceq}$ is substitutable. Path-independence of $\mathcal{F}^{w,\alpha}_{\preceq}$ follows the same way as we proved it for $\mathcal{F}^{w}_{\preceq}$: the value of the $\left[\mathcal{D}^{w,\alpha}_{\preceq}(f)\right](v)$ depends only on the $\mathcal{F}^{w}_{\preceq}(f)$-values of elements $u$ with $v \prec u$, hence $\mathcal{D}^{w,\alpha}_{\preceq}(f) = \mathcal{D}^{w,\alpha}_{\preceq}(\mathcal{F}^{w,\alpha}_{\preceq}(f))$ holds for any $f \in \mathcal{L}^w$, and $\mathcal{F}^{w,\alpha}_{\preceq}$ is path-independent by Lemma 1.3.6. $\square$

We encourage the motivated reader to construct further nontrivial examples of substitutable path-independent choice functions on $\mathcal{L}^w$.

## 2.5 Matroid kernels

We can extend the notion of stability for matroids. In this section we will show a proof for that matroid kernels always exist, and give a simple algorithm for finding one.

**Definition 1.** $\mathcal{M} = (E, \mathcal{I})$ is a *matroid* on a ground set $E$ if $E$ is a finite set and $I \subseteq 2^E$ is a nonempty family satisfying the following two conditions.

1. If $X \subseteq Y \in \mathcal{I}$, then $X \in \mathcal{I}$.

2. If $X, Y \in I$ and $|X| < |Y|$, then $X + e \in I$ for some $e \in Y \setminus X$.

A subset $X \subseteq E$ is called *independent* if $X \in \mathcal{I}$.

We say $C \subseteq E$ is a *circuit* in the matroid $\mathcal{M} = (E, \mathcal{I})$, if $C$ is not independent, but for every $C' \subsetneq C$, $C'$ is independent. The set of all circuits is denoted by $\mathcal{C}$.

The rank function $r_M : 2^E \to \mathbb{Z}$ is defined by $r_M(X) = max\{|Y| : Y \in \mathcal{I}, Y \subseteq X\}$. For a subset $X \subseteq E$, its superset $\text{span}_M(X)$ is defined by $\text{span}_M(X) = \{e \in E \mid r_M(X) = r_M(X + e)\}$.

**Observation 2.5.1.** *For $X \in \mathcal{I}$, $span_M(X) = X \cup \{e \in E \mid X + e \notin \mathcal{I}\}$.*

For a given independent set $I$ and element $u \in span_M(I)$, there is a unique circuit in $I + u$ containing $u$, which is denoted by $C(I, u)$ and called the *fundamental circuit* of $u$. For each element $v \in C(I, u)$, the set $I - v + u$ is also independent in $M$. If $I$ was a basis in $M$, then $I - v + u$ is also a basis.

If there is a cost function $c : E \to \mathbb{R}$ on the matroid, one can use the greedy algorithm to find the maximum or minimum cost basis.

The following is a well-known property of the maximal weight basis of a matroid:

**Statement 2.5.2.** *Basis $B$ of a matroid is a minimal cost basis if and only if for all $y \in E \setminus B$ and $x \in C(B, y)$, $c(y) \geq c(x)$.*

We call a triple $(E, \mathcal{I}, \succ)$ an *ordered matroid* on $E$ if $(E, \mathcal{I})$ is a matroid and $\succ$ is a total order on $E$. It can be presented as a preference list of an agent, i.e. for $a, b \in E$ we write $a \succ b$ if the agent prefers $a$ to $b$.

For an ordered matroid $(E, \mathcal{I}, \succ)$, we say that a subset $X \in \mathcal{I}$ $\succ$*-dominates* an element $e \in E \setminus X$ if the following two conditions hold:

- $X + e \notin \mathcal{I}$
- $\forall e' \in X$ if $X + e - e' \in \mathcal{I}$ then $e' \succeq e$.

We call subset $K$ of $E$ an $\mathcal{M}_1\mathcal{M}_2$*-kernel* if it is a common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$ and every $e \in E \setminus K$ is $\succ_1$-dominated or $\succ_2$-dominated by $K$.

In the following, represent the preferences $\succ_1$ and $\succ_2$ with cost functions $c_1$ and $c_2$ such that $c_i(a) \leq c_i(b)$ if and only if $a \succeq_i b$. For any given $i \in \{1, 2\}$, if $\succ_i$ is a strict preference order, all the $c_i(e)$ costs are different, if $\succ_i$ is a weak order, $c_i$ allows equalities.

In the following we will use the greedy algorithm to define a choice function over an ordered matroid. Let $\mathcal{M} = (E, \mathcal{C})$ be a matroid on ground set $E$ and let $c : E \to \mathbb{R}_+$ be a cost function on $E = \{e_1, e_2, \dots e_n\}$ such that $c(e_i) \leq c(e_{i+1})$ for $1 \leq i < n$. Define the set $K_n(E')$ recursively for any subset $E'$ of $E$ by $K_0(E') = \emptyset$ and for $0 < i \leq n$

$$K_i(E') = \begin{cases} K_{i-1}(E') & \text{if } e_i \notin E' \text{ or } K_{i-1}(E') + e_i \notin \mathcal{I} \\ K_{i-1}(E') + e_i & \text{otherwise} \end{cases} \tag{2.17}$$

**Statement 2.5.3** (Fleiner). *[17] The above defined function $K_n : 2^E \to 2^E$ is substitutable, and set $K_n(E')$ is a minimum cost subset of $E'$ that spans $E'$. Moreover, if $c$ is injective then this minimum cost spanning set is unique for any subset $E'$ of $E$.*

*Proof.* [17] Let $C$ denote a circuit in the matroid, $\overline{K}_n(E') = E' \setminus K_n(E') = \{e_i \in E' : C \subseteq \{e_1, e_2, \ldots, e_{i-1}\}$ with $C + e_i \in \mathcal{C}\}$ is a monotone function. Hence $K_n$ is substitutable. The other facts are well-known. □

For matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ and cost functions $c_1, c_2 : E \to \mathbb{R}$, we say that $(E_1, E_2)$ is an $\mathcal{M}_1\mathcal{M}_2$*-stable* pair of $E$ if $E_1 \cup E_2 = E$ and $E_1 \cap E_2$ is a minimum $c_i$-cost spanning set of $E_i$ in $M_i$ for $i \in \{1, 2\}$.

**Theorem 2.5.4.** *[17] For matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ let $c_1, c_2 : E \to \mathbb{R}$ be cost functions on their common ground set. Then there is an $\mathcal{M}_1\mathcal{M}_2$-stable pair $(E_1, E_2)$ of $E$ and an $\mathcal{M}_1\mathcal{M}_2$-kernel $K$.*

Fleiner [17] gave a proof for Theorem 2.5.4, using choice functions and Tarski's fixed point theorem, but now we present an alternative, choice function-free proof for it.

*Proof.* Suppose $\succ_1$ is a strict preference order, so all the $c_1(e)$ costs are different. If $\succ_1$ was not strict, we break the ties and modify $c_1$ costs by a small amount.

We use induction by the size of $|E|$. For $|E| = 1$ the largest common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the $\mathcal{M}_1\mathcal{M}_2$-kernel. (If $\{e\}$ is independent in both matroids, this is the kernel, otherwise $\emptyset$.)

Let $B_1$ be the minimum $c_1$-cost basis in matroid $\mathcal{M}_1$. (We can find it with the greedy algorithm.) If $B_1 \in \mathcal{I}_2$, then it is a common independent set, and every $e \notin B_1$ is $\succ_1$-dominated, therefore $B_1$ is a $\mathcal{M}_1\mathcal{M}_2$-kernel.

If $B_1 \notin \mathcal{I}_2$, there is an $\mathcal{M}_2$-circuit $C$ in it, with $C \subseteq B_1$ and $C \in \mathcal{C}_2$. Let $f$ be one of the maximum $c_2$-cost elements of $C$. Now delete $f$ from the system, and define new ordered matroids $\mathcal{M}'_1 = \mathcal{M}_1|_{E-f}$ and $\mathcal{M}'_2 = \mathcal{M}_2|_{E-f}$. By the induction hypothesis, this scheme has an $\mathcal{M}'_1\mathcal{M}'_2$-kernel $K$.

When we return to our original matroids, every $e \in (E \setminus K) - f$ is $\succ_1$-dominated or $\succ_2$-dominated.

Case 1: $C - f \subseteq K$. Then $f$ is $\succ_2$-dominated, since $c_2(a) \le c_2(f)$ for every $a \in C - f$.

Case 2: There exists a $g \in (C - f) \setminus K$. Since $C \subseteq B_1$, $g \in B_1$. Because of Statement 2.5.2, $g$ cannot be $\succ_1$-dominated, therefore it is $\succ_2$-dominated.

Let $(C-f) \setminus K = \{g_1, g_2, \dots, g_k\}$. For every $g_i$, there is a circuit $C_i = C_2(K, g_i)$ such that for every $c \in C_i - g_i$, $c \succeq_2 g_i$.

Since $f \in C \setminus C_i$ for every $1 \leq i \leq k$, we will use the strong circuit axiom multiple times. There is a circuit $C_1' \in \mathcal{C}_2$ such that $f \in C_1' \subseteq (C \cup C_1 - g_1)$. If $\{g_2, g_3, \dots, g_k\} \cap C_1' = \emptyset$, then let $C'' := C_1'$. If $C_1'$ contains one of $\{g_2, g_3, \dots, g_k\}$ we can suppose it is $g_2$. Since $g_2 \in C_1' \cap C_2$ and $f \in C_1' \setminus C_2$ there is a circuit $C_2' \in \mathcal{C}_2$ such that $f \in C_2' \subseteq (C_1' \cup C_2 - g_2)$. Note that $g_1 \notin C_1'$ and $g_1 \notin C_2$, therefore $g_1 \notin C_2'$. With each step our circuits contain fewer $g_i$-elements. Repeating this method, we will reach a $\mathcal{C}_2$-circuit $C''$ such that $f \in C''$, $\{g_2, g_3, \dots, g_k\} \cap C'' = \emptyset$ and $C'' \subseteq (C \cup \bigcup_{i=1}^{k} C_i)$. Therefore $C'' \subseteq (K+f)$. For every $c \in C''-f$, if $c \in C_i$ then $c \succeq_2 g_i \succeq_2 f$, and if $c \in C$ then $c \succeq_2 f$ since $f$ was one of the $\succ_2$-worst elements in the circuit. Therefore $f$ is $\succ_2$-dominated, which means $K$ is an $\mathcal{M}_1\mathcal{M}_2$-stable kernel in the original instance. □

Note that if $\succeq_1$ was a weak ordering, this solution still gives a $\mathcal{M}_1\mathcal{M}_2$-stable kernel, since every node which is strongly $\succeq_1$-dominated here is weakly $\succeq_1$-dominated in the original instance.

Using this proof, we can create an algorithm to find a kernel.

1. Find the $c_1$-minimal $\mathcal{M}_1$-basis, $B_1$.
2. If this is $\mathcal{M}_2$ independent, stop. Otherwise find a $\mathcal{M}_2$-circuit $C$ in $B_1$ and let $f$-be a $c_2$-maximal element of $C$. Delete $f$.
3. In $\mathcal{M}_1-f$, if $B_1-f$ is a basis, then let $B_2 = B_1-f$. Otherwise let $B_2 = B_1-f+x$, where $x$ is the $c_1$-cheapest element such that $B_1-f+x$ is an $\mathcal{M}_1$-basis.
4. Repeat.

**Lemma 2.5.5.** *Consider a matroid $\mathcal{M} = (E, \mathcal{I})$ with cost function $c : E \to \mathbb{R}$ and suppose the costs of all elements are different. Let $B$ be a minimum cost $\mathcal{M}_1$-basis and $f$ be an arbitrary element of $B$. If $B-f$ is not a basis in $\mathcal{M}_1-f$, then let $B' = B-f+x$ where $x$ is the $c$-cheapest element such that $B-f+x$ is a $\mathcal{M}_1-f$-basis. Then this $B_1-f+x$ is a minimum cost $\mathcal{M}_1-f$-basis.*

*Proof.* Suppose $c(e_i) \leq c(e_{i+1})$ for $1 \leq i < n$, $f = e_i$ and $x = e_j$. Let $B^k = B \cap \{e_1, \dots, e_k\}$. For any $k < i$ if $e_k \notin B$, then $B^{k-1} + e_k \notin \mathcal{I}$ therefore $j > i$. Since $x \notin B$, there is a fundamental circuit $C = C(B^{j-1}, x)$ and $f \in C$.

Imagine removing $f$ and doing the greedy algorithm. Let $B''$ be the minimal cost basis in $\mathcal{M}-f$.

For every $k > j$ such that $e_k \notin B$, $B^{k-1}+e_k \notin \mathcal{I}$. If $f \notin C(B^{k-1}, e_k)$ then trivially $e_k \notin B''$. Otherwise $f \in C(B^{k-1}, e_k)$ and from the strong circuit axiom there is a circuit $C' \subseteq C \cup C(B^{k-1}, e_k)$ such that $e_k \in C'$ and $f \notin C'$ therefore $e_k \notin B''$.

So $B'' \subseteq B-f+x$ and they are of the same size, thus $B'' = B-f+x$.

□

# Chapter 3

# Trading Networks

In the previous chapter, we examined two-sided markets or the union of two posets, which can be also represented with two choice functions. Now we will show a model of supply chains. In our model, firms have heterogeneous preferences over bilateral contracts with other firms. Contracts may encode many dimensions of a relationship including the quantity of a good traded, time of delivery, quality, and price. The universe of possible relationships between firms is described by a *contract network* – a multi-sided matching market in which firms form downstream contracts to sell outputs and upstream contracts to buy inputs.

We build on a seminal contribution by Ostrovsky [38], who introduced a matching model of *supply chains*. In a supply chain, there are agents, who only supply inputs (e.g. farmers); agents, who only buy final outputs (e.g. consumers); while the rest of the agents are intermediaries, who buy inputs and sell outputs (e.g. supermarkets). All agents are partially ordered along the supply chain: downstream (upstream) firms cannot sell to (buy from) firms upstream (downstream) i.e. the contract network is *acyclic*. His key assumption about the market, which we retain in his paper, was that firms' choice functions over contracts satisfy *same-side substitutability* and *cross-side complementarity* (Hatfield and Kominers [31] later called these two conditions together*full substitutability*). This assumption requires that firms view any downstream or any upstream contracts as substitutes, but any downstream and any upstream contract as complements.Ostrovsky [38] showed that any supply chain has a *chain-stable* outcome for which there are no blocking downstream chains of contracts. Hatfield and Kominers [31] further showed that, in the presence of network acyclicity, chain-stable outcomes are equivalent to (what we call) *set-stable* outcomes i.e. those that are immune to deviations by arbitrary sets of firms. Even under full substitutability, chain-stable/set-stable outcomes in general supply chains may be Pareto inefficient.

While a supply chain may be a good model of production in certain industries [5], in general, firms simultaneously supply inputs to *and* buy outputs from other firms (possibly through intermediaries). If this is the case, we say a contract network contains a contract cycle. For example, the sectoral input-output network of the U.S. economy, illustrated by [1, Figure 3], shows that American firms are very interdependent and the contract network contains many cycles. Consider a coal mine that supplies coal to a steel factory. The factory uses coal to produce steel, which is an input for a manufacturing firm that sells mining equipment back to the mine. This creates a contract cycle. However, Hatfield and Kominers [31] showed that if a contract network has a contract cycle then set-stable outcomes may fail to exist. Our first result shows that checking whether an outcome is in fact set-stable is computationally hard. We then show that, even in the presence of contract cycles, outcomes that satisfy a weaker notion of stability – *trail stability* – can still be found. A trail of contracts is a sequence of distinct contracts in which a seller (buyer) in one contract is a buyer (seller) in the subsequent one. We argue that trail stability is a useful and intuitive equilibrium concept for the analysis of matching markets in networks. Along a blocking trail, firms make unilateral offers to their neighbors while conditionally accepting a sequence of previous pairwise blocks. Firms can receive several offers along the trail. Trail-stable outcomes rule out any sequence of such consecutive pairwise blocks. Trail stability is equivalent to chain stability (and therefore to set stability under our assumptions) in acyclic contract networks and to pairwise stability in two-sided many-to-many matching markets with contracts. Unsurprisingly, therefore, trail-stable outcomes may also be Pareto inefficient [11].

In order to analyze properties of trail-stable outcomes, we introduce another stability notion, called *full trail stability*, which does not force intermediary firms to accept all the contracts along a trail, but rather only sign upstream/downstream pairs. We argue that this could also be seen as a useful stability notion for short-run contract relationships. But studying full trail stability also allows us to use familiar fixed-point theorems and other techniques from the matching literature. Fully trail-stable outcomes correspond to the fixed-points of an operator and form a particular lattice structure for terminal agents, who can sign only upstream or only downstream contracts. The lattice reflects the classic opposition-of-interests property of two-sided markets, but between terminal buyers and terminal sellers. In addition to this strong lattice property, we extend previous results on the existence of buyer- and seller-optimal stable outcomes, the rural hospitals theorem [43],[31], strategyproofness [29],[31] as well as comparative statics on firm entry and exit [38],[30] that have only been studied in a supply-chain or two-sided setting under general choice functions. Fully trail-stable and trail-stable coincide under *separability*, a condition that ensures that decisions over certain pairs of upstream and downstream contracts are taken independently from others. We provide a complete description of the relationships between all stability notions – set stability, chain stability, trail stability, full trail stability – that we use here.

Our work complements a recent paper by Hatfield et. al. [33] on the properties of set-stable outcomes in general contract networks. They show that in general contract networks, under certain conditions, set-stable outcomes coincide with (what we call) *strongly trail-stable* outcomes i.e. those immune to coordinated deviations by a set of firms which is simultaneously signing a trail of contracts. This model is also related to the stability of (continuous and discrete) network flows discussed by Fleiner [18]. In these models, agents choose the amount of "flow" they receive from upstream and downstream agents and have preferences over who they receive the "flow" from. The network flow model allows for cycles. However, the choice functions in the network flow models are restricted by Kirchhoff's law (the total amount of incoming (current) flow is equal to the total amount of outgoing flow) and in the discrete case, these choice functions are special cases of Ostrovsky [38]. We generalize both of the supply chain and the network flow models, while offering two appealing new stability concepts.

All Theorems, Lemmas and Prepositions in this Chapter, up to Section 3.9 are results of Fleiner, Jankó, Tamura, and Teytelboym [22], unless stated otherwise.

| | General networks | General choice functions | Existence and structure | New stability concepts used | Corresponding name in this work |
|---|---|---|---|---|---|
| Ostrovsky [38] | ✗ acyclic | ✓ | ✓ | Chain-stable | Chain-stable |
| Westkamp [48] | ✗ acyclic | ✓ | ✓ | Group-stable or Setwise-stable, Core | – |
| Hatfield and Kominers[31] | ✗ acyclic | ✓ | ✓ | Stable or Weak Setwise-stable | Set-Stable |
| Hatfield et. al.[32], Hatfield and Kominers[27] | ✓ | ✗ quasilinear | ✓ | Strong group-stable | – |
| Hatfield et. al.[33] | ✓ | ✓ | ✗ | Chain-stable | Strong trail-stable |
| This work | ✓ | ✓ | ✓ | Trail-stable, Fully trail-stable | Trail-stable, Fully trail-stable |

Figure 3.1: Relationship to previous work.

| Paper | Theorem | Description | Generalization in this Chapter |
|---|---|---|---|
| [38] | Theorem 1 | Existence of stable outcomes | Theorem 3.4.1 and Theorem 3.5.1 |
| [31] | Theorem 4 | Buyer- and seller-optimality | Lemma 3.8.1 and Lemma 3.8.2 |
| [31] | Theorem 8 | Rural hospitals theorem | Proposition 3.9.1 |
| [31] | Theorem 10 | Strategy-proofness | Proposition 3.9.2 |
| [38] | Theorem 3 | Firm entry | Proposition 3.9.3 |
| [30] | Theorem | Vacancy chain dynamics | Proposition 3.9.4 |

Figure 3.2: Previous results generalized here to a trading network setting with general choice functions.

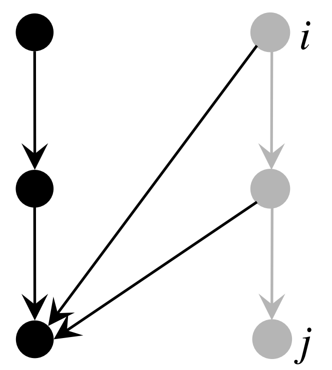



Figure 3.3: Supply chain

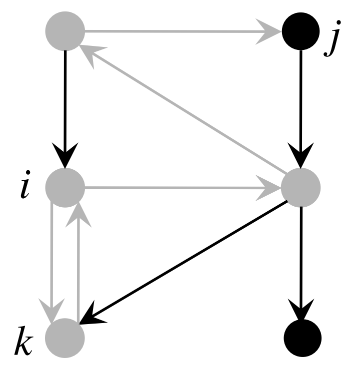



Figure 3.4: Contract network

## 3.1 Model

There is finite set of agents (firms or consumers) $F$ and a finite set of contracts (contract network) $X$.[1] A contract $x \in X$ is a bilateral agreement between a buyer $b(x) \in F$ and a seller $s(x) \in F$. Hence, $F(x) := \{s(x), b(x)\}$ is the set of firms associated with contract $x$ and, more generally, $F(Y)$ is the set of firms associated with contract set $Y \subseteq X$. Call $X_f^B := \{x \in X | b(x) = f\}$ and $X_f^S := \{x \in X | s(x) = f\}$ the sets of $f$'s upstream and downstream contracts – for which $f$ is a buyer and a seller, respectively. Clearly, $Y_f^B$ and $Y_f^S$ form a partition over the set of contracts $Y_f := \{y \in Y | f \in F(y)\}$ which involve $f$, since an agent cannot be a buyer and a seller in the same contract.

We can show graphically that our structure is more general than the setting described by [38] or [31]. Each firm $f \in F$ is associated with a vertex of a directed multigraph $(F, X)$ and each contract $x \in X$ is a directed edge of this graph. For any $f$, $X_f^B$ is represented by a set of incoming edges and $X_f^S$ is represented by outgoing edges. In Figure 3.3, we illustrate a three-level supply chain with two producers, two intermediaries and two final consumers. Supply chains require a partial order on the firms' positions in the chain although firms may sell to (buy from) any downstream (upstream) level. Hence, in Figure 3.3, the right producer sells directly to the left consumer bypassing the intermediary. In our model, we consider general contract networks, which may contain contract cycles (i.e. directed cycles on the graph), illustrated in Figure 3.4.

Every firm has a choice function $C^f$, such that $C^f(Y_f) \subseteq Y_f$ for any $Y_f \subseteq X_f$. (Since firms only care about their own contracts, we can write $C^f(Y)$ to mean $C^f(Y_f)$. )

[1] The standard justification for this assumption is given by [40, p. 49]: "elements of a [contract] can take on only discrete values; salary cannot be specified more precisely than to the nearest penny, hours to the nearest second, etc." In fact, the finiteness assumption is not necessary for our proofs. We only require that the set of contracts between any two agents forms a lattice.

For any $Y \subseteq X$ and $Z \subseteq X$, define the *chosen* set of upstream contracts

$$C_B^f(Y|Z) := C^f(Y_f^B \cup Z_f^S) \cap X_f^B \tag{3.1}$$

which is the set of contracts $f$ chooses as a buyer when $f$ has access to upstream contracts $Y$ and downstream contracts $Z$. Analogously, define the chosen set of downstream contracts

$$C_S^f(Z|Y) := C^f(Z_f^S \cup Y_f^B) \cap X_f^S \tag{3.2}$$

Hence, we can define *rejected* sets of contracts $R_B^f(Y|Z) := Y_f \setminus C_B^f(Y|Z)$ and $R_S^f(Z|Y) := Z_f \setminus C_S^f(Z|Y)$. An *outcome* $A \subseteq X$ is a set of contracts.

A set of contracts $A \subseteq X$ is *individually rational* for an agent $f \in F$ if $C^f(A_f) = A_f$. We call set $A$ *acceptable* if $A$ is individually rational for all agents $f \in F$. For sets of contracts $W, A \subseteq X$, we say that $A$ is $(W, f)$*-rational* if $A_f \subseteq C^f(W_f \cup A_f)$ i.e. if the agent $f$ chooses all contracts from set $A_f$ whenever she is offered $A$ alongside $W$. Set of contracts $A$ is $W$*-rational* if $A$ is $(W, f)$-rational for all agents $f \in F$. Note that contract set $A$ is individually rational for agent $f$ if and only if it is $(\emptyset, f)$-rational. If $y \in X_f^B$ and $z \in X_f^S$ then $\{y, z\}$ is a $(W, f)$*-rational pair* if neither $x$ nor $z$ is $(W, f)$-rational but $\{y, z\}$ is $(W, f)$-rational. Note that any rational pair consists of exactly one upstream and one downstream contract.

## 3.2 Assumptions on Choice Functions

We can now state our key assumption on choice functions introduced by Ostrovsky [38].

Choice functions of $f \in F$ satisfy *full substitutability* if for all $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$ they are:

1. *Same-side substitutable* (SSS):
   (a) $R_B^f(Y'|Z) \subseteq R_B^f(Y|Z)$
   (b) $R_S^f(Z'|Y) \subseteq R_S^f(Z|Y)$
2. *Cross-side complementary*(CSC):
   (a) $R_B^f(Y|Z) \subseteq R_B^f(Y|Z')$
   (b) $R_S^f(Z|Y) \subseteq R_S^f(Z|Y')$

Contracts are fully substitutable if every firm regards any of its upstream or any of its downstream contracts as substitutes, but its upstream and downstream contracts as complements. Hence, rejected downstream (upstream) contracts continue to be rejected whenever the set of offered downstream (upstream) contracts expands or whenever the set of offered upstream (downstream) contracts shrinks.

**Remark 3.2.1.** *When a choice function is substitutable, we have seen in Lemma 1.1.10 that IRC and path-independence are equivalent. However, with full substitutability, this is not the case. As in Example 1.1.11, for contracts $a$ and $b$ of firm $f$, where one is a downstream, the other one is an upstream contract, let $C^f(\{a\}) = C^f(\{b\}) = \emptyset$ and*

*$C^f(\{a,b\}) = \{a,b\}$. This choice function is IRC and fully substitutable, but it is not path-independent, $C^f(\{a,b\}) = \{a,b\} \neq \emptyset = C^f(C^f(\{a\}) \cup \{b\})$.*

We also introduce a new restriction on choice functions that will play a major role in linking together various stability concepts.

Choice functions of $f \in F$ satisfy *separability* if for any $A, W \subseteq X$ and $y \in X_f^B \setminus A$ and $z \in X_f^S \setminus A$, whenever $A$ is $(W, f)$-rational, and $\{y, z\}$ is a $(W, f)$-rational pair, then $A \cup \{y, z\}$ is $(W, f)$-rational.

Separable choice functions impose a kind of independence on choices of pairs of upstream and downstream contracts. It says that whenever the firm chooses $A$ alongside some set $W$ and $\{y, z\}$ alongside $W$ (but $y$ and $z$ would not be chosen separately alongside $W$ since $\{y, z\}$ is a $(W, f)$-rational pair), then it would choose $A \cup \{y, z\}$ alongside $W$. Suppose signing $A$ and $\{y, z\}$ are decisions made by separate units of the firm. Separable choice functions say that it can delegate the joint input-output decisions to the units because its overall choices do not require any coordination between the units. One natural example of separable choice functions is the following: suppose each firms totally orders individual upstream contracts and individual downstream contracts. Whenever a firm is offered $k$ downstream and $l$ upstream contracts, it chooses the $z$ best upstream and the $z$ best downstream contracts where $z = \min(k, l)$. As the example shows, separability is closely related "responsiveness" in the contract network setting as was described by Roth [41]. It is worth noting, however, that separability, unlike responsiveness, does not imply full substitutability.

### 3.2.1 Laws of Aggregate Demand and Supply

We first re-state the familiar Laws of Aggregate Demand and Supply (LAD/LAS) [28],[31]. LAD (LAS) states that when a firm has more upstream (downstream) contracts available (holding the same downstream (upstream) contracts), the number of downstream (upstream) contracts the firms chooses does not increase more than the number of upstream (downstream) contracts the firm chooses. Intuitively, an increase in the availability of contracts on one side, does not increase the difference between the number of contracts signed on either side.

Choice functions of $f \in F$ satisfy the Law of Aggregate Demand if for all $Y, Z \subseteq X$ and $Y' \subseteq Y$

$$|C_B^f(Y|Z)| - |C_B^f(Y'|Z)| \geq |C_S^f(Z|Y)| - |C_S^f(Z|Y')|$$

and the Law of Aggregate Supply if for all $Y, Z \subseteq X$ and $Z' \subseteq Z$

$$|C_S^f(Z|Y)| - |C_S^f(Z'|Y)| \geq |C_B^f(Y|Z)| - |C_B^f(Y|Z')|$$

We can easily show that LAD/LAS imply IRC, extending the result by [7].

**Lemma 3.2.2.** *In any contract network $X$ if choice functions of $f \in F$ satisfy full substitutability and LAD/LAS then the choice functions of $f$ satisfy IRC.*

*Proof.* Consider $Y \subseteq X_f$ and $z \in X_f^B \setminus Y$ such that $z \notin C^f(Y \cup \{z\})$. Then, from SSS, $C_B^f(Y \cup \{z\}) \subseteq C_B^f(Y)$ and from CSC $C_S^f(Y \cup \{z\}) \supseteq C_S^f(Y)$. If choice functions satisfy LAD/LAS then $|C_B^f(Y)| - |C_S^f(Y))| \leq |C_B^f(Y \cup \{z\})| - |C_S^f(Y \cup \{z\})|$ so there must be equality, so $C^f(Y \cup \{z\}) = C^f(Y)$. □

### 3.2.2 Stability Concepts

We start off by defining two stability notions that have appeared in previous work.

An outcome $A \subseteq X$ is *set-stable*[2] if:

1. $A$ is acceptable.

2. There exist no non-empty blocking set of contracts $Z \subseteq X$, such that $Z \cap A = \emptyset$ and $Z$ is $(A, f)$-rational for all $f \in F(Z)$.

Set-stable outcome are immune to deviations by *sets* of firms, which can re-contract freely among themselves. Set-stable outcomes always exist in acyclic networks. In order to study more general contract networks, we first introduce trails of contracts.

A non-empty sequence of different contracts $T = \{x_1, \ldots, x_M\}$ is a *trail* if $b(x_m) = s(x_{m+1})$ holds for all $m = 1, \ldots, M-1$.

While a trail may not contain the same contract more than once, it may include the same agents any number of times. Figure 3.4 illustrates a trail that starts from firm $i$ to firm $j$ via firm $k$. A trail $T$ is a *chain* if all the agents $F(T)$ involved in the trail are distinct.[3] A chain from firm $i$ to firm $j$ is illustrated in Figure 3.3.

An outcome $A \subseteq X$ is *strongly trail-stable* if

1. $A$ is acceptable.

2. There is no trail $T$, such that $T \cap A = \emptyset$ and $T$ is $(A, f)$-rational for all $f \in F(T)$.

Hatfield et. al. [33] showed that in general contract networks set-stable outcomes are equivalent to strongly trail-stable outcomes whenever choice functions satisfy full substitutability and Laws of Aggregate Demand and Supply.[4] However, Fleiner [18] and Hatfield-Kominers [31] showed that a set-stable outcomes may not exist in general

[2] Klaus and Walzl [35] call set-stable outcomes "weak setwise stable" and Hatfield and Kominers [31] call them "stable", we take the middle ground.

[3] The chains here are called paths in graph theory.

[4] [33] call trails "chains" and strong trail stability "chain stability". We use our terminology to avoid the confusion with the original definition of "chains" and "chain stability" in [38]. Our distinction between "trails" and "chains" (or "paths") is used in most graph theory textbooks.

contract networks (see Example 3.3.1 below). Moreover, our first result demonstrates that set stability is computationally intractable. Let us define decision problem GS as follows. An instance of GS is a trading network with a set of agents $F$ and set of contracts $X$ (with choice functions that satisfy full substitutability and IRC) and an outcome $A$. The answer for an instance of GS is YES if the particular outcome $A$ is not set-stable (that is, if there is a set of contracts $Z$ that blocks $A$), otherwise the answer is NO.

**Theorem 3.2.3.** *Problem GS is NP-complete. Moreover, if choice functions are represented by oracles then finding the right answer for an instance of GS might need an exponential number of oracle calls.*

*Proof.* Problem GS is clearly belongs to complexity class NP as a blocking set $Z$ is a polynomial time proof of non-set-stability.

To show that GS is NP-hard we reduce the NP-complete partition problem to GS. An instance of the partition problem is given by a $k$-tuple $A = (a_1, a_2, \ldots, a_k)$ of positive integers such that $a_1 \leq a_2 \leq \ldots \leq a_k$ holds. The answer to this problem is YES if and only if there is a subset $I$ of $\{1, 2, \ldots, k\}$ such that $\sum_{i \in I} a_i = s$ where $2s = \sum_{i=1}^{k} a_i$. So assume that the partition problem is given by $\mathcal{A} = (a_1, a_2 \ldots a_k)$. Construct a trading network with firms $f$ and $g$ and with contracts $y$ and $x_i$ such that $f = s(y) = b(x_i)$ and $g = b(y) = s(x_i)$ for $i \in \{1, 2, \ldots, k\}$. Define choice function $C^f_{\mathcal{A}}$ with the help of $s := \frac{1}{2}\sum_{i=1}^{k} a_i$ by

$$C^f_{\mathcal{A}}(X|Y) = \begin{cases} (X|Y) & \text{if } \sum\{a_i : x_i \in X\} \geq s \\ (X|\emptyset) & \text{if } \sum\{a_i : x_i \in X\} < s \end{cases}$$

It is easy to check that $C^f_{\mathcal{A}}$ satisfies full substitutability and IRC. Define $C^g_{\mathcal{A}}$ as follows:

$$C^g_{\mathcal{A}}(Y|X) = \begin{cases} (\emptyset|\emptyset) & \text{if } Y = \emptyset \\ (Y|X) & \text{if } Y = \{y\} \text{ and } \sum\{a_i : x_i \in X\} \leq s \\ (Y|X \cap \{x_1, x_2, \ldots, x_t\}) & \begin{array}{l}\text{if } Y = \{y\} \text{ and } \sum\{a_i : x_i \in X, i \leq t\} \leq s \\ < \sum\{a_i : x_i \in X, i < t+1\}\end{array} \end{cases}$$

One can readily check that $C^g_{\mathcal{A}}$ also satisfies full substitutability and IRC. That is, based on the partition problem instance, we have determined a trading network. To define our GS instance, define an outcome $A = \emptyset$. We have to show that the answer to the partition problem is YES if and only if $A = \emptyset$ is not set-stable.

Assume now that the answer to our partition problem instance is YES, that is $\sum_{i \in I} a_i = s$. Define $X_I := \{x_i : i \in I\}$ and $Y = \{y\}$. By the above definitions, $C^f_{\mathcal{A}}(X|Y) = (X|Y)$ and $C^g_{\mathcal{A}}(Y|X) = (Y|X)$, hence $X \cup Y$ blocks $A = \emptyset$, so $A$ is not set-stable.

Assume now that $A = \emptyset$ is not set-stable. This means that there is a blocking set $Z$ to $A$ and define $I = \{i : x_i \in Z\}$, $X_I := \{x_i : x_i \in Z\}$ and $Y := Z \cap \{y\}$. As $Z$

is blocking, we have $C^f_{\mathcal{A}}(X_I|Y) = (X_I|Y)$ and $C^g_{\mathcal{A}}(Y|X_I) = (Y|X_I)$. If $Y = \emptyset$ then $(Y|X_I) = C^g_{\mathcal{A}}(Y|X_I) = C^g_{\mathcal{A}}(\emptyset|X_I) = (\emptyset, \emptyset)$, so $Z = X_I \cup Y = \emptyset \cup \emptyset = \emptyset$, and hence $Z$ is not blocking. Otherwise, $Y = \{y\}$, and from $C^g_{\mathcal{A}}(Y|X_I) = (Y|X_I)$ we get that $\sum_{i \in I} a_i \leq s$. Moreover, from $y \in C^f_{\mathcal{A}}(X_I, Y)$ we get that $\sum_{i \in I} a_i \geq s$. Consequently $\sum_{i \in I} a_i = s$, and the answer to the partition problem is YES.

To prove the second part of the theorem, define a contract network with firms $f$ and $g$ and with contracts $y$ and $x_i$ such that $f = s(y) = b(x_i)$ and $g = b(y) = s(x_i)$ for for $1 \leq i \leq 2n$. Define the following choice function

$$C^f_0(X|Y) = \begin{cases} (X|Y) & \text{if } |X| \geq n+1 \\ (X|\emptyset) & \text{if } |X| \leq n \end{cases} \tag{3.3}$$

For $I \subseteq \{1, 2, \ldots, n\}$ define $X_I := \{x_i : i \in I\}$. For $|I| = n$ let

$$C^f_I(X|Y) = \begin{cases} (X|Y) & \text{if } |X| \geq n+1 \text{ or if } X = X_I \\ (X|\emptyset) & \text{if } |X| \leq n \text{ and } X \neq X_I \end{cases}$$

It is straightforward to check that choice functions $C^f_0$ and $C^f_I$ above satisfy full substitutability and IRC. Define the following choice function for $g$

$$C^g(Y|X) = \begin{cases} (\emptyset|\emptyset) & \text{if } Y = \emptyset \\ (Y|X) & \text{if } Y = \{y\} \text{ and } |X| \leq n \\ (Y|X \cap \{x_1, x_2, \ldots, x_t\}) & \text{if } Y = \{y\} \text{ and } |\{x_i \in X : i \leq t\}| = n \end{cases} \tag{3.4}$$

As $C^g = C^g_{\mathcal{A}}$ for $\mathcal{A} = (1, 1, \ldots, 1)$, $C^g$ also satisfies full substitutability and IRC.

Now assume that an instance of problem GS is given by the above network and an outcome $A = \emptyset$. Assume that the choice functions are not given explicitly, but by value-returning oracles. Moreover, we know exactly that the choice function of $g$ is the one defined in (3.4) and we know that the choice function of $f$ is either $C^f_0$ or $C^f_I$ for some $I$. It is easy to check that $A$ is not set-stable if and only if $C^f = C^f_I$ and in this case the only blocking set is $Z = X_i \cup \{y\}$. So if one has to decide set stability of $A$, then one must determine the $C^f(Z)$ values for all such possible $Z$, and this means $\binom{2n}{n}$ oracle calls. □

## 3.3 Trail Stability

The non-existence of set-stable outcomes and their computational intractability motivates us to define a less restrictive stability notions.

We first define *trail stability*, which coincides with pairwise stability in a two-sided many-to-many matching market with contracts [40] and with chain stability in supply chains [38, p. 903]. Define $T^{\leq m}_f = \{x_1, ..., x_m\} \cap T_f$ to be firm $f$'s contracts out of the first $m$ contracts in the trail and $T^{\geq m}_f = \{x_m, ..., x_M\} \cap T_f$ to be firm $f$'s contracts out of the last $M - m + 1$ contracts in the trail.

An outcome $A \subseteq X$ is *trail-stable* if

1. $A$ is acceptable.
2. There is no trail $T = \{x_1, x_2, \ldots, x_M\}$, such that $T \cap A = \emptyset$ and
   (a) $x_1$ is $(A, f_1)$-rational for $f_1 = s(x_1)$ and
   (b) At least one of the following two options holds:
      i. $T_{f_m}^{\leq m}$ is $(A, f_m)$-rational for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$, or
      ii. $T_{f_m}^{\geq m-1}$ is $(A, f_m)$-rational for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$
   (c) $x_M$ is $(A, f_{M+1})$-rational for $f_{M+1} = b(x_M)$.

   The above trail $T$ is called a *blocking trail to $A$*.

Trail stability is a natural stability concept when firms interact mainly with their buyers and suppliers and deviations by arbitrary sets of firms are difficult to arrange. In a trail-stable outcome, no agent wants to drop his contracts and there exists no set of *consecutive* bilateral contracts comprising a trail preferred by all the agents in the trail to the current outcome. First, $f_1$ makes an unilateral offer of $x_1$ (the first contract in the trail) to the buyer $f_2$. At this stage seller $f_1$ does not consider whether he may act as a buyer or a seller in the trail again (in that sense the deviations are pairwise and consecutive). The buyer $f_2$ then either unconditionally accepts the offer (forming a blocking trail) or conditionally accepts the seller's offer while looking to offer a contract ($x_2$) to another buyer $f_3$. If $f_2$'s buyer in $x_2$ happens to be $f_1$, then $f_1$ considers the offer of $x_2$ together with $x_1$ (which he has already offered). If $f_1$ accepts, we have a blocking trail. If $f_2$'s buyer is not $f_1$, then his buyer either accepts $x_2$ unconditionally or looks for another seller $f_4$ after a conditional acceptance of $x_2$. The trail of "conditional" contracts continues until the last buyer $f_{M+1}$ in the trail unconditionally accepts the upstream contract offer $x_M$.[5] Note that as the blocking trail grows, we ensure that each intermediate agent wants to choose all his contracts along the trail.

In general, trail stability is a weaker stability notion than set stability. The following example illustrates that trail-stable outcomes are not necessarily set-stable.[6]

[5]The trail and the order of conditional acceptances can, of course, be reversed with $f_{M+1}$ offering the first upstream contract to seller $f_M$ and so on.

[6]This is similar to examples in [18] and [31, Fig. 3, p. 13].

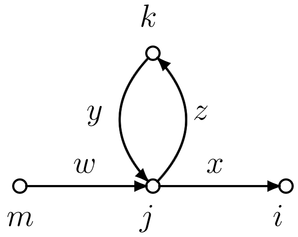



Figure 3.5: Example of a network that is trail-stable, but not set-stable

**Example 3.3.1** (Trail-stable outcomes are not necessarily set-stable). *Consider four contracts $x$, $y$, $z$ and $m$. Assume that $i = b(x)$, $j = s(x) = s(z) = b(y) = b(w)$, $k = b(z) = s(y)$ and $m = s(w)$ (see Figure 3.5). Agents have the following preferences that induce fully substitutable choice functions:*[7]

$\succ_i$: $\{x\} \succ_i \emptyset$

$\succ_m$: $\{w\} \succ_m \emptyset$

$\succ_j$: $\{x, y, w\} \succ_j \{z, y, w\} \succ_j \{x, y\} \succ_j \{z, y\} \succ_j \{w\} \succ_j \emptyset$

$\succ_k$: $\{z, y\} \succ_k \emptyset$.

*Hence, a trail-stable outcome exists: $A = \{w\}$.*[8] *The trail-stable outcome is Pareto inefficient as $\{w\}$ is the least preferred outcome for all agents. There is, however, no set-stable outcome.*[9]

To illustrate trail stability further, let us drop agents $i$ and $m$ and their corresponding contracts from the example above. The new preferences of $j$ are $\{y, z\} \succ_j \emptyset$. There is one set-stable outcome $\{y, z\}$. There are, however, two trail-stable outcomes: $\emptyset$ and $\{y, z\}$. Is $\emptyset$ a reasonable possible outcome of this market? We argue that, in a variety of richer economic models of contracts, it may well be. Suppose that firms are unable to have a joint meeting and must resort to making a unilateral offers. Either firm may be reluctant to make the first offer because in absence of the counteroffer it could end up revealing sensitive information about its costs. Therefore, firms are unable to coordinate $\{y, z\}$ and are stuck in the "inefficient equilibrium". As such, trail stability provides a natural solution concept for matching markets in which firms have limited ability to coordinate their decisions in the contract network.

[7] In all our examples, $\succ$ denotes a strict preference relation. Choice function induced by strict preferences satisfy IRC. We say that $f \in F$ "prefers" outcome $A$ to outcome $A'$ if $C^f(A \cup A') = A_f$

[8] An outcome $A$ is chain-stable if $A$ is acceptable and there are no blocking chains [38]. Therefore, $\{w\}$ is also the unique chain-stable outcome.

[9] Because $\{w\} \succ_j \{x, w\} \succ_k \{x, z, w\} \succ_{i,j} \{z, y, w\} \succ_{j,k} \{w\}$ and other outcomes are not acceptable.

## 3.4 Existence and Properties of Stable Outcomes

We can now state the key result about trail-stable outcomes.

**Theorem 3.4.1** (Fleiner, Jankó, Tamura, Teytelboym). *[22] In any contract network $X$ if choice functions of $F$ satisfy full substitutability and IRC then there exists a trail-stable outcome $A \subseteq X$.*

This theorem establishes a positive existence result for stable outcomes in general contract networks: under the usual assumptions, trail-stable outcomes always exist.[10] To prove this theorem, first we have to look at fully trail-stable outcomes.

## 3.5 Fully Trail-Stable Outcomes

In order to examine the structure of trail-stable outcomes, we need to introduce another stability notion.

An outcome $A \subseteq X$ is *fully trail-stable* if

1. $A$ is acceptable.
2. There is no trail $T = \{x_1, x_2, \ldots, x_M\}$, such as $T \cap A = \emptyset$ and
   (a) $x_1$ is $(A, f_1)$-rational for $f_1 = s(x_1)$, and
   (b) $\{x_{m-1}, x_m\}$ is $(A, f_m)$-rational for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$ and
   (c) $x_M$ is $(A, f_M)$-rational for $f_{M+1} = b(x_M)$.

The above trail $T$ is called a *locally blocking trail to $A$*.

Full trail stability may, at first glance, appear to be an unappealing stability concept. While it rules out (locally) blocking trails, it does not require, as trail stability, that agents accept all their contracts along such blocking trails. More formally, a locally blocking trail may not be an acceptable blocking trail. However, full trail stability has an interesting and important economic interpretation. Suppose contracts only need to be fulfilled sequentially i.e. once a firm's upstream contract has been fulfilled, it immediately fulfils its downstream contract.[11] This is a natural assumption in sequential production networks as production may not be able to continue without inputs and inputs would not be bought without a standing order. Then firms do not need to worry about being involved in multiple chains of contracts along the trail since they

[10]Since trail stability is, in general, stronger than chain stability, Theorem 3.4.1 also implies than any contract network has a chain-stable outcome. Our results do not contradict Theorem 5 on the non-existence of set-stable outcomes in [31] since Theorem 3.4.1 only considers the existence of trail-stable outcomes.

[11]Alternatively, contracts further down the trail could be specified to be fulfilled later.

never need to be fulfilled together. As such full trail stability can be a useful stability concept in production networks in which production is sequential rather than (possibly) simultaneous. Full trail stability may be a better stability concept for a short-run prediction of network stability whereas trail stability is more suitable for the long run. It turns out that fully trail-stable outcomes also exist in general production networks.

**Theorem 3.5.1.** *In any contract network $X$ if choice functions of $F$ satisfy full substitutability and IRC then there exists a fully trail-stable outcome $A \subseteq X$.*

In order to prove Theorem 3.5.1, we use tools familiar to matching theory, such as the Tarski fixed-point theorem.

Consider $Y^B$ and $Z^S$, which are subsets of $X$, and represent sets of available upstream and downstream contracts for all agents, respectively. Define a lattice $\mathcal{L}$ with the ground set $X \times X$ with an order $\sqsubseteq$ such that $(Y^B, Z^S) \sqsubseteq (Y'^B, Z'^S)$ if $Y^B \subseteq Y'^B$ and $Z^S \supseteq Z'^S$.

Furthermore, define a mapping $\Phi$ as follows:

$$\begin{aligned}\Phi_B(Y^B, Z^S) &= X \setminus R_S(Z^S|Y^B)\\ \Phi_S(Y^B, Z^S) &= X \setminus R_B(Y^B|Z^S)\\ \Phi(Y^B, Z^S) &= (\Phi_B(Y^B, Z^S), \Phi_S(Y^B, Z^S))\end{aligned}$$

where $R_S(Z^S|Y^B) := \cup_{f\in F} R^f_S(Z^S|Y^B)$ and $R_B(Y^B|Z^S) := \cup_{f\in F} R^f_B(Y^B|Z^S)$. Clearly, $\Phi$ is isotone [17],[38],[31] on $\mathcal{L}$.

*Proof of Theorem 3.5.1.* Existence of fixed-points of $\Phi$ follows from Theorem 1.2.1 (Tarski's fixed point theorem) since $(X \times X, \sqsubseteq)$ is a complete lattice.

We claim that every fixed point $(\dot{X}^B, \dot{X}^S)$ of $\Phi$ corresponds to an outcome $\dot{X}^B \cap \dot{X}^S = A$ that is fully trail-stable. First, we show that $A$ is individually rational. Observe that if $(\dot{X}^B, \dot{X}^S)$ is a fixed point then $\dot{X}^S \cup \dot{X}^B = X$. To see this suppose for contradiction that there is a contract $x \notin \dot{X}^S \cup \dot{X}^B$. Then $x \notin R_S(\dot{X}^S|\dot{X}^B)$ therefore $x \in X \setminus R_S(\dot{X}^S|\dot{X}^B) = \dot{X}^B$. So $x$ is has to be in $\dot{X}^S \cup \dot{X}^B$ This implies that $R_S(\dot{X}^S|\dot{X}^B) = X \setminus \dot{X}^B = \dot{X}^S \setminus A$ so $C_S(\dot{X}^S|\dot{X}^B) = A$ and similarly $C_B(\dot{X}^B|\dot{X}^S) = A$ From this, we can see that $A$ is individually rational.

Second, we show that $A$ is fully trail-stable. This is similar to Step 1 of the Proof of Lemma 1 in [38]. Suppose that $T = \{x_1, ...x_m\}$ is a locally blocking trail and assume towards a contradiction that $T \cap A = \emptyset$. Since we have that $x_1 \in C^{s(x_1)}_S(A + x_1|A)$, we must have that $x_1 \in C^{s(x_1)}_S(\dot{X}^S + x_1|A)$ Since if $C^{s(x_1)}_S(\dot{X}^S + x_1|A) \subseteq \dot{X}^S$ then by IRC $C^{s(x_1)}_S(\dot{X}^S + x_1|A) = A$, therefore $C^{s(x_1)}_S(A + x_1|A) = A$ Also, $x_1 \in C^{s(x_1)}_S(\dot{X}^S + x_1|\dot{X}^B)$ (by CSC). If $x_1 \in \dot{X}^S$, then $x_1 \in \dot{X}^B = X \setminus R_S(\dot{X}^S|\dot{X}^B)$. But we assumed that $x_1 \notin A$, so $x_1 \in \dot{X}^B$. Now, consider $x_2$.

By definition of a locally blocking trail, we have that $x_2 \in C^{s(x_2)}_S(A + x_2|A + x_1)$. Once again by full substitutability and IRC, we obtain that and $x_2 \in C^{s(x_2)}_S(\dot{X}^S +$

$x_2|\dot{X}^B + x_1)$. If $x_2 \in \dot{X}^S$, then $x_2 \in \dot{X}^B = X \setminus R_S(\dot{X}^S|\dot{X}^B)$. But we assumed that $x_2 \notin A$, so $x_2 \in \dot{X}^B$. Now proceed by induction, we show that every $x \in T$ is in $\dot{X}^B$. Consider the last contract $x_m$. Since $x_m \in C_B^{b(x_m)}(A+x_m|A)$, using the same argument we had for $x_1$, we get that $x_m \in \dot{X}^S$. A contradiction.

Now we show that every fully trail-stable outcome corresponds to a fixed point: Suppose $A$ is fully trail-stable. For every $x_i \notin A$, if there exists a trail $\{x_1x_2\dots x_i\}$ such that $x_1$ is $(A, s(x_1))$-rational, and $\{x_{m-1}, x_m\}$ is $(A, f_m)$-rational for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq i$, then let $x_i \in X_0^B$, otherwise $x_i \in X_0^S$. Let $\dot{X}^B = A \cup X_0^B$ and $\dot{X}^S = A \cup X_0^S$. Clearly $\dot{X}^S \cup \dot{X}^B = X$.

Outcome $A$ is individually rational, so $C^f(A) = A_f$ for all $f \in F$. For every firm $f$, if $f = s(x)$ and $x \in \dot{X}^S \setminus A$ then $x \notin C^f(A+x)$ otherwise $x$ would be in $X_B$. From SSS, $C_S^f(\dot{X}^S|A) = A$. And if $f = b(y)$ and $y \in \dot{X}^B \setminus A$ then $y \notin C^f(A+y)$ otherwise the trail ending in $y$ would be a locally blocking trail. From SSS, $C_B^f(\dot{X}^B|A) = A$. Moreover, $\{x, y\} \nsubseteq C(A \cup \{x, y\})$ otherwise $x$ would be in $\dot{X}^B$. These together imply that $C_S(\dot{X}^S|\dot{X}^B) = A$ and $C_B(\dot{X}^B|\dot{X}^S) = A$. Therefore $R_S(\dot{X}^S|\dot{X}^B) = \dot{X}^S \setminus A$, $R_B(\dot{X}^B|\dot{X}^S) = \dot{X}^B \setminus A$, so $X \setminus R_S(\dot{X}^S|\dot{X}^B) = \dot{X}^B$, $X \setminus R_B(\dot{X}^B|\dot{X}^S) = \dot{X}^S$. So this $(\dot{X}^B, \dot{X}^S)$ pair is a suitable fixed point for $A$. We will call it the *canonical stable pair* for $A$.

□

## 3.6 Relationships Between Stability Concepts

In this section, we link together all the stability concepts discussed above. We first show that set stability implies full trail stability, which in turn implies trail stability. We also link set stability and trail stability via intensity. We then link trail stable and fully trail-stable outcome via separability. Finally, we explore chain stability [38].
The follow lemma ties three key stability concepts together.

**Lemma 3.6.1.** *In any contract network $X$ if choice functions of $F$ satisfy full substitutability and IRC then the following holds for an outcome $A \subseteq X$.*
*(i) If $A$ is a fully trail-stable outcome then $A$ is also trail-stable.*
*(ii) If $A$ is a set-stable outcome then $A$ is fully trail-stable.*

To prove Lemma 3.6.1 the following two Lemmata come in handy.

In the proofs, we will use the concept of a circuit, which is a closed trail. A non-empty sequence of different contracts $Q = \{x_1, \dots, x_M\}$ is a *circuit* if $b(x_m) = s(x_{m+1})$ holds for all $m = 1, \dots, M-1$, and $b(x_M) = s(x_1)$.

**Lemma 3.6.2.** *Let $F$ be the set of agents and $X$ be the set of contracts in a contract network with fully substitutable choice functions. If $Y$ and $Z$ are disjoint sets of contracts and $f$ is an agent such that $Z_f$ is $(Y, f)$-rational then for any contract $z \in Z_f^B$*

*one of the following options hold:*
*(1) $z$ is $(Y, f)$-rational or*
*(2) there exists some $z' \in Z_f^S$ such that $\{z, z'\}$ is a $(Y, f)$-rational pair or*
*(3) there are $z_1, z_2, \ldots, z_k \in Z_f^S$ such that both $\{z, z_1, z_2, \ldots, z_k\}$ and $\{z_i\}$ (for $1 \leq i \leq k$) are $(Y, f)$-rational. For $z \in Z_f^S$ an analogous statement holds.*

*Proof.* We can suppose without loss of generality that $z \in X_f^B$. From the SSS property, it follows that $z \in C^f(Y_f \cup Z_f^S \cup \{z\})$.

Assume that $C^f(Y_f \cup Z_f^S \cup \{z\}) \cap Z_f^S = \emptyset$. Therefore $C^f(Y_f \cup Z_f^S \cup \{z\}) \subseteq (Y_f \cup \{z\}) \subseteq (Y_f \cup Z_f^S \cup \{z\})$ so from IRC $z \in C^f(Y_f \cup \{z\})$, so $z$ is $(Y, f)$-rational, we get option (1).

If $z$ is not $(Y, f)$-rational then there are some contracts $\{z_1, z_2 \ldots z_k\} = C^f(Y_f \cup Z_f^S \cup \{z\}) \cap Z_f^S$. Using SSS again, we have $z_i \in C^f(Y_f \cup \{z, z_i\})$ for every $z_i \in \{z_1, z_2 \ldots z_k\}$. If there exists an $z_i$ such that $z_i$ is not $(Y, f)$-rational, then suppose $z \notin C^f(Y_f \cup \{z, z_i\})$, so $C^f(Y_f \cup \{z, z_i\}) \subseteq (Y_f \cup \{z_i\})$, and from IRC we have $C^f(Y_f \cup \{z, z_i\}) = C^f(Y_f \cup \{z_i\})$. But since $z_i$ is not $(Y, f)$-rational this is impossible, therefore $\{z, z_i\} \subseteq C^f(Y_f \cup \{z, z_i\})$, we achieved a $(Y, f)$-rational pair.

If all of $\{z_1, z_2 \ldots z_k\}$ are $(Y, f)$-rational, we get option (3). □

In the following lemma, consider a firm $f$ with some contracts $x_1, x_2, \ldots, x_k \in X_f^B$ and $z_1, z_2, \ldots, z_k \in X_f^S$. When we say that $x_1$ is *void*, we mean that $x_1$ is empty, every "$\{x_1, z_j\}$ is an $(Y, f)$-rational pair" translates to "$\{z_j\}$ is an $(Y, f)$-rational contract."

**Lemma 3.6.3.** *Let $F$ be the set of agents and $f$ be an agent in a contract network with fully substitutable choice functions. Assume that $Y$ is acceptable and $x_1, x_2, \ldots, x_k \in X_f^B$ and $z_1, z_2, \ldots, z_k \in X_f^S$ such that $\{x_i, z_i\}$ is a $(Y, f)$-rational pair for any $1 \leq i \leq k$ but $\{x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_k\}$ is not $(Y, f)$-rational. Then $\{x_i, z_j\}$ is a $(Y, f)$-rational pair for some $i \neq j$.*

*The above statement remains true if $x_1$ or $z_k$ or both are void. If both $x_1$ and $z_k$ are void, there is $(Y, f)$-rational pair $\{x_i, z_j\}$, $i \neq j$ such that $\{x_i, z_j\} \neq \{x_k, z_1\}$.*

*Proof.* Suppose for example $z_j \notin C^f(Y \cup \{x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_k\})$ for some $j$ such that both $x_j$ and $z_j$ exist. Then from CSC, $z_j \notin C^f(Y \cup \{x_j, z_1, z_2, \ldots, z_k\})$. But $x_j \in C^f(Y \cup \{x_j z_j\})$ so from CSC $x_j \in C^f(Y \cup \{x_j, z_1, z_2, \ldots, z_k\})$. Since $x_j$ is not $(Y, f)$-rational, there is a $z_l \in C^f(Y \cup \{x_j, z_1, z_2, \ldots, z_k\})$ therefore $\{x_j, z_l\}$ is $(Y, f)$-rational and $l \neq j$.

In the case that $x_1$ is void and $z_1 \notin C^f(Y \cup \{x_2, \ldots, x_k, z_1, z_2, \ldots, z_k\})$, from CSC, $z_1 \notin C^f(Y \cup \{z_1, z_2, \ldots, z_k\})$. This is impossible when $z_1$ is $(Y, f)$-rational but none of the other $z_j$ contracts are $(Y, f)$-rational. Therefore if we have found $(Y, f)$-rational pair $\{x_i, z_j\}$, then at least one of $x_i$ and $z_j$ was not $(Y, f)$-rational by itself. □

A consequence of Lemma 3.6.2 is that full trail stability is a stronger property than trail stability, as we will show now:

*Proof of Lemma 3.6.1.* Without the loss of generality, we may assume that (b)ii holds in the definition of trail-stability. The other case when (b)i holds can be proved analogously. Consider a fully trail-stable outcome $A$. Suppose that $A$ is not trail-stable, i.e. there exists a blocking trail $T$ for it. If this trail reaches firm $f$ multiple times, let $T_f^B = \{a_1, a_2 \dots a_k\}$ and $T_f^S = \{b_1, b_2 \dots b_k\}$ where $a_i, b_i$ are two consecutive contracts in the trail. In the notation above some of contracts can be void, if the trail starts at $f$ then $a_1$ is void, if the trail ends at $f$ then $b_k$ is void.

We will show that there exist some $1 \leq i_1 \leq i_2 \leq \dots \leq i_l \leq k$ and $1 \leq j_1 \leq j_2 \leq \dots \leq j_l \leq k$ such that

- $i_{r+1} = j_r + 1$ for every $1 \leq r < l$
- either $\{a_{i_1}, b_{j_1}\}$ or $b_{j_1}$ is $(A, f)$-rational and
- $\{a_{i_r}, b_{j_r}\}$ is $(A, f)$-rational for all $1 < r < l$ and
- either $a_{i_l}$ or $\{a_{i_l}, b_{j_l}\}$ is $(A, f)$-rational.

If none of the contracts $\{b_1, b_2 \dots b_k\}$ is $(A, f)$-rational, let $i_1 = 1$, so $a_{i_1} = a_1$. If some $b_n$ is $(A, f)$-rational, choose the last one in the trail, i.e. $b_n$ is $(A, f)$-rational but for any $m > n$, $b_m$ is not $(A, f)$-rational. Then let $j_1 = n$. (In this case $i_1$ is not needed at all.)
Suppose we have already found $i_1 \dots i_r, j_1 \dots j_r$ that satisfies our requirements. If $a_{i_{r+1}} = a_{j_r+1}$ is $(A, f)$-rational, we end the trail there, and let $l = r + 1$, $a_{i_l} = a_{i_{r+1}}$. From the definition of blocking trails, $\{a_{i_{r+1}}, b_{i_{r+1}} \dots a_k, b_k\}$ is $(A, f)$-rational. If $a_{i_r+1}$ is not $(A, f)$-rational, using Lemma 3.6.2, there is a $b_{j_{r+1}}$ such that $j_{r+1} \geq i_{r+1}$ and $\{a_{i_{r+1}}, b_{j_{r+1}}\}$ is $(A, f)$ rational. Using this method, we created a cutoff at firm $f$ such that we have a shorter trail, and the consecutive $a_{i_r}, b_{j_r}$ passing through it are $(A, f)$-rational. Using this method for every firm in the trail, we get a locally blocking trail, therefore $A$ is not fully trail-stable.

To show that every set-stable outcome is fully trail-stable, consider an outcome $A$ which is not fully trail-stable, and choose the shortest locally blocking trail $T$ for it. For every firm involved in $T$, if $T_f \nsubseteq C^f(A \cup T_f)$, then using Lemma 3.6.3 there is a upstream-downstream contract-pair $x_j \in T_f^B$ and $z_l \in T_f^S$ such that $j \neq l$ and $\{x_j, z_l\}$ is $(A, f)$-rational. This way we get a shorter locally blocking trail or circuit. Since this was the shortest trail, it must be a circuit. Repeat finding shortcuts until we get a circuit $Z$ such that $Z_f \subseteq C^f(A \cup Z_f)$ for every firm $f$, so this a blocking set. Since $T \cap A = \emptyset$ and $Z \subseteq T$, $Z \cap A = \emptyset$. □

Theorem 3.5.1 and Lemma 3.6.1 immediately imply Theorem 3.4.1. We now pin down the role of separability for trail-stable and fully trail-stable outcomes.

**Proposition 3.6.4.** *In any contract network $X$ whenever choice functions of $F$ satisfy full substitutability, IRC and separability, an outcome $A \subseteq X$ is fully trail-stable if and only if it is trail-stable.*

*Proof of Proposition 3.6.4.* Lemma 3.6.1 implies that if outcome $A$ is fully trail-stable then $A$ is also trail-stable. So assume that outcome $A$ is trail-stable. If $A$ is not fully trail-stable then there is a locally blocking trail $T$ to $A$. The separable property of the choice functions imply that $T$ is a blocking trail, contradicting the trail stability of $A$. So $A$ is fully trail-stable. □

Under separability all properties of fully trail-stable outcomes apply to trail-stable outcomes. This is summarized in the following corollary.

**Corollary 3.6.5.** *Suppose that in a contract network $X$, choice functions of $F$ satisfy full substitutability, IRC and separability. Then all properties of fully trail-stable outcomes, described in Lemma 3.8.1, Lemma 3.8.2, Proposition 3.9.1, Proposition 3.9.2, Proposition 3.9.3 and Proposition 3.9.4, apply to trail-stable outcomes.*

Separability is crucial for the correspondence between fully trail-stable and trail-stable outcomes. Separability ensures that all blocking trails are locally blocking trails. An example below shows that full trail stability is strictly stronger that trail stability.

**Example 3.6.6** (Trail-stable outcomes are not always fully trail-stable)**.** *Consider agents and contracts described in Example 1 and Figure 3.5. Agents have the following preferences that induce fully substitutable choice functions:*
$\succ_m$: $\{w\} \succ_m \emptyset$
$\succ_i$: $\{x\} \succ_i \emptyset$
$\succ_k$: $\{z, y\} \succ_k \emptyset$
$\succ_j$: $\{z, y\} \succ_j \{w, z\} \succ_j \{y, x\} \succ_j \emptyset$.
*The empty set is preferred to any other set of contracts.*
*For outcome $A = \emptyset$, the trail $\{w, z, y, x\}$ is locally blocking trail but not trail-blocking. Therefore, trail-stable outcomes are $\emptyset$ and $\{z, y\}$ but the only fully trail-stable outcome is $\{z, y\}$. Note that $j$'s choice function induced by the preference is not separable.*

Finally, we explore the relationship between trail stability, set stability and chain stability.

Choice functions of $f \in F$ satisfy *simplicity* if there exists an "intensity" mapping $w : X_f \to \mathbb{R}$ such that whenever $A$ is a $(W, f)$-rational set for some acceptable set $A$ of contracts, then for every $y \in X_f^B \cap A$ there exists $z \in X_f^S \cap A$ such that $w(y) > w(z)$ holds.

One example of choice functions which are simple are the following: if the agent is offered a set of contracts, he picks the upstream contract $y$ with the highest intensity and a downstream contract $z$ with the lowest intensity (as long as the intensity of the $y$

is greater than of $z$, otherwise he picks nothing). For example, if the intensity mapping $w$ represents the per-unit price of the contract, then the condition says that the firm only signs a pair of contracts if the price in the downstream contract is greater than the price in the upstream contract, while picking the highest-price downstream contract and the lowest-price upstream contract.

**Proposition 3.6.7.** *In any contract network $X$ whenever choice functions of $F$ satisfy full substitutability, IRC and simplicity then an outcome $A \subseteq X$ is set-stable if and only if it is trail-stable.*

*Proof of Proposition 3.6.7.* Lemma 3.6.1 implies that if outcome $A$ is set-stable then $A$ is also fully trail-stable. Assume that outcome $A$ is fully trail-stable, but not set-stable, it has a blocking set $Z$.

If for every $z \in Z$, contract $z$ is neither $(A, s(z))$-rational nor $(A, b(z))$-rational, then using Lemma 3.6.2 we can find a circuit $Q = \{z_1, z_2, \dots z_k\} \subseteq Z$ such that $\{z_i, z_{i+1}\}$ is an $(A, b(z_i))$-rational pair for every $1 \leq i \leq k$ and $\{z_k, z_1\}$ is an $(A, b(z_k))$-rational pair. Since every $\{z_i, z_{i+1}\}$ an $(A, b(z_i))$-rational set by itself, as choice functions are simple, intensity function $w$ must strictly decrease along circuit $Q$, which is impossible.

If some of the contracts are $A$-rational: Suppose that $z_1$ is $(A, s(z_1))$-rational. From Lemma 3.6.2 we can find a trail $\{z_2, z_3 \dots z_k\} \subseteq Z$ such that for every $z_i$, either $\{z_i, z_{i+1}\}$ is a $(A, b(z_i))$-rational pair, (therefore $w(z_i) > w(z_{i+1})$) or there are some $y_1 \dots y_l$ such that $b(y_j) = s(z_i)$ for all $1 \leq j \leq l$ and $\{z_i, y_1 \dots y_l\}$ is $(A, b(z_i))$-rational. From the simplicity property there is a $y_j$ such that $w(z_i) > w(y_j)$, this $y_j$ contract will be $z_{i+1}$. The trail terminates at the first occasion when $z_i$ is $(A, b(z_i))$-rational.

Since the intensity strictly decreases, we cannot get back to a contract used earlier in the trail, so the trail must terminate. Let us pick a contract $z_i$ in the trail such that it is the last one which is $(A, s(z_i))$-rational, and then choose the smallest $j$ such that $j \geq i$ and $z_j$ is $(A, b(z_j))$-rational. From Lemma 3.6.2, the trail from $z_i$ to $z_j$ is locally blocking, so outcome $A$ is not fully trail-stable. $\square$

We now formally define chain stability, introduced by [38]. To recap, a chain $C$ is a trail in which all the agents are distinct. Chain-stable outcomes rule out consecutive pairwise deviations along chains. While this stability concept was introduced in the context of acyclic trading network, it could also be applicable to general trading networks in which firms only have one opportunity to recontract during a deviation.

An outcome $A \subseteq X$ is *chain-stable* if

1. $A$ is acceptable.

2. There is no chain $C = \{x_1, x_2, \dots, x_M\}$, such as $C \cap A = \emptyset$ and

   (a) $x_1$ is $(A, f_1)$-rational for $f_1 = s(x_1)$, and

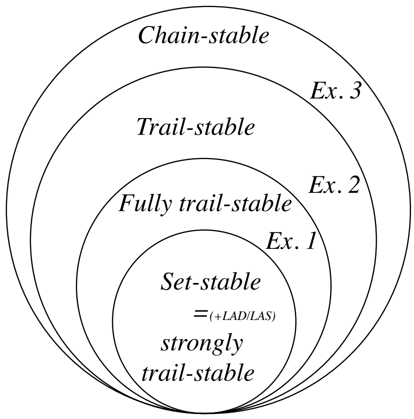



Figure 3.6: Relationship between stability concepts in general contract networks

(b) $\{x_{m-1}, x_m\}$ is $(A, f_m)$-rational for $f_m = b(x_{m-1}) = s(x_m)$ whenever $1 < m \leq M$ and

(c) $x_M$ is $(A, f_M)$-rational for $f_{M+1} = b(x_M)$.

Since every chain is a trail, every trail-stable outcome is chain-stable. In acyclic networks every trail is also chain, so chain-stable, trail-stable and fully trail-stable outcomes coincide with set-stable outcomes [31]. However, as the example below shows, chain stability is weaker than trail stability (and hence weaker than full trail stability) in general contract networks.

**Example 3.6.8** (Chain-stable outcomes are not necessarily trail-stable)**.** *Consider agents and contracts described in Examples 1 and 2, and Figure 3.5. Agents have the following fully substitutable preferences:*
$\succ_m$: $\{w\} \succ_m \emptyset$
$\succ_i$: $\{x\} \succ_i \emptyset$
$\succ_k$: $\{z, y\} \succ_k \emptyset$
$\succ_j$: $\{w, x, z, y\} \succ_j \{w, z\} \succ_j \{y, x\} \succ_j \{y, z\} \succ_j \emptyset$
*The empty set is preferred to any other set of contracts.*
*Now, for outcome $\emptyset$, the trail $\{w, z, y, x\}$ is trail-blocking, but there is no blocking chain for $A = \emptyset$. Outcome $\{z, y\}$ is, however, blocked by chain $\{w, x\}$. Therefore the only trail-stable outcome is $\{w, z, y, x\}$ and the chain-stable outcomes are $\emptyset$ and $\{w, z, y, x\}$.*

This is intuitive because chains allows the firms to appear in the blocking set only once therefore they rule out fewer possible blocks. Figure 3.6 summarizes the relationships between various stability concepts in general contract networks. Set-stable and strongly trail-stable outcomes may not exist and they are only equivalent under the Laws of Aggregate Demand and Supply as Example 1 in [33] shows.

## 3.7 Terminal Agents and Terminal Superiority

We now introduce some terminology that describes contracts of agents, who only act as buyers or only act as sellers. A firm $f$ is a *terminal seller* if there are no upstream contracts for $f$ in the network and $f$ is a *terminal buyer* if the network does not contain any downstream contracts for $f$. An agent who is either a terminal buyer or a terminal seller is called a *terminal agent*. Let $\mathcal{T}$ denote the set of terminal agents in $F$ and for a set $A$ of contracts let us denote the *terminal contracts of* $A$ by $A_{\mathcal{T}} := \bigcup\{A_f : f \in \mathcal{T}\}$. A set $Y$ of contracts is *terminal-acceptable* if there is an acceptable set $A$ of contracts such that $Y = A_{\mathcal{T}}$.

If $Y$ is a set of contracts bought or sold by terminal agents, i.e., $Y \subseteq X_{\mathcal{T}}$, we say that $Y$ is *terminal-trail-stable* if there is a trail-stable outcome $A \subseteq X$ such that $Y = A_{\mathcal{T}}$. Similarly, $Y$ is *terminal-fully-trail-stable* if there is a fully trail-stable outcome $A \subseteq X$ such that $Y = A_{\mathcal{T}}$.

If $A$ and $W$ are terminal-acceptable sets of contracts then we say that $A$ is *seller-superior* to $W$ (denoted by $A \succeq^S W$) if $C^f(A_f \cup W_f) = A_f$ for each terminal seller $f$ and $C^g(A_g \cup W_g) = W_g$ for each terminal buyer $g$. Similarly, $A$ is *buyer-superior* to $W$ (denoted by $A \succeq^B W$) if $C^f(A_f \cup W_f) = W_f$ for each terminal seller $f$ and $C^g(A_g \cup W_g) = A_g$ for each terminal buyer $g$. Clearly, these relations are opposite, that is, $W \succeq^S A$ if and only if $A \succeq^B W$ holds. Whenever either relation holds, we call this partial order on outcomes *terminal superiority.* Terminal agents are going to play a key role when we describe the structure of outcomes in contract networks.

Recall that in the marriage model of Gale and Shapley, the existence of man-optimal and woman-optimal stable matchings follow from the well-known lattice structure of stable matchings. The key to extending this result to contract networks is to consider only terminal agents. We say that a fully trail-stable outcome $A_{max}$ is *buyer-optimal* if any terminal buyer prefers it to any other outcome, so for any fully trail-stable $Z \subseteq X$, we have that $A_{max} \succeq^B Z$.

A fully trail-stable outcome $A_{min}$ that is *seller-optimal* if any terminal seller prefers it to any other outcome, so $A_{min} \succeq^S Z$ so for any fully trail-stable $Z$.

Since terminal agents have only upstream or only downstream contracts, for their choice functions same-side substitutability is equivalent to substitutability. Recall that a if a choice function $C : 2^X \to 2^X$ satisfies substitutability and IRC then $C$ is also path-independent, that is, $C(Y \cup Z) = C(Y \cup C(Z))$ holds for any subsets $Y, Z$ of $X$. This was shown in Lemma 1.1.10.

**Lemma 3.7.1.** *If choice functions are fully substitutable in a trading network then both buyer-superiority and seller-superiority are is a partial order on terminal-acceptable outcomes.*

*Proof of Lemma 3.7.1.* For seller-superiority, we need to prove that $\preceq^S$ is reflexive, antisymmetric and transitive. Assume that $A, A'$ and $A''$ are acceptable outcomes. As $C^f(A_f \cup A_f) = C^f(A_f) = A_f$ holds for each agent (and hence for each terminal seller) $f$, relation $\preceq^S$ is reflexive. If $A \preceq^S A' \preceq^S A$ then we have $A_f = C^f(A_f \cup A'_f) = A'_f$ holds for any terminal agent $f$, hence $A = A'$ and $\preceq^S$ is antisymmetric. For transitivity, assume that $A \succeq^S A' \succeq^S A''$. Using this and path-independence, we get for any terminal agent $f$ that

$$C^f(A_f \cup A''_f) = C^f(C^f(A_f \cup A'_f) \cup A''_f) = C^f(A_f \cup A'_f \cup A''_f) = \\ C^f(A_f \cup C^f(A'_f \cup A''_f)) = C^f(A_f \cup A'_f) = A_f \ ,$$

hence $A \succeq^S A''$ holds, indeed. Similarly for buyer-superiority. □

Now we consider only the contracts sold by the terminal sellers. For any $Y \subseteq X$, let $Y_{\mathcal{S}} = \{x \in Y : s(x) \text{ is a terminal seller }\}$.

Given two fully trail-stable outcomes $A$ and $A'$, let us denote the canonical stable pair, defined as at the end of Proof of Theorem 3.5.1 for $A$ with $\dot{X}^B$ and $\dot{X}^S$, and the canonical stable pair for $A'$ with $\dot{X}'^B$ and $\dot{X}'^S$.

**Lemma 3.7.2.** *Given two fully trail-stable outcomes $A$ and $A'$, $C^f(A_f \cup A'_f) = A_f$ for each terminal seller if and only if $\dot{X}^S_{\mathcal{S}} \supseteq \dot{X}'^S_{\mathcal{S}}$ and $\dot{X}^B_{\mathcal{S}} \subseteq \dot{X}'^B_{\mathcal{S}}$ holds. A similar statement holds for terminal buyers.*

*Proof.* If $f$ is a terminal seller, $C^f(\dot{X}^S) = A_f$ and $C^f(\dot{X}'^S) = A'_f$. Suppose that $\dot{X}^S_{\mathcal{S}} \supseteq \dot{X}'^S_{\mathcal{S}}$. By IRC, $A_f \subseteq A_f \cup A'_f \subseteq \dot{X}^S_f$ implies that $C^f(A_f \cup A'_f) = A_f$.

For the opposite direction, take a contract $x \in X_f$ such that $x \notin C^f(A'_f \cup x)$. We use Lemma 3.7.1, $A \succeq^S A' \succeq^S x$, therefore $A \succeq^S x$, so $x \notin C^f(A_f \cup x)$. When we define the stable pairs for $A$ and $A'$, if $x \in C^f(A'_f \cup x)$ then $x \in \dot{X}^B$, if $x \notin C^f(A'_f \cup x)$ then $x \in \dot{X}^S$. From the previous observation we can see that $\dot{X}^S_{\mathcal{S}} \supseteq \dot{X}'^S_{\mathcal{S}}$ and $\dot{X}^B_{\mathcal{S}} \subseteq \dot{X}'^B_{\mathcal{S}}$. The proof for terminal buyers is analogous. □

## 3.8 Lattice Property of Fully Trail-Stable Outcomes

**Lemma 3.8.1.** *In any contract network $X$ if choice functions of $F$ satisfy full substitutability and IRC then the set of fully trail-stable outcomes contains buyer-optimal and seller-optimal outcomes.*

Lemma 3.8.1 extends Theorem 4 by [31], which establishes the existence of buyer- and seller-optimal outcomes in acyclic trading networks.[12] We say that $Y \subseteq X$ is *terminal-fully-trail-stable* if there is a fully trail-stable outcome $A \subseteq X$ such that $Y = A_{\mathcal{T}}$.

[12] This is a common property of stable outcomes in two-sided markets with substitutable choice functions, however, it typically fails in richer matching models [39],[3],[4].

**Lemma 3.8.2.** *In any contract network $X$ if choice functions of $F$ satisfy full substitutability and LAD/LAS then the terminal-fully-trail-stable contract sets form a lattice under terminal superiority.*

Lemma 3.8.2 shows that whenever LAD/LAS holds choice functions of terminal agents define a natural partial order on outcomes and the terminal-fully-trail-stable contract sets form a lattice under this order. Note that for the lattice and the opposition-of-interests structure, only terminal agents play a role: two outcomes are equivalent if all the terminal agents have the same set of contracts. Indeed, if $A^1$ and $A^2$ are fully trail-stable outcomes then there is a fully trail-stable outcome $A^+$ such that all terminal buyers prefer $A^+$ to both $A^1$ and $A^2$ and all sellers prefer any of $A^1$ and $A^2$ to $A^+$.[13] This establishes full "polarization of interests" in trail-stable outcomes in the sense of [42] and immediately implies the existence of buyer-optimal ($A_{max}$) and seller-optimal ($A_{min}$) fully trail-stable outcomes. Therefore, our result substantially strengthens and generalizes the previous results by [42],[11],[14] and [31].[14]

### 3.8.1 The sublattice property of fixed points

We can rephrase the definitions of the Laws of Aggregate Demand and Supply (LAD/LAS) in the following way:

If the choice functions of firm $f$ satisfy LAD and LAS, for sets of contracts $Y' \subseteq Y \subseteq X_f^B$, and $Z \subseteq Z' \subseteq X_f^S$ (i.e. $(Y', Z') \sqsubseteq (Y, Z)$) then $|C_B^f(Y'|Z')| - |C_S^f(Z'|Y')| \leq |C_B^f(Y|Z)| - |C_S^f(Z|Y)|$.

For every firm $f$ we define a weight function on the contracts in $X_f$, namely let $w(x) = 1$ if $x \in X_f^B$ and $w(x) = -1$ if $x \in X_f^S$. So $w(C^f(Y, Z)) = |C_B^f(Y|Z)| - |C_S^f(Z|Y)|$. Therefore if $C^f$ is LAD-LAS, then $(Y', Z') \sqsubseteq (Y, Z)$ implies $w(C^f(Y', Z')) \leq w(C^f(Y, Z))$.

Let $Y$ and $Y'$ be subsets of $X_f^B$, $Z$ and $Z'$ are subsets of $X_f^S$. We denote the complement of $Z$ in $X_f^S$ with $\overline{Z} = X_f^S \setminus Z$. Define the operation $(Y, Z)\widetilde{\setminus}(Y', Z') = (Y \setminus Y', \overline{Z' \setminus Z})$. For a given firm $f$, we call a set function $R : 2_f^X \to 2_f^X$ a *w-contraction* if for every $(Y', Z') \sqsubseteq (Y, Z)$ pair, $w(R(Y, Z)\widetilde{\setminus}R(Y', Z')) \leq w((Y, Z)\widetilde{\setminus}(Y', Z'))$

Let us describe some properties of this $\widetilde{\setminus}$ operation:

**Lemma 3.8.3.** *For a firm $f$, let $Y$ and $Y'$ be subsets of $X_f^B$, $Z$ and $Z'$ are subsets of $X_f^S$ such that $(Y', Z') \sqsubseteq (Y, Z)$. Then the following holds:*

1. $w((Y, Z)\widetilde{\setminus}(Y', Z')) = w((Y, Z)) - w((Y', Z')) - |X_f^S|$.
2. *For any $(A, B)$ pair,* $w((A, B)\widetilde{\setminus}(Y, Z)) \leq w((A, B)\widetilde{\setminus}(Y', Z'))$.

[13] Of course, the same holds for if we exchange the role of buyers and sellers.

[14] Theorem 4 in [18], which states that any two stable flows agree on terminal contracts, is a further strengthening of Lemma 3.8.2 in the special case of network flows.

3. *If $(Y, Z) \sqsubseteq (A, B)$ then the $w((A, B)\widetilde{\setminus}(Y, Z)) = w((A, B)\widetilde{\setminus}(Y', Z'))$ equality implies $(Y', Z') = (Y, Z)$.*

*Proof.* 1. $w((Y, Z)\widetilde{\setminus}(Y', Z')) = |Y \setminus Y'| - |\overline{Z' \setminus Z}| = |Y| - |Y'| - |X_f^S| + |Z'| - |Z| = w((Y, Z)) - w((Y', Z')) - |X_f^S|$.

2. Since $Y \supseteq Y'$, this implies $A \setminus Y \subseteq A \setminus Y'$, and similarly $Z \subseteq Z'$ gives us $Z \setminus B \subseteq Z' \setminus B$, so $\overline{Z \setminus B} \supseteq \overline{Z' \setminus B}$, therefore $w((A, B)\widetilde{\setminus}(Y, Z)) = |A \setminus Y| - |\overline{Z \setminus B}| \leq |A \setminus Y'| - |\overline{Z' \setminus B}| = w((A, B)\widetilde{\setminus}(Y', Z'))$

3. If $w((A, B)\widetilde{\setminus}(Y, Z)) = w((A, B)\widetilde{\setminus}(Y', Z'))$ then equality must hold at $|A \setminus Y| = |A \setminus Y'|$ and $|\overline{Z \setminus B}| = |\overline{Z' \setminus B}|$. Since $Y' \subseteq Y \subseteq A$ and $Z' \supseteq Z \supseteq B$, we get that $Y = Y'$ and $Z = Z'$.

□

**Lemma 3.8.4.** *For a given firm $f$, if the firm's choice functions satisfy full substitutability and LAD/LAS, then the rejection function $R^f$ is $\sqsubseteq$-isotone and a $w$-contraction.*

*Proof.* Let $Y$ and $Y'$ be subsets of $X_f^B$, and $Z$ and $Z'$ are subsets of $X_f^S$ moreover $(Y', Z') \sqsubseteq (Y, Z)$.
We have seen earlier that $R^f$ is $\sqsubseteq$-isotone, so $R^f(Y', Z') \sqsubseteq R^f(Y, Z)$. To prove that it is $w$-contraction, $w(R^f(Y, Z)\widetilde{\setminus}R^f(Y', Z')) + |X_f^S| = w(R^f(Y, Z)) - w(R^f(Y', Z')) = w((Y, Z) \setminus C^f(Y, Z)) - w((Y', Z') \setminus C^f(Y', Z')) = w(Y, Z) - w(C^f(Y, Z)) - w(Y', Z') + w(C^f(Y', Z')) \leq w(Y, Z) - w(Y', Z') = w((Y, Z)\widetilde{\setminus}(Y', Z')) + |X_f^S|$.

We used that $w(C^f(Y', Z')) \leq w(C^f(Y, Z))$. If we subtract $|X_f^S|$ from both sides, we get that
$w(R^f(Y, Z)\widetilde{\setminus}R^f(Y', Z')) \leq w((Y, Z)\widetilde{\setminus}(Y', Z'))$, so $R^f$ is indeed a $w$-contraction. □

We will work on the $(2^{(X,X)}, \widetilde{\cup}, \widetilde{\cap})$ lattice. We can imagine it as a network that contains exactly two (unrelated) copies of each contract, one for the buyer and one for the seller of the contract.

Now the $C^f$ choice functions of the firms are defined over disjoint set of contracts, so for every $Y \subseteq (X, X)$ we can define $C(Y) = \bigcup C^f(Y_f)$. Similarly $R(Y) = \bigcup R^f(Y_f)$. On this whole network, we call a set function $R : 2^{(X,X)} \to 2^{(X,X)}$ a *$w$-contraction* if for every firm $f$ the corresponding $R_f$ was a $w$-contraction.

Let us denote the set of the starting half-contracts (seller's side) with $X_F^S = \bigcup_{f \in F} X_f^S$, and the set of ending half-contracts (buyer's side) with $X_F^B = \bigcup_{f \in F} X_f^B$. Now $|X_F^S| = |X_F^B| = |X|$.

Let $Y \subseteq X_F^B$ and $Z \subseteq X_F^S$. The *dual* of $(Y, Z)$ is what we get by switching the two parts. We denote it with $(Y, Z)^* = (Z, Y)$.

In this model let all the contracts in $X_F^S$ have weight $w = -1$ and all contracts in $X_F^B$ have weight $w = 1$.

**Lemma 3.8.5.** *If $F : 2^{(X,X)} \to 2^{(X,X)}$ is $\sqsubseteq$-isotone and a $w$-contraction then fixed points of $F$ form a nonempty sublattice of $(2^{(X,X)}, \tilde{\cup}, \tilde{\cap})$*

*Proof.* By Tarski's fixed-point theorem, the set of fixed points is nonempty. Now let $U \subseteq (X,X)$ and $V \subseteq (X,X)$. Assume that $F(U) = U$ and $F(V) = V$. By monotonicity, $U \tilde{\cap} V = F(U) \tilde{\cap} F(V) \sqsupseteq F(U \tilde{\cap} V)$ and $U \tilde{\cup} V = F(U) \tilde{\cup} F(V) \sqsubseteq F(U \tilde{\cap} V)$. From the $w$-contraction property and Lemma 3.8.3

$$w(U \tilde{\setminus} (U \tilde{\cap} V)) \geq w(F(U) \tilde{\setminus} F(U \tilde{\cap} V)) \geq w(U \tilde{\setminus} (U \tilde{\cap} V))$$

$$w((U \tilde{\cup} V) \tilde{\setminus} U) \geq w(F(U \tilde{\cup} V) \tilde{\setminus} F(U)) \geq w((U \tilde{\cup} V) \tilde{\setminus} U)$$

hence there must be equality throughout. Using the third part of Lemma 3.8.3 we can see that $(U \tilde{\cap} V) = F(U \tilde{\cap} V)$ and $(U \tilde{\cup} V) = F(U \tilde{\cup} V)$ so they are also fixed points of $F$. $\square$

**Observation 3.8.6.** *Consider two intensity schemes $(Y,Z)$ and $(Y',Z')$ , where $Y, Y' \subseteq X_F^B$ and $Z, Z' \subseteq X_F^S$ and $(X,X) \setminus (Y,Z) = (X \setminus Y, X \setminus Z)$. If $(Y',Z') \sqsubseteq (Y,Z)$, then* $((X \setminus Y, X \setminus Z)^* \tilde{\setminus} (X \setminus Y', X \setminus Z')^*) = ((X \setminus Z) \setminus (X \setminus Z'), \overline{(X \setminus Y') \setminus (X \setminus Y)}) = ((Z' \setminus Z), \overline{(Y \setminus Y')}) = ((X,X) \setminus ((Y,Z) \tilde{\setminus} (Y',Z'))^*$

**Theorem 3.8.7.** *If the choice functions of all agents are fully substitutable and satisfy LAD and LAS, then the fixed points of $\Phi(Y,Z) = (X \setminus R_S(Z|Y), X \setminus R_B(Y|Z))$ form a nonempty, complete sublattice of $(2^X \times 2^X, \tilde{\cup}, \tilde{\cap})$.*

*Proof.* The $\Phi(Y,Z) = (X \setminus R_S(Z|Y), X \setminus R_B(Y|Z))$ function can be also written as $\Phi(Y) = ((X,X) \setminus R(Y,Z))^*$. Since $R$ is $\sqsubseteq$-isotone, $\Phi$ is also $\sqsubseteq$-isotone. We need to show that $\Phi$ is a $w$-contraction. Suppose that $(Y',Z') \sqsubseteq (Y,Z)$ . Using Observation 3.8.6, $w(\Phi(Y,Z) \tilde{\setminus} \Phi(Y',Z')) = w(((X,X) \setminus R(Y,Z))^* \tilde{\setminus} ((X,X) \setminus R(Y',Z'))^*) = w(((X,X) \setminus (R(Y,Z) \tilde{\setminus} R(Y',Z')))^*) = w(R(Y,Z) \tilde{\setminus} R(Y',Z')) \leq w((Y,Z) \tilde{\setminus} (Y',Z'))$ because in Lemma 3.8.4 we showed that $R$ is a $w$-contraction.

Since $\Phi$ is $\sqsubseteq$-isotone and a $w$-contraction, Lemma 3.8.5 gives that the fixed points of $\Phi$ form a sublattice of $(2^{(X,X)}, \tilde{\cup}, \tilde{\cap})$. $\square$

### 3.8.2 Lattice on the terminals

**Theorem 3.8.8.** *If $L$ is a nonempty complete sublattice of $(2^X \times 2^X, \tilde{\cup}, \tilde{\cap})$ then $L_{\mathcal{T}} = \{(Y_{\mathcal{T}}, Z_{\mathcal{T}}) : (Y,Z) \in L\}$ is a sublattice of $(2^{\mathcal{T}} \times 2^{\mathcal{T}}, \tilde{\cup}, \tilde{\cap})$.*

*Proof.* For a given $(A_{\mathcal{T}}, B_{\mathcal{T}})$ there can be more than one inverse image in the original lattice. Let $(A^*, B^*) = \tilde{\bigcup}\{(Y,Z) \in L : (Y_{\mathcal{T}}, Z_{\mathcal{T}}) \sqsubseteq (A_{\mathcal{T}}, B_{\mathcal{T}})\}$ since $L$ is a complete lattice with lattice operations $\tilde{\cup}, \tilde{\cap}$, this $(A^*, B^*) \in L$ and $(A^*_{\mathcal{T}}, B^*_{\mathcal{T}}) = (A_{\mathcal{T}}, B_{\mathcal{T}})$. We call it the *canonical inverse image* of $(A_{\mathcal{T}}, B_{\mathcal{T}})$, since this is the $\sqsubseteq$-greatest among all inverse images.

If $(A_\mathcal{T}, B_\mathcal{T})$ and $(C_\mathcal{T}, D_\mathcal{T}) \in L_\mathcal{T}$ let us consider $(Y,Z) = (A^*, B^*) \,\widetilde{\cap}\, (C^*, D^*)$. Since $(Y,Z) \sqsubseteq (A^*, B^*)$ this implies $(Y_\mathcal{T}, Z_\mathcal{T}) \sqsubseteq (A^*_\mathcal{T}, B^*_\mathcal{T}) = (A_\mathcal{T}, B_\mathcal{T})$. Similarly $(Y_\mathcal{T}, Z_\mathcal{T}) \sqsubseteq (C_\mathcal{T}, D_\mathcal{T})$. We want to show that $(Y_\mathcal{T}, Z_\mathcal{T})$ is the greatest lower bound of $(A_\mathcal{T}, B_\mathcal{T})$ and $(C_\mathcal{T}, D_\mathcal{T})$ in $L_\mathcal{T}$. We can see that $(Y^*, Z^*) \sqsubseteq (A^*, B^*)$ and $(Y^*, Z^*) \sqsubseteq (C^*, D^*)$ because $(A^*, B^*)$ is defined by the union of a greater set. Therefore $(Y^*, Z^*) = (Y, Z)$.

Suppose there exists a $(E_\mathcal{T}, F_\mathcal{T}) \in L_\mathcal{T}$ such that $(E_\mathcal{T}, F_\mathcal{T}) \sqsubseteq (A_\mathcal{T}, B_\mathcal{T})$ and $(E_\mathcal{T}, F_\mathcal{T}) \sqsubseteq (C_\mathcal{T}, D_\mathcal{T})$ but $(E_\mathcal{T}, F_\mathcal{T}) \not\sqsubseteq (Y_\mathcal{T}, Z_\mathcal{T})$. Then in the original lattice $(E^*, F^*) \sqsubseteq (A^*, B^*)$ and $(E^*, F^*) \sqsubseteq (C^*, D^*)$ but $(E^*, F^*) \not\sqsubseteq (Y^*, Z^*)$. But this is impossible since $(Y, Z) = (A^*, B^*) \,\widetilde{\cap}\, (C^*, D^*)$. So we have found a unique greatest common lower bound of $(A_\mathcal{T}, B_\mathcal{T})$ and $(C_\mathcal{T}, D_\mathcal{T})$.

Similar argument can be made in order to find the lowest common upper bound of $(A_\mathcal{T}, B_\mathcal{T})$ and $(C_\mathcal{T}, D_\mathcal{T})$. Let $(Y,Z) = (A^*, B^*) \,\widetilde{\cup}\, (C^*, D^*)$. Since $(Y,Z) \sqsupseteq (A^*, B^*)$ this implies $(Y_\mathcal{T}, Z_\mathcal{T}) \sqsupseteq (A^*_\mathcal{T}, B^*_\mathcal{T}) = (A_\mathcal{T}, B_\mathcal{T})$. Similarly $(Y_\mathcal{T}, Z_\mathcal{T}) \sqsupseteq (C_\mathcal{T}, D_\mathcal{T})$.

Suppose there exists a $(E_\mathcal{T}, F_\mathcal{T}) \in L_\mathcal{T}$ such that $(E_\mathcal{T}, F_\mathcal{T}) \sqsupseteq (A_\mathcal{T}, B_\mathcal{T})$ and $(E_\mathcal{T}, F_\mathcal{T}) \sqsupseteq (C_\mathcal{T}, D_\mathcal{T})$ but $(E_\mathcal{T}, F_\mathcal{T}) \not\sqsupseteq (Y_\mathcal{T}, Z_\mathcal{T})$. Then in the original lattice $(E^*, F^*) \sqsupseteq (A^*, B^*)$ and $(E^*, F^*) \sqsupseteq (C^*, D^*)$ therefore $(E^*, F^*) \sqsupseteq (Y, Z)$, so $(E^*_\mathcal{T}, F^*_\mathcal{T}) = (E_\mathcal{T}, F_\mathcal{T}) \sqsupseteq (Y_\mathcal{T}, Z_\mathcal{T})$, which is a contradiction.

So we have found a unique lowest common upper bound of $(A_\mathcal{T}, B_\mathcal{T})$ and $(C_\mathcal{T}, D_\mathcal{T})$, so $(L_\mathcal{T}, \widetilde{\cup}, \widetilde{\cap})$ is indeed a lattice.

□

*Proof of Lemma 3.8.1.* In the proof of Theorem 3.5.1 we have seen that any fixed point $(\dot{X}^B, \dot{X}^S)$ of isotone mapping $\Phi$ on lattice $L$ determines a fully-trail-stable outcome $A^X$. Moreover, each stable outcome $A$ corresponds to at least one fixed point $(\dot{X}^B, \dot{X}^S)$ of $\Phi$. From Theorem 1.2.1, it follows that fixed points of $\Phi$ form a lattice, hence there is a $\sqsubseteq$-minimal fixed point $(\dot{Y}^B, \dot{Y}^S)$ and a $\sqsubseteq$-maximal one $(\dot{Z}^B, \dot{Z}^S)$. We show that fully-trail-stable outcome $A^Y$ is seller-optimal and $A^Z$ is buyer optimal. So assume that $A = A^X$ is a fully-trail-stable outcome. As $(\dot{Y}^B, \dot{Y}^S) \sqsubseteq (\dot{X}^B, \dot{X}^S) \sqsubseteq (\dot{Z}^B, \dot{Z}^S)$, we have $\dot{Y}^B \subseteq \dot{X}^B \subseteq \dot{Z}^B$ and $\dot{Y}^S \supseteq \dot{X}^S \supseteq \dot{Z}^S$. Lemma 3.7.2 implies that $C^f(A_f \cup A^Y_f) = A^Y_f$ and $C^f(A_f \cup A^Z_f) = A_f$ for any terminal seller $f$ and $C^g(A_g \cup A^Y_g) = A_g$ and $C^g(A_g \cup A^Z_g) = A^Z_g$ for any terminal buyer $g$. So, by definition $A$ is seller-superior to $A^Y$ and $A^Z$ is seller-superior to $A$. □

*Proof of Lemma 3.8.2.* In the proof of Theorem 3.5.1 we have seen that $A$ is fully trail-stable if and only if there is canonical pair $(\dot{X}^B, \dot{X}^S)$ of such that $(\dot{X}^B, \dot{X}^S)$ is a fixed point of isotone mapping $\Phi$ and $A = \dot{X}^B \cap \dot{X}^S$. Moreover, if the choice functions are LAD/LAS, fixed points of $\Phi$ form a sublattice $L$ of $(2^X \times 2^X, \widetilde{\cup}, \widetilde{\cap})$. From Lemma 3.8.8, the projection of the above lattice to the terminals, $L_\mathcal{T}$ is also a lattice under $\sqsubseteq$ and from Lemma 3.7.2 this partial order coincides with $\preceq^S$. Therefore, the stable outcomes form a lattice under terminal-superiority. □

**Conjecture 3.8.9.** *Let $F$ be the set of agents and $X$ be the set of contracts in a contract network. Preferences of $F$ are fully substitutable, IRC and LAD/LAS. For every trail-stable outcome $A$ there exist a fully trail-stable outcome $A'$ such that $A_t = A'_t$ for all the terminal agents. (That is, $A_{\mathcal{T}} = A'_{\mathcal{T}}$.)*

Since we know form Lemma 3.6.1 that every fully trail-stable outcome is also trail-stable, this Conjecture states that the terminal-fully-trail-stable and the terminal-trail-stable outcomes are the same. The corollary of this Conjecture and Theorem 3.8.8 is the following:

**Conjecture 3.8.10.** *In any contract network $X$ if choice functions of $F$ satisfy full substitutability and* only *LAD/LAS then the terminal-trail-stable contract sets form a lattice under terminal superiority.*

**Example 3.8.11.** *If the choice functions of the agents are not LAD/LAS, we can show an example where trail-stable and fully trail stable solutions differ on the terminal agents. The network $(F, X)$ consists of firms $F = \{s,\ z,\ v,\ w\}$ and there are four possible contracts: $a = sv,\ d = zv,\ b = vw,\ c = wv$ (So the contracts $b$ and $c$ involve the same firms, but with opposite roles.) Firm $s$ and $z$ are terminal sellers. The preferences of each agent are as follows:*
$\prec_s$: $\{a\} \succ_s \emptyset$ *In other words, $a$ is rational.*
$\prec_w$: $\{bc\} \succ_w \emptyset$
$\prec_v$: $\{b, c\} \succ_v \{a, d, b\} \succ_v \{a, b\} \succ_v \{c\} \succ_v \{d\} \succ_v \emptyset$
$\prec_z$: $\{d\} \succ_z \emptyset$ *The preference order of agent $v$ is fully substitutable and IRC, however it is not separable and do not satisfy the Laws of Aggregate Demand and Supply. Preferences of other agents satisfy everything. We can show that $\{b, c\}$ is both trail-stable and fully trail-stable. $\{d\}$ is trail-stable but not fully trail-stable, because $\{a, b, c\}$ is a locally blocking trail.*

*Therefore for the terminal seller $z$, the trail-stable and fully trail stable solutions are different.*

## 3.9 Rural Hospitals and Market Rearrangements

The lattice structure of fully-trail stable allows us to straightforwardly extend two well-known properties of stable outcomes that have been known in two-sided matching markets and acyclic contract networks. One such property is the classic "rural hospitals theorem", which shows that in every stable allocation of a two-sided many-to-one doctor-hospital matching market, the same number of doctors are matched to every hospital [43]. In buyer-seller networks, we can instead consider the difference between the number of upstream and downstream contracts that firms sign [31]. The following proposition gives the most general rural hospital theorem result.

**Proposition 3.9.1.** *Suppose that in a contract network $X$ choice functions of $F$ satisfy full substitutability, IRC and LAD/LAS. Then, for each firm, the difference between the number of upstream contracts and the number of downstream contracts is invariant across fully trail-stable allocations.*

Its proof, which we omit, follows the proof of Theorem 8 in [31] word-for-word, only replacing "stable" with "fully trail-stable".

The lattice structure of fully trail-stable outcomes also gives a (somewhat weak) mechanism design result. A mechanism $\mathcal{M}$ is a mapping from a profile of agents' choice functions, $\mathbf{C}^F = (C^f)_{f \in F}$, to the set of outcomes. A mechanism is group strategy-proof for a group of agents if they cannot jointly manipulate their choice functions and obtain an outcome that is better for all of them.

A mechanism is *group strategy-proof* for $G \subseteq F$ if for any $\bar{G} \subseteq G$, there does not exist a choice function profile $\bar{\mathbf{C}}^{\bar{G}}$ such that for outcomes $\bar{A} = \mathcal{M}(\bar{\mathbf{C}}^{\bar{G}}, \mathbf{C}^{F \setminus \bar{G}})$ and $A = \mathcal{M}(\mathbf{C}^F)$ we have that $C^f(\bar{A} \cup A) = \bar{A}$ for every $f \in \bar{G}$.

Like Hatfield and Kominers [31], we are only going to consider group strategy-proofness for terminal agents. We generalize their Theorem 10 with the following result.

**Proposition 3.9.2.** *Suppose that in a contract network $X$ choice functions of $F$ satisfy full substitutability, IRC, LAD/LAS, and, additionally, all terminal buyers (terminal sellers) demand at most one contract, then any mechanism that selects the buyer-optimal (seller-optimal) fully trail-stable allocation is group strategy-proof for all terminal buyers.*

As is well known, the assumptions that underpin Proposition 3.9.2 – unit demands and extreme one-sidedness – cannot be substantially relaxed.[15]

The second set of properties of fully trail-stable outcomes concerns the effect of entry and exit of new firms in the trading network. This type of comparative static analysis is well-studied in two-sided matching markets [25],[13],[28]. More recently, [38] and [30] extended these results the case of supply chains.

First, let us consider what happens when a terminal seller is added to the market. More formally, let $F' = F \cup f'$ and let $A'_{max}$ and $A'_{min}$ be the buyer-optimal and the seller-optimal fully trail-stable outcomes in $F'$ respectively.

**Proposition 3.9.3.** *Suppose that in a contract network $X$, a new terminal seller $f'$ comes, choice functions of $F \cup f'$ satisfy full substitutability and IRC, then every terminal seller $f \neq f'$ is at least as well off in $A_{max}$ as in $A'_{max}$ and at least as well off in $A_{min}$ as in $A'_{min}$, and each terminal buyer $f$ is at most as well off in $A_{max}$ as in $A'_{max}$ and at most as well off in $A_{min}$ as in $A'_{min}$.*
*The opposite holds when $f'$ is terminal buyer.*

[15] Its proof, which we omit, just follows the proof of Theorem 1 in [29] (which was pointed out by [31]).

The proposition says that with a new seller, the seller-optimal outcome $A_{min}$ and the buyer-optimal outcome $A_{max}$ move in the direction favorable to terminal buyers and unfavorable to terminal sellers. Symmetrically, when a terminal buyer is added, $A_{min}$ and $A_{max}$ move in the opposite direction. In other words, more competition on one end of an industry is bad for the agents on that end and good for the agents on the other end. This proposition generalizes Theorem 3 in [38].

Now consider the following *market readjustment process*: When the new terminal seller $f'$ enters, and we already have a fully trail-stable outcome $A$ with corresponding fixed point $(\dot{X}^B, \dot{X}^S)$ then let $X$ be the set of all contracts in the new network, and let us define $(\dot{X}'^B, \dot{X}'^S) = (\dot{X}^B, \dot{X}^S \cup X_{f'})$. Operator $\Phi'$ acts on $(X'^B, X'^S)$ using choice functions of $F'$. Let $(\hat{X}^B, \hat{X}^S)$ be the fixed point of the iteration of fuction $\Phi$, with associated outcome $\hat{A} = \hat{X}^B \cap \hat{X}^S$. This $\hat{A}$ be the result of the market readjustment process.

**Proposition 3.9.4.** *Suppose that all firms' choice functions are fully substitutable and that $A$ is a fully trail-stable outcome with associated buyer and seller offer sets $X^B$ and $X^S$. Suppose that an terminal seller $f'$ enters the market, and let $\hat{A}$ be the result of the market readjustment process. Then, all terminal sellers weakly prefer $A$ to $\hat{A}$ and all terminal buyers (other than $f'$) weakly prefer $\hat{A}$ to $A$.*
*The opposite holds when $f'$ is terminal buyer.*

An analogous result can be obtained when terminal buyers and terminal sellers exit the market so this proposition generalizes the Theorem in [30].

Our proof is similar to Ostrovsky's proof. First we investigate the restabilized outcome from $A$, which we play part in the proofs of both Propositions 3.9.3 and 3.9.4. Let $A$ be an arbitrary fully trail-stable outcome in the original network, with a corresponding canonical pair $(\dot{X}^B, \dot{X}^S)$. After the new terminal seller $f'$ arrives, let $X$ be the set of all contracts in the new network, and let us define $(X^{*B}, X^{*S}) = (\dot{X}^B, \dot{X}^S \cup X_{f'})$. In the following, we will use $\Phi$ according to the choice fuctions on the new network, so $(\dot{X}^B, \dot{X}^S)$ does not need to be a fixed point of $\Phi$ anymore.

Since $X_{f'} \cap X^{*B} = \emptyset$, for every firm $f \neq f'$, $R^f_S(X^{*S}|X^{*B}) = R^f_S(\dot{X}^S|\dot{X}^B)$ and $R^f_B(X^{*B}|X^{*S}) = R^f_B(\dot{X}^B|\dot{X}^S)$. For example, if $f$ has a conctracts with $f'$, contract $x = f'f$ was not offered for firm $f$ in $X^{*B}$ so it does not get rejected.

For firm $f'$, $R^{f'}_S(X^{*S}|X^{*B}) = X_{f'} \setminus C^{f'}_S(X_{f'})$ and $R^{f'}_B(X^{*B}|X^{*S}) = \emptyset$.

Therefore $\Phi(X^{*B}, X^{*S}) = (\dot{X}^B \cup C^{f'}(X_{f'}), \dot{X}^S \cup X_{f'})$.

So $(X^{*B}, X^{*S}) \sqsubseteq \Phi(X^{*B}, X^{*S})$, and $\Phi$ is $\sqsubseteq$-isotone, so $\Phi(X^{*B}, X^{*S}) \sqsubseteq \Phi(\Phi(X^{*B}, X^{*S}))$ and so on. The lattice of all possible subset-pairs is finite, so there is a $k$ such that $\Phi^k(X^{*B}, X^{*S}) = (\hat{X}^B, \hat{X}^S)$ is a fixed point. So $(X^{*B}, X^{*S}) \sqsubseteq \Phi(X^{*B}, X^{*S}) \sqsubseteq \Phi^k(X^{*B}, X^{*S}) = (\hat{X}^B, \hat{X}^S)$. Outcome $\hat{A} = \hat{X}^B \cap \hat{X}^S$ is fully trail-stable, and this is what we call the *restabilized outcome* from $A$.

*Proof of Proposition 3.9.3.* If $f'$ is a terminal seller, and we start from outcome $A_{max}$

and the $\sqsubseteq$-maximal pair $(\dot{Z}^B, \dot{Z}^S)$. Using the previous method, outcome $\hat{A} = \hat{Z}^B \cap \hat{Z}^S$ is the restabilized outcome from $A$. In the new network there exists a $\sqsubseteq$-maximal fixed point of $\Phi$, namely $(Z'^B, Z'^S)$, therefore $(\dot{Z}^B, \dot{Z}^S \cup X_{f'}) = (Z^{*B}, Z^{*S}) \sqsubseteq (\hat{Z}^B, \hat{Z}^S) \sqsubseteq (Z'^B, Z'^S)$. The fully trail-stable outcome corresponding to the maximal fixed point is $A'_{max} = Z'^B \cap Z'^S$. We have to show that $A'_{max}$ is better for terminal buyers and worse for terminal sellers than the original $A_{max}$. If $f$ is a terminal buyer, since $(Z'^B, Z'^S)$ is fixed point of $\Phi$ and $(\dot{Z}^B, \dot{Z}^S)$ was fixed before the new agent arrived, $C^f(Z'^B) = A'_{f,max}$ and $C^f(Z^{*B}) = A_{f,max}$ and $Z^{*B} \subseteq Z'^B$ so from $C^f(Z'^B) \subseteq (A_{f,max} \cup A'_{f,max}) \subseteq Z'^B$ by IRC we obtain $C^f(A_{f,max} \cup A'_{f,max}) = A'_{f,max}$ so $A'_{f,max}$ is better for terminal buyers.

Similarly, if $f$ is a terminal seller outside $f'$, $C^f(Z'^S) = A'_{f,max}$ and $C^f(Z^{*S}) = A_{f,max}$ and $Z'^S \subseteq Z^{*S}$ so from $C^f(Z^{*S}) \subseteq (A_{f,max} \cup A'_{f,max}) \subseteq Z^{*S}$ by IRC we obtain $C^f(A_{f,max} \cup A'_{f,max}) = A_{f,max}$ so $A_{f,max}$ is better for terminal buyers.

If $f'$ is a terminal buyer then we can use the same proof with reversing the roles of buyers and sellers.

$\square$

*Proof of Proposition 3.9.4.* If $f'$ is a terminal seller, and $A$ is any fully trail-stable outcome in the original network, with canonical pair $(\dot{X}^B, \dot{X}^S)$, then $(X^{*B}, X^{*S}) = (\dot{X}^B, \dot{X}^S \cup X_{f'}) \sqsubseteq (\hat{X}^B, \hat{X}^S)$. The restabilized outcome is $\hat{A} = \hat{X}^B \cap \hat{X}^S$, and similarly to the proof of Proposition 3.9.3 one can show that initial producers weakly prefer $A$ to $\hat{A}$ and all end consumers (other than $f'$) weakly prefer $\hat{A}$ to $A$.
If $f'$ is a terminal buyer, preferences are the the opposite.

$\square$

## 3.10 Trading Networks on Lattices

Consider a new model based on a directed graph $D = (F, X)$, just like the one in Section 3.1. Vertices of $F$ represent the firms (selfish agents) and each arc $x = uv$ of $X$ indicates a contract between agents $u$ and $v$.

Now, differently from the previous sections, each contract is specified with a certain "intensity". Intensity can represent the amount of the good traded. (If we consider a market with workers and firms, the intensity can show the number of hours worked.) We assume that for each contract $x$, the set of possible intensities form a distributive complete lattice $\mathcal{L}_x$. The simplest (and most typical) example is $\mathcal{L}_x = (\{0,1\}, \leq)$ where a 0-contract means no trade, and a 1-contract is just a single contract between the agents. This is what we covered in the previous sections. A more complex lattice is for example $\mathcal{L}_x = (\{0,2,3,7\}, \leq)$, which means that the only realizable possibilities along arc $x = uv$ are that $u$ sells $0, 2, 3$ or $7$ units of the particular good to $v$. Note that intensity-lattice $\mathcal{L}_x$ might even be infinite: $\mathcal{L}_x = (\{\sqrt{2}, 4\} \cup [5, 9], \leq)$ describes a situation where $u$ can sell $\sqrt{2}$, 4 or any (possibly non integral) amount between 5 and

9 from the particular good to $v$, and no other option (like abandoning the contract and selling nothing) is allowed.

Each firm $f \in F$ has certain preferences over the set $X_f$ of contracts that involve $f$, that is, over the arcs entering or leaving $f$. Well, not exactly. Indeed, this is the situation if $\mathcal{L}_x = (\{0,1\}, \leq)$ holds for each contract $x$. However, in the general case, these preferences are not over the set of contracts but over the intensity-vectors. For this reason, we define lattice $\mathcal{L}_f := \bigoplus_{x \in X_f} \mathcal{L}_x$ as the direct sum of contract-describing intensity-lattices for agent $f$. Now elements of $\mathcal{L}_f$ that we shall call *intensity-vectors* are the exact descriptions of all the possible situations that firm $f$ can experience. If there are $k$ contracts that involve agent $f$ then $(\ell_1, \ell_2, \ldots, \ell_k)$ describes the situation where the first contract is available for $f$ up to intensity $\ell_1$, the second one up to intensity $\ell_2$, and so on. The preference of $f$ is represented by a choice function $C^f : \mathcal{L}_f \to \mathcal{L}_f$. The intended meaning of $C^f(\ell_1, \ell_2, \ldots, \ell_k) = (\ell'_1, \ell'_2, \ldots, \ell'_k)$ is that if firm $f$ picks its favorite contract-intensities in a selfish manner such that $\ell_i$ is an upper bound for the intensity of the $i^{th}$ contract then it sets the intensity of contract $i$ to $\ell'_i$ for each $1 \leq i \leq k$. (In the 0/1 case, $\mathcal{L}_f$ is isomorphic to $(2^{X_f}, \subseteq)$, and a choice function on this lattice is basically a "traditional" choice function, that is, a subset mapping.) An intensity-vector $l \in \mathcal{L}_f$ is *rational for* $f$ if it is rational for $C^f$, that is, if $C^f(l) = l$, or in other words, if $f$ is happy to accept the intensities described by $l$. We assume that choice function $C^f$ is IRC for each firm $f$.

Define $\mathcal{L}_D := \bigoplus_{x \in X} \mathcal{L}_x$. Essentially, an element $(\ell_1, \ell_2, \ldots, \ell_m)$ of $\mathcal{L}_D$ assigns an intensity to each of the $m$ contracts, where $m$ is the size of contract set $X$. We call elements of $\mathcal{L}_D$ *contract schemes*. (In the 0/1 case, a contract scheme is essentially a subset of the contracts.) For a contract scheme $l$ of $\mathcal{L}_D$ and contract $x$ of $X$, $l_x$ denotes the corresponding coordinate of $l$, that is the intensity of $x$ in $l$. For an agent $f$, $l_f$ denotes the projection of $l$ to $X_f$, that is $l_f$ contains only those intensities in $l$ that correspond to a contract involving $f$. From now on, we shall cruelly abuse notation according to the following convention. We compare intensities, intensity-vectors and contract schemes and interpret "mixed" operations on them. For example for intensity-vector $l$ of $\mathcal{L}_f$ and intensity $\ell$ of contract $x$ involving $f$ we say that $l \leq \ell$ if $l_x \leq \ell$ holds and $l \vee \ell$ means the intensity-vector of $\mathcal{L}_f$ that we get from $l$ by replacing its $x$-coordinate $l_x$ by $l_x \vee \ell$, whereas for contract scheme $L$, contract scheme $L \wedge \ell$ is the contract scheme that we get from $L$ by replacing $L_x$ by $L_x \wedge \ell$ .

We say that a contract scheme $l \in \mathcal{L}_D$ is *acceptable* if for each firm $f$, the $f$-projection $l_f$ of $l$ (which is an intensity-vector itself) is rational for $C^f$, that is, if $C^f(l_f) = l_f$. This means that no agent is interested in decreasing any of the intensities on her contracts.

**Example 3.10.1.** *Assume that $D = (E \cup W, X)$ is a directed graph such that $E$ and $W$ are disjoint sets of employers and workers, respectively, and each arc $ew \in X$ is oriented from an employer $e$ to a worker $w$, moreover each vertex $f$ of $D$ has a linear* *order $\leq_f$ on the set $X_f$ of arcs incident to $f$ and lastly, each employer $e$ has a quota $q(e)$ on the workers that it can employ. Let $\mathcal{L}_x := (\{0,1\}, \leq)$ for each contract $X$. Then elements of $\mathcal{L}_f$ bijectively correspond to subsets of $X_f$. Define choice function $C^f : \mathcal{L}_f \to \mathcal{L}_f$ such that $C_w(Y)$ is the $\leq_w$-best arc of $Y$ for any worker $w$ of $W$ and $C_e(Y)$ is the $\leq_e$-best $q(e)$ arcs of $Y$ for any employer $e$.*

*In this situation, a rational contract scheme corresponds to a set of arcs of $D$ such that each worker has at most one contract and each employer $e$ is involved in at most $q(e)$ contracts.*

In what follows, we generalize the so called "same side substitutable" and "cross side complementary" properties of traditional choice functions defined by Ostrovsky in [38]. Observe that these two properties can be stated in a more compact way as follows. Choice function $C^f$ is same side substitutable and cross side complementary if the set of ignored incoming arcs is monotone in the set of offered incoming arcs and antitone in the set of offered outgoing arcs and the set of ignored outgoing arcs is monotone in the set of offered outgoing arcs and antitone in the set of offered incoming arcs. This latter approach allows us to generalize these notions.

Recall that $X_f^B$ and $X_f^S$ denote the set of incoming and outgoing arcs of $f$, respectively, that is, $X_f = X_f^B \cup X_f^S$. We say that mapping $D^f : \mathcal{L}_f \to \mathcal{L}_f$ is *same side antitone* if the following properties hold. If elements $l = (\ell_1, \ell_2, \ldots, \ell_k)$ and $l' = (\ell'_1, \ell'_2, \ldots, \ell'_k)$ of $\mathcal{L}_f$ are such that $\ell_i = \ell'_i$ for any $i \in X_f^S$ and $\ell_j \leq_j \ell'_j$ for any $j \in X_f^B$ then $(D^f(l))_j \geq_j (D^f(l'))_j$ holds for any $j \in X_f^B$. Moreover, if elements $l = (\ell_1, \ell_2, \ldots, \ell_k)$ and $l' = (\ell'_1, \ell'_2, \ldots, \ell'_k)$ of $\mathcal{L}_f$ are such that $\ell_i \leq_i \ell'_i$ for any $i \in X_f^S$ and $\ell_j = \ell'_j$ for any $j \in X_f^B$ then $(D^f(l))_i \geq_i (D^f(l'))_i$ holds for any $i \in X_f^S$. (Here the $i$ subscript denotes the $i$th coordinate of the corresponding intensity-vector.)

The above mapping $D^f : \mathcal{L}_f \to \mathcal{L}_f$ is called *cross side monotone* if the following properties hold. If elements $l = (\ell_1, \ell_2, \ldots, \ell_k)$ and $l' = (\ell'_1, \ell'_2, \ldots, \ell'_k)$ of $\mathcal{L}_f$ are such that $\ell_i = \ell'_i$ for any $i \in X_f^S$ and $\ell_j \leq_j \ell'_j$ for any $j \in X_f^B$ then $(D^f(l))_i \leq_i (D^f(l'))_i$ holds for any $i \in X_f^S$. Moreover, if elements $l = (\ell_1, \ell_2, \ldots, \ell_k)$ and $l' = (\ell'_1, \ell'_2, \ldots, \ell'_k)$ of $\mathcal{L}_f$ are such that $\ell_i \leq_i \ell'_i$ for any $i \in X_f^S$ and $\ell_j = \ell'_j$ for any $j \in X_f^B$ then $(D^f(l))_j \leq_j (D^f(l'))_j$ holds for any $j \in X_f^B$. Clearly, in the 0/1 case, the above properties require a certain monotonicity and antitonicity property of the corresponding set-function.

We say that choice function $C^f : \mathcal{L}_f \to \mathcal{L}_f$ has the SSS and CSC properties (or in other words, fully substitutable) if there exists a same side antitone and cross side monotone determinant $\mathcal{D}^f : \mathcal{L}_f \to \mathcal{L}_f$ of $C$. Observe that in the 0/1 case, an SSS and CSC choice function is exactly a "traditional" choice function that has the same side substitutability and cross side complementarity properties.

**Example 3.10.2.** *Ostrovsky's model in [38] consists of a finite number of agents in a supply chain and certain possible contracts between pairs of these agents. There is also given a 2-partition of the contracts of each agent into incoming and outgoing contracts,* *and these partitions are given in an acyclic manner. Furthermore, each agent $f$ has a choice function $C^f$ on its contracts that has the same side substitutability and cross side complementarity properties. A set $Y$ of contracts is individually rational in Ostrovsky's model if $C^f(Y_f) = Y_f$ holds for each agent $f$ where $Y_f$ denotes the set of contracts of $Y$ that involve agent $f$.*

*To see that Ostrovsky's above model fits in ours, define digraph $D$ with vertices corresponding Ostrovsky's agents and arcs of $D$ are along the contracts and the orientation is from the outgoing to the incoming. For each $x$ contract, $\mathcal{L}_x := (\{0,1\}, \leq)$. Clearly, the above choice functions in Ostrovsky's model define choice functions on $\mathcal{L}_f$ for each agent $f$ and these choice functions have the SSS and CSC properties. Furthermore, elements of $\mathcal{L}_D$ correspond to subsets of contracts, and individual rationality of an element of $\mathcal{L}_D$ means that the corresponding subset of contracts is individually rational in Ostrovsky's model.*

**Example 3.10.3.** *In Fleiner's model in [18], agents are the vertices of a directed graph $D$, and an arc $uv$ represents the contract that $u$ sells some universal good to $f$. Each contract $x$ has a capacity $c(x)$ determining the maximum amount that can be sold along contract $x$. To represent this in our model, $\mathcal{L}_x = ([0, c(x)], \leq)$ or $\mathcal{L}_x = (\{0, 1, 2, \ldots, c(x)\} \leq)$, depending whether the traded good is divisible or not. Individual rationality of a trading scheme in this model is nothing but requiring that the trading scheme is a network flow, that is, no more good is traded along an arc than its capacity and Kirchhoff's conservation law holds for all agents except for $s$ and $t$: they have to buy the same amount as the amount they sell. Moreover, each agent $f$ has a linear preference order on the set $X_f$ of arcs. These linear preferences determine a choice function for $f$ on $\mathcal{L}_f$: agent $f$ maximizes the amount of goods she trades and this throughput is achieved along the most preferred arcs. (Special agents $s$ and $t$ have the identity choice function: $C_s(l) = l$ for each $l \in \mathcal{L}_s$, and similar holds for $t$.) It is easy to check that individually rational contract schemes correspond to network flows of the underlying network.*

The key to our result is understanding the relation between the SSS and CSC properties of a choice function and comonotonicity. To do so, we look through certain glasses that we construct from an unusual lattice, and all of a sudden the picture becomes clear. So let us start with the lattice first. For each agent $f$ of our model define lattice

$$\widetilde{\mathcal{L}}_f := \bigoplus_{x \in X_f^B} \mathcal{L}_x \oplus \bigoplus_{x \in X_f^B} \mathcal{L}_x^{-1} .$$

Observe that $\widetilde{\mathcal{L}}_f$ and $\mathcal{L}_f$ have the same ground set (so elements of $\widetilde{\mathcal{L}}_f$ are intensity-vectors), the difference is that the partial order on the outgoing arcs are opposite. E.g., in the 0/1 case a set $Z \sqsupseteq Y$ of such arcs if $Z_f^B \supseteq Y_f^B$ and $Z_f^S \subseteq Y_f^S$ To distinguish the lattice order and operations of lattices $\widetilde{\mathcal{L}}_f$ and $\mathcal{L}_f$, $\sqsubseteq_f$, $\widetilde{\wedge}$ and $\widetilde{\vee}$ denote it for the former and $\leq_f$, $\wedge$ and $\vee$ for the latter one (just as we did so far). We can now formulate the promised key observation.

**Observation 3.10.4.** *Mapping $\mathcal{D}_f$ is same side antitone and cross side monotone on $\mathcal{L}_f$ if and only if $\mathcal{D}_f$ is antitone on $\widetilde{\mathcal{L}}_f$.*

*Proof.* Straightforward from the definition. □

Recall that by definition, if choice function $C^f : \mathcal{L}_f \to \mathcal{L}_f$ has the SSS and CSC properties then there is a same side antitone and cross side monotone determinant $\mathcal{D}_f$ of $C^f$. By Observation 3.10.4, $\mathcal{D}_f$ is antitone in $\widetilde{\mathcal{L}}_f$.

Note that an SSS CSC, IRC choice function $C$ might have several antitone determinants $\mathcal{D}$. Recall that we can find a canonical determinant:

**Lemma 3.10.5.** *Assume that $C^f : \mathcal{L}_f \to \mathcal{L}_f$ is an SSS CSC choice function on complete and infinitely distributive lattice $L_f$ and $l \in \mathcal{L}_f$ is an intensity vector, then define*

$$\mathcal{D}_{C^f}(l) := \bigvee\{\ell \vee 0_f : x \in X_f, \ell \in \mathcal{L}_x, \ell \leq C^f(l \vee \ell)(x)\} \tag{3.5}$$

*as the join of all $\ell$-rational intensities. Then $\mathcal{D}_{C^f}$ is an $\widetilde{\mathcal{L}}_f$-antitone determinant of $C^f$. Moreover, this is the smallest antitone determinant of $C^f$.*

The proof of this Lemma is the same as the proof of Lemma 1.3.7, the only difference is that we include $0_f$, the minimal element of lattice $\mathcal{L}_f$ in the definition, while in Lemma 1.3.7 it was automatically 0.

Antitone determinant $D_{C^f}$ in Lemma 3.10.5 is called the canonical determinant of $C^f$. The definition in case of 0/1 lattices states that $D_{C^f}(Y)$ is the set of all contracts a that are selected if $f$ can choose only from $Y \cup a$. Sometimes these contracts are referred as the ones that are undominated by contract set $Y$. In the more general case of lattices we can also say that $D_{C^f}$ is the join of all contract intensities that are undominated by the input of the determinant. A consequence of Lemma 3.10.5 that is that if $\mathcal{D}$ is some $\widetilde{\mathcal{L}}_f$-antitone determinant of $C^f$ then $\mathcal{D}_{C^f} \leq \mathcal{D}$ holds, that is, $\mathcal{D}_{C^f}$ is the smallest determinant of $C^f$. We remark moreover, that $\widetilde{\mathcal{L}}_f$-antitone determinants of SSS CSC, IRC choice function $C$ form a complete lattice with lattice operations $\wedge$ and $\vee$. This follows that $\mathcal{D}_{C^f}$ is the meet of all determinants of $C^f$.

**Example 3.10.6.** *Lemma 3.10.5 shows that for choice functions in the flow model seen in Example 3.10.3 we can calculate the canonical determinant the following way. Suppose intensity vector $l$ is offered for agent $f$. The agent calculates the total amount of incoming and outgoing offers, say $w_{in} = \sum_{x \in X_f^B} l_x$ and $w_{out} \sum_{x \in X_f^S} l_x$. The value of the determinant at an incoming arc $x$ will be the minimum of the capacity of $x$ and the positive part of the difference of $w_{out}$ and the total offer in the incoming arcs that are preferred to $x$ by $f$. Similarly, the determinant on an outgoing arc $y$ will be the minimum of the capacity of $y$ and the positive part of the difference of*

*$w_{in}$ and the total offer on outgoing arcs better than $y$. In particular, for the concrete case of $\ell = (0,3,0,2;0,0,1,0,3,0)$ , if the capacities are $(2,4,2,2;1,7,2,2,5,3)$ then we have $\mathcal{D}_{C^f}(\ell) = (2,4,1,1;1,5,2,2,4,1)$. Clearly, $C^f(0,3,0,2;0,0,1,0,3,0) = (0,3,0,2;0,0,1,0,3,0) \wedge (2,4,1,1;1,5,2,2,4,1) = (0,3,0,1;0,0,1,0,3,0)$, just as we have seen before.*

Recall that contract scheme $l \in \mathcal{L}_D$ is acceptable if for each agent $f$, the $f$-projection $l_f$ of $l$ is rational for $C^f$, that is, if $C^f(l_f) = l_f$.

For an intensity $\ell \in \mathcal{L}_x$ of contract $x$ involving $f$, we say that $\ell$ is $(0,f)$-*rational* if $f$ is happy to accept intensity $\ell$ for contract $x$ when for all other contracts the minimum intensity is offered. Or, formally, if $\ell = C^f(0_f \vee \ell)_x$ holds, where $0_f$ denotes the minimum element of lattice $\mathcal{L}_f$ and by abusing notation, we denote by $0_f \vee \ell$ that intensity-vector of $\mathcal{L}_f$ that has the corresponding lattice-minimum on each coordinate except for $x$ where it has intensity $\ell$.

If $\ell$ is an intensity of a contract $x$ of firm $f$, and we have a contract scheme $l$, then we say $\ell$ is $(l,f)$-*rational* if $\ell \geq C^f(l \vee \ell)_x$.

Now assume that $\ell_1 \in \mathcal{L}_{x_1}$ and $\ell_2 \in \mathcal{L}_{x_2}$ are intensities of contracts $x_1 = uv$ and $x_2 = vw$. We say that pair $(\ell_1, \ell_2)$ is an $(l,f)$-*rational pair* if $f$ does accept neither $\ell_1$ nor $\ell_2$ if they are offered individually along with $l$ (and for all other contracts the minimum intensity is offered), but $f$ is happy to accept both intensities if for all other contracts the minimum is offered. The formal definition is that neither $\ell_1$ nor $\ell_2$ is $(l,v)$-rational for $f$ but $\ell_1 = C^f(0_f \vee \ell_1 \vee \ell_2)_{x_1}$ and $\ell_2 = C^f(0_f \vee \ell_1 \vee \ell_2)_{x_2}$ holds.

Contract schemes play the role what outcomes played in the 0/1 model. In the 0/1 model, outcome $A$ was a subset of the contracts, $A \subseteq X$. Its correspondings contract scheme $l$ is defined as $l(x) = 1$ for $x \in A$, and $l(x) = 0$ for $x \notin A$.

Our next goal is to define stability of a rational contract scheme $l$ in our model. We can generalize the concept of full trail-stability here. Namely, a contract scheme $l \in \mathcal{L}_D$ is *fully trail-stable* if

1. $l$ is acceptable i.e., $C^f(l_f) = l_f$ for every firm $f$.

2. There is no trail $T = \{x_1, x_2, \ldots, x_M\}$ with intensity $\ell_i$ on contract $x_i = f_i f_{i+1}$ such that

   (a) $\ell_i \nleq l(x_i)$ for every $1 \leq i \leq M$.

   (b) $x_1$ with intensity $\ell_1$ is $(l, f_1)$-rational for $f_1 = s(x_1)$ i.e., $(C^{f_1}(l \vee \ell_1))_{x_1} \geq \ell_1$

   (c) $(C^{f_i}(l \vee \ell_i \vee \ell_{i+1}))_{x_i} \geq \ell_i$ and $(C^{f_i}(l \vee \ell_i \vee \ell_{i+1}))_{x_{i+1}} \geq \ell_{i+1}$ for every $1 \leq i \leq M-1$.

   (d) $x_M$ with intensity $\ell_M$ is $(l, f_{M+1})$-rational for $f_{M+1} = b(x_M)$ i.e. $(C^{f_{M+1}}(l \vee \ell_M))_{x_M} \geq \ell_M$

The above trail $T$ is called a *locally blocking trail to $l$*.

So if a contract scheme $l$ is not fully trail-stable then there is some agent $f_1$ that would like to increase the intensity of her selling to agent $f_2$. If agent $f_2$ accepts this increase then we already have the blocking walk. Otherwise, $f_2$ asks around whether some other agent $f_3$ would be happy if the intensity of a $f_2f_3$ contract would be increased to an amount that follows from the increased volume of the $f_1f_2$ contract. There might be an agent $f_3$ who accepts it right away or on the condition that she can increase the intensity of a certain $f_3f_4$ selling contract. As contract scheme $l$ is unstable, this sequence must end such that some $f_M$ accepts the intensity increase without looking for some further agent to whom she can sell.

| **Notation** | **defined over** | **name** |
|---|---|---|
| $\ell \in \mathcal{L}_x$ | a contact | intensity |
| $l \in \mathcal{L}_f$ | contacts involving $f$ | intensity vector |
| $l \in \mathcal{L}_D$ | all contacts | contract scheme |
| $l \in \widetilde{\mathcal{L}}_f$ | contacts involving $f$ | intensity vector [16] |
| $\widetilde{l} \in \widetilde{\mathcal{L}}_D$ | all contacts, twice | intensity scheme |

Table 3.1: Names

### 3.10.1 Existence of a Stable Contract Scheme

Our main result below states that there always exists a fully trail-stable contract scheme in our model.

**Theorem 3.10.7.** *[Fleiner, Jankó, Tamura For any digraph $D = (F, X)$, for any complete lattices $\mathcal{L}_x$ for each contract $x \in X$ and for any IRC, SSS and CSC choice functions $C^f : \mathcal{L}_f \to \mathcal{L}_f$ for each $f \in F$ there exists a fully trail-stable contract scheme $l \in \mathcal{L}_D$.*

To prove Theorem 3.10.7, we construct a tricky lattice and a monotone mapping on it. Define lattice $\widetilde{\mathcal{L}}_D := \bigoplus_{f\in F} \widetilde{\mathcal{L}}_f$ and call the elements of $\widetilde{\mathcal{L}}_D$ *intensity schemes.* Observe that each intensity scheme $l$ is a vector with coordinates $l_f$ indexed by agents $f$. Each coordinate $l_f \in \mathcal{L}_f$ is an intensity-vector itself, coordinate $(l_f)_a$ corresponding to contract $a$ is a single intensity of $\mathcal{L}_a$. This means that each intensity scheme contains exactly two (unrelated) intensities for each contract, one for the agent that the corresponding arc leaves and another one where this arc enters. Observe that $\widetilde{\mathcal{L}}_D$ is a lattice, but the lattice order is unusual as it comes from the lattice orders of $\widetilde{\mathcal{L}}_f$ that are opposite on the outgoing arcs. For any contract scheme $l$, we can define an intensity scheme $ll$ by "duplicating" the coordinates, that is, $(ll_f)_a = l_a$ holds for each agent $f$ and each contract $a$ involving $f$.

We define two operations on $\widetilde{\mathcal{L}}_D$. For an intensity scheme $l$, define intensity scheme

$$\mathcal{D}(l) := \bigoplus_{f\in F}(\mathcal{D}_f(l_f))$$

where $\mathcal{D}_f$ is the $\sqsubseteq$-antitone determinant of $C^f$. By Observation 3.10.4, $l \mapsto \mathcal{D}(l)$ is an $\sqsubseteq$-antitone mapping. It is easy to see that $\mathcal{D}$ is a determinant of choice function $C := \bigoplus_{f\in F} C^f$.

The other operation is the following. The *dual* of intensity scheme $\widetilde{l} \in \widetilde{\mathcal{L}}_D$ is the intensity scheme $\widetilde{l}^*$ of $\widetilde{\mathcal{L}}_D$ that we get by switching the two intensities of each contract. Formally, for each arc $x = uv$ we have $(\widetilde{l}^*_u)_x = (\widetilde{l}_v)_x$ and $(\widetilde{l}^*_v)_x = (\widetilde{l}_u)_x$. (According to our notation, $\widetilde{l}_v$ is the $v$-coordinate of $\widetilde{l}$ that is itself an intensity-vector, with $x$-coordinate $(\widetilde{l}_v)_x$.)

**Observation 3.10.8.** *If $\widetilde{l} \in \widetilde{\mathcal{L}}_D$ is an intensity scheme and each choice function $C^f$ is SSS and CSC then both $\widetilde{l} \mapsto \mathcal{D}(\widetilde{l})$ and $\widetilde{l} \mapsto \widetilde{l}^*$ are $\sqsubseteq$-antitone mappings.*

*Proof.* We have already seen that the first mapping is $\sqsubseteq$-antitone. To prove the same for the dual, assume that $\widetilde{l} \sqsubseteq \widetilde{l}'$ holds for intensity schemes $\widetilde{l}$ and $\widetilde{l}'$. This means that intensities are greater in $\widetilde{l}$ than in $\widetilde{l}'$ on outgoing arcs and less in $\widetilde{l}$ than in $\widetilde{l}'$ on incoming arcs. Dualization means that we switch intensities along the two vertices of each arcs, so these relations hold the opposite way for $\widetilde{l}^*$ and $(\widetilde{l}')^*$, and this is exactly we had to prove. □

The key to the proof of Theorem 3.10.7 is the following lemma.

**Lemma 3.10.9** (Fleiner,Jankó,Tamura)**.** *[21] If there exists an intensity scheme $\widetilde{l}$ such that $ll = \widetilde{l} \wedge \widetilde{l}^*$ and $\widetilde{l} = (\mathcal{D}(\widetilde{l}))^*$ then intensity scheme $ll$ is a duplication of a fully trail-stable contract scheme.*

*Proof of Lemma 3.10.9.* Suppose $\widetilde{l}$ is a fixed point. Observe that $ll = \widetilde{l} \wedge \widetilde{l}^* = \widetilde{l} \wedge (\mathcal{D}(\widetilde{l})^*)^* = \widetilde{l} \wedge \mathcal{D}(\widetilde{l}) = C(\widetilde{l})$. Using the notation $b = \mathcal{D}(\widetilde{l})$, $b^* = \widetilde{l}$, and $b \wedge b^* = \widetilde{l}^* \wedge \widetilde{l} = ll$

Assume indirectly that there is locally blocking trail for $l$, namely $T = (f_0, x_1, f_1, x_2 \ldots x_k, f_k)$ and the contracts $x_1, x_2 \ldots x_k$ have intensities $\ell_{x_1}, \ell_{x_2}, \ldots, \ell_{x_k}$ respectively.

Let us use the following names: arc $x_i = f_{i-1}f_i$ of the trail is a *starter* if $(\widetilde{l}_{f_{i-1}})_{x_i} \geq \ell_i$. Arc $x_i$ is an *ender* if $(\widetilde{l}_{f_i})_{x_i} \geq \ell_i$ and let us call $x_i$ *neutral* if $(\widetilde{l}_{f_{i-1}})_{x_i} \not\geq \ell_i$ and $(\widetilde{l}_{f_{i-1}})_{x_i} \not\geq \ell_i$. It is easy to see that every $x_i$ falls in at least one of these categories, and might be a starter and ender at the same time.

Note that $l$ is defined over all contracts, and $\widetilde{l}$ has a value for all $(\widetilde{l}_v)_a$ if $v$ is a firm and $a$ is a contract involving $v$ (i.e. $\widetilde{l}$ has two unrelated intensities for each contract). However, $\ell$ is only defined on the arcs of the trail, so every time an arc $x'$ is not on this trail, then $((\widetilde{l} \vee \ell)_v)_{x'} = (\widetilde{l}_v)_{x'}$ and $(l \vee \ell)_{x'} = l_{x'}$.

*(i) We will show that if any of the following is true, then trail $T$ with intensity $\ell$ cannot block the contract scheme.*

1. $x_1$ is a starter.
2. $x_k$ is an ender.
3. For some $1 \leq i \leq k-1$, arc $x_i$ is an ender and $x_{i+1}$ is a starter.

To check these three possibilities in one setting, let $f_i$ be the firm where the two consecutive contract involving it are a starter and ender. If $x_1$ is a starter, then $i = 0$, and if $x_k$ is an ender, then $i = k$.

In this case, $l_{x_i} \leq (l \vee \ell)_{x_i} \leq (\widetilde{l}_{f_i})_{x_i}$ (if $i \geq 1$) and $l_{x_{i+1}} \leq (l \vee \ell)_{x_{i+1}} \leq (\widetilde{l}_{f_i})_{x_{i+1}}$ (if $i < k$). Since $C(\widetilde{l}) = l$, from the IRC property we get $C(l \vee \ell_{x_i} \vee \ell_{x_{i+1}}) = l$, so this $\ell$ cannot be locally blocking.

*(ii) Also, we will show that if trail $T$ with intensity $\ell$ is a locally blocking trail, then every $x_i$ is an ender.* (i.e. $(\widetilde{l}_{f_i})_{x_i} \geq \ell_i$ for all $1 \leq i \leq k$.) We are going to prove it by induction. Suppose that $i = 0$ or $x_i$ is an ender. Let $\widetilde{k}$ be the intensity scheme we get from $\widetilde{l}$ by replacing $(\widetilde{l}_{f_i})_{x_{i+1}}$ with $(\widetilde{l}_{f_i})_{x_{i+1}} \vee \ell_{x_{i+1}}$. In other words, $\widetilde{k}_{f_i} = \widetilde{l}_{f_i} \vee \ell_{x_{i+1}}$ and $\widetilde{k}_{v'} = \widetilde{l}_{v'}$ for every $v' \neq f_i$. (We may use the notation $\widetilde{k} = \widetilde{l} \vee \ell_{x_{i+1}}$ but it is ambiguous, since there are two intensities of $x_{i+1}$ in $\widetilde{l}$.)

Since we only increased intesity on an arc leaving $f_i$, for the twisted partial ordering $\widetilde{k} \sqsubseteq \widetilde{l}$.

Since $\mathcal{D}$ is $\sqsupseteq$-antitone, $\mathcal{D}(\widetilde{k}) \sqsupseteq \mathcal{D}(\widetilde{l}) = b$, so at contract $x_{i+1}$, $\mathcal{D}(\widetilde{k}_{f_i})_{x_{i+1}} \leq (b_{f_i})_{x_{i+1}}$. Looking at contract $x_{i+1}$ only, $C_{f_i}(\widetilde{l} \vee \ell_{x_{i+1}})_{x_{i+1}} = (\widetilde{l}_{f_i} \vee \ell_{x_{i+1}})_{x_{i+1}} \wedge \mathcal{D}_{f_i}(\widetilde{l} \vee \ell_{x_{i+1}})_{x_{i+1}} \leq (\widetilde{l}_{f_i} \vee \ell_{x_{i+1}})_{x_{i+1}} \wedge (b_{f_i})_{x_{i+1}} = ((\widetilde{l}_{f_i})_{x_{i+1}} \vee \ell_{x_{i+1}}) \wedge (\widetilde{l}_{f_{i+1}})_{x_{i+1}}$ In the last step, we use that $b = \widetilde{l}^*$. From now on, let us use the notation $x = ((\widetilde{l}_{f_i})_{x_{i+1}} \vee \ell_{x_{i+1}}) \wedge (\widetilde{l}_{f_{i+1}})_{x_{i+1}}$.

Using that the lattice is distributive:
$x = (\ell_{x_{i+1}} \vee (\widetilde{l}_{f_i})_{x_{i+1}}) \wedge (\widetilde{l}_{f_{i+1}})_{x_{i+1}} = (\ell_{x_{i+1}} \wedge (\widetilde{l}_{f_{i+1}})_{x_{i+1}}) \vee ((\widetilde{l}_{f_i})_{x_{i+1}} \wedge (\widetilde{l}_{f_{i+1}})_{x_{i+1}})$
From definition, $((\widetilde{l}_{f_i})_{x_{i+1}} \wedge (\widetilde{l}_{f_{i+1}})_{x_{i+1}}) = l_{x_{i+1}}$
therefore $x = (\ell_{x_{i+1}} \wedge (\widetilde{l}_{f_{i+1}})_{x_{i+1}})) \vee ((\widetilde{l}_{f_i})_{x_{i+1}} \wedge (\widetilde{l}_{f_{i+1}})_{x_{i+1}}) \leq \ell_{x_{i+1}} \vee l_{x_{i+1}}$

Let $\widetilde{m} = \widetilde{l}|l \vee \ell_{x_{i+1}}$ denote the intensity scheme we get from $\widetilde{l}$ if for all $e \in X^S_{f_i}$ (i.e. contract $e$ whose seller is $f_i$), we replace $(\widetilde{l}_{f_i})_e$ with $l_e$ and on contract $x_{i+1}$ we replace $(\widetilde{l}_{f_i})_{x_{i+1}}$ with $l_{x_{i+1}} \vee \ell_{x_{i+1}}$. Therefore, $\widetilde{m} \mathrel{\widetilde{\leq}} \widetilde{k}$ and $\widetilde{m}_{f_i} \sqsupseteq (l \vee \ell_{x_{i+1}})_{f_i}$.

For contract $x_{i+1}$ we have seen that $C_{f_i}(\widetilde{l}_{f_i} \vee \ell_{x_{i+1}})_{x_{i+1}} \leq x \leq \ell_{x_{i+1}} \vee l_{x_{i+1}}$
For contract $e$ whose seller is $f_i$, buyer is $v'$ and $e \neq x_{i+1}$, $C_{f_i}(\widetilde{l}_{f_i} \vee \ell_{x_{i+1}})_e = (\widetilde{l}_{f_i} \vee \ell_{x_{i+1}})_e \wedge \mathcal{D}_{f_i}(\widetilde{l} \vee \ell_{x_{i+1}})_e \leq (\widetilde{l}_{f_i})_e \wedge (b_{f_i})_e = ((\widetilde{l}_{f_i})_e) \wedge (\widetilde{l}_{v'})_e = l_e$

All together, at firm $f_i$, $C_{f_i}(\widetilde{l}_{f_i} \vee \ell_{x_{i+1}}) \leq \widetilde{m}_{f_i} \leq (\widetilde{l}_{f_i} \vee \ell_{x_{i+1}})$, in other words $C_{f_i}(\widetilde{k}_{f_i}) \leq \widetilde{m}_{f_i} \leq \widetilde{k}_{f_i}$ and $C_{f_i}$ is IRC, so $C_{f_i}(\widetilde{k}) = C_{f_i}(\widetilde{m})$.

Since we supposed that $x_i$ is an ender, $\widetilde{m}_{x_i} = \widetilde{l}_{x_i} \geq (l \vee \ell)_{x_i}$, so $\widetilde{m}_{f_i} \sqsupseteq (l \vee \ell_{x_i} \vee \ell_{x_{i+1}})_{f_i}$. If $i = 0$, then $\widetilde{m}_{f_0} \sqsupseteq (l \vee \ell_{x_1})_{f_0}$.

The determinant is antitone so $\mathcal{D}_{f_i}(\widetilde{m})_{x_{i+1}} \geq \mathcal{D}_{f_i}(l \vee \ell_{x_i} \vee \ell_{x_{i+1}})_{x_{i+1}} \geq \ell_{x_{i+1}}$ (because we supposed that $\ell$ is a locally blocking trail) so $C_{f_i}(\widetilde{m})_{x_{i+1}} \geq \ell_{x_{i+1}}$. Thus $C_{f_i}(\widetilde{k})_{x_{i+1}} \geq \ell_{x_{i+1}}$. We know that $\widetilde{k}_{f_i}$ differs from $\widetilde{l}_{f_i}$ only on the value on contract $x_{i+1}$. From the definition of the canonical determinant, $C_{f_i}(\widetilde{k})_{x_{i+1}} \geq \ell_{x_{i+1}}$ implies $\mathcal{D}_{f_i}(\widetilde{l})_{x_{i+1}} \geq \ell_{x_{i+1}}$ so $(\widetilde{l}_{f_{i+1}})_{x_{i+1}} \geq \ell_{x_{i+1}}$ therefore $x_{i+1}$ is an ender.

At the end of the trail, $x_k$ is an ender, and by $(i)$ this implies that the trail with $\ell$ intensities cannot be blocking.

□

**Example 3.10.10.** *If the lattice of intensities on a contract, $\mathcal{L}_x$, is not distributive then this fixed-point method may not lead to a fully trail-stable contraxt scheme, as we show in this counterexample.*

*The market consists of only one contract. $F = \{s,t\}, X = \{x\} = st$. Lattice $\mathcal{L}_x = (\{0,a,b,c,1\}, \leq)$ where $0 < a < 1, 0 < b < 1, 0 < c < 1$, and $a,b,c$ are incomparable. The choice functions and their determinants are:*

| | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| $C^s$ | 0 | 0 | b | c | b |
| $\mathcal{D}_s$ | 1 | b | b | 1 | b |
| $C^t$ | 0 | a | 0 | c | a |
| $\mathcal{D}_t$ | 1 | a | a | 1 | a |

*One can check that these $\mathcal{D}$ functions are antitone, $C^s$ and $C^t$ are IRC.*

*Let $(\widetilde{l}_s)_x = a, (\widetilde{l}_t)_x = b$, so $\mathcal{D}(\widetilde{l}) = (b,a)$ and $(\mathcal{D}(\widetilde{l}))^* = (a,b)$. Therefore $(a,b)$ is a fixed point. Here $l = \widetilde{l} \wedge \widetilde{l}^* = (0,0)$. Thus $0$ should be fully trail-stable, but contract $x$ with intensity $\ell = c$ is a blocking trail, since both ends accept $c$. The only fully trail-stable solution is $c$.*

*Proof of Theorem 3.10.7.* Define $\varphi : \widetilde{\mathcal{L}}_D \to \widetilde{\mathcal{L}}_D$ by $\varphi(\widetilde{l}) = (D(\widetilde{l}))^*$ as a composition of two mappings that are $\sqsubseteq$-antitone by Observation 3.10.8. So $\varphi$ is monotone on lattice $\widetilde{\mathcal{L}}_D$, hence it has a fixed point $\widetilde{l}$ by Theorem 1.2.1 of Tarski. Lemma 3.10.9 implies that intensity scheme $\widetilde{l} \wedge \widetilde{l}^*$ is the duplication of a stable contract scheme. □

## 3.11 Conclusion

We summarize here the results of the Thesis. In the first part, we dealt with two-sided markets (where contracts form an general undirected bipartite graph) and supply chains (where contracts form a directed graph). In both scenarios, we defined various models of stability and compared them with each other. For three- four- and score-stability we can generalize the Gale-Shapley algorithm to get the optimal and pessimal matchings for one side of the market.

We can generalize Blair's theorem to show that the four-stable outcomes form a lattice for a natural partial order. This remains true even with lattice choice functions. Using this, we can show an alternative proof for the Theorem of Aharoni, Berger, and Gorelik about the existence of weighted kernels [20].

For contract networks, we can show set-stable outcomes do not always exist, moreover for a given outcome, deciding whether a blocking set exist is NP-hard. Trail-stable and fully trail-stable outcomes always exists, moreover fully trail-stable outcomes form a lattice for a preference ordering of the terminal agents. We conjecture that terminal-superiority on the trail-stable outcomes has the lattice property [22].

If a new agent (a terminal seller) enters or leaves the market, after a market rearrangement, the new outcome will be worse for all terminal sellers, and better for all terminal buyers.

The thesis is based on three and a half papers of the candidate.

- [19] Tamás Fleiner and Zsuzsanna Jankó. Choice function-based two-sided markets: Stability, lattice property, path independence and algorithms. *Algorithms*, 7(1):32, 2014.

  This paper yields the base of Sections 2.1, 2.2 and 2.3.

- [20] Tamás Fleiner and Zsuzsanna Jankó. On weighted kernels of two posets. *Order*, 33(1):51–65, 2016.

  (Some parts of Section 1.1 and Section 2.3 concerning the lattice choice functions, and all of Section 2.4.)

- [22] Tamás Fleiner, Zsuzsanna Jankó, Akihisa Tamura, and Alexander Teytelboym Trading networks with bilateral contracts. *Working paper, Oxford University*, 2015.

  The results of this working paper are contained in Chapter 3 (except Section 3.10 and 3.11.)

- [21] Tamás Fleiner, Zsuzsanna Jankó, and Akihisa Tamura: Stability of generalized network flows, *Proceedings of the 8th Japasese-Hungarian Symposium on Discrete Mathematics and Its Applications*, Veszprém, Hungary , 133-141, 2013

(Section 3.10.)

Finally, the following paper has a topic different from the others, namely house exchange markets, thus it is not included in this thesis.

- [12] Katarína Cechlárová, Tamás Fleiner, and Zsuzsanna Jankó. House-swapping with divorcing and engaged pairs. *Discrete Applied Mathematics*, 206:1–8, 2016.

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# Index

# Abstract


The Thesis consists of two main parts, in the first, we deal with two-sided markets, which can be represented with a bipartite graph, every vertex corresponds to an agent, who has a preference ordering, or more generally a choice function over all the edges (contracts) incident to them.

- We deal with various concepts of stability, such that dominating stability, three-stability, four-stability and explore their relation to each other.
- We define generalized choice functions over arbitrary lattices and show the usefulness of the determinant.
- We generalize Blair's Theorem [11], and show that four-stable sets form a lattice if the choice function of one side is IRC, even for lattice choice functions.
- We show that in the Hungarian college entrance model, the stable score limits form a lattice.
- We give an alternative proof for Aharoni, Berger and Gorelik's Theorem [2] and show the lattice property of weighted kernels.
- We show and alternative proof and an algorithm for finding a stable kernel defined in two ordered matroids.

In the second part of the thesis, we work with trading networks, which can be represented with a directed graph, every vertex correspont s to a firm, and every arc represents a bilateral contract between them. A set contracts is called an *outcome*.

- We introduce the concept of trail-stability and fully trail-stability, and show that trail-stable and fully trail-stable outcomes always exist.
- We show that the decision problem whether a given outcome is set-stable or has a blocking set is NP-complete.
- If we define a partial order concerning only the preferences of terminal buyers and sellers, we can show that trail-stable outcomes include a buyer-optimal and a seller-optimal one, and fully trail-stable outcomes form a lattice.
- Given a fully trail-stable outcome, if a new terminal seller enters the market, after a market rearrangement period, the new fully trail-stable outcome will be better for all terminal buyers, and worse for all terminal buyers than before.
- In the last section we describe a more complex model: in a trading network, every contract has an intensity, (which is an element of a lattice on this contract). Choice functions can be defined on the direct sum of intensities, and there always exists a fully trail-stable contract scheme.

# Összefoglalás

A tézis két fő részből áll, az első részben kétoldalú piacokkal foglalkozunk. Egy kétoldalú piacot egy páros gráffal írhatunk le, a csúcsok a játékosok, az élek a lehetséges kapcsolatok. A szereplők preferenciáit a lehetséges kapcsolatokon kiválasztási függvényekkel írhatjuk le.

- Megvizsgáljuk a dominálásos, háromrészes, négyrészes és vonalhúzásos stabilitást, és ezek kapcsolatát.
- Általánosítjuk a kiválasztási függvényeket hálókra, és a megmutatjuk, hogy a determináns segítségével szebben lehet definiálni a komonotonitást.
- Blair tételét [11] általánosítjuk négyrészes stabilitásra, a mindkét oldal kiválasztási függvénye komonoton, és legalább egyiké IRC.
- A magyar felvételi rendszert és a stabil vonalhúzásokat vizsgáljuk. A stabil vonalhúzások hálót alkotnak.
- Egy alternatív bizonyítást adunk Aharoni, Berger és Gorelik tételére [2], miszerint súlyozott kernel mindig létezik. Egy erősebb állítást is belátunk, a súlyozott kernelek hálótulajdonságát.
- Mutatunk egy alternatív bizonyítást és egy algoritmust két rendezett matroid esetén stabil kernel megtalálására.

A dolgozat második felében kereskedési rendszereket modellezünk egy irányított gráffal, ahol a csúcsok a cégek, az élek a lehetséges kereskedések.

- Definiáljuk a trail-stabilitást és a teljes trail-stabilitást, és megmutatjuk, hogy trail-stabil és a teljes trail-stabil megoldások mindig léteznek.
- Egy adott kereskedési rendszerre az eldöntési feladat, hogy egy adott kimenetel halmaz-stabil, vagy létezik hozzá blokkoló élhalmaz, NP-teljes.
- Csak a végső eladók és végső vevők (azaz források és nyelők) preferenciáit figyelembe véve definiálhatunk egy részbenrendezést a kimeneteleken. A trail-stabil megoldások között találhatunk vevő-optimális és eladó-optimális megoldást. A teljesen trail-stabil élhalmazok hálót alkotnak.
- Ha adott egy teljesen trail-stabil kimenetel, és érkezik egy új végső eladó a piacra, egy átrendeződési időszak után kapunk egy olyan, új teljesen trail-stabil kimenetelt, ahol minden végsó vevő jobban jár, és minden végső eladó rosszabbul jár mint az eredeti megoldásban.
- Az utolsó fejezetben még egy ennél is általánosabb modellt írunk le: az irányított éleken adottak hálók, ebből választhatják ki a cégek, hogy mekkora mennyiséget küldenek rajta. Az éleken lévő hálók direkt összegén definiálhatók kiválasztási függvények, és ezek segítségével találhatunk teljesen trail-stabil megoldásokat.