Convergence Rates for Hölder-Windows in Filtered Back Projection

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***Abstract*—In this paper we consider the approximation of bivariate functions by using the well-established filtered back projection (FBP) formula from computerized tomography. We establish error estimates and convergence rates for the FBP reconstruction method for target functions f from a Sobolev space $\mathrm{H}^\alpha(\mathbb{R}^2)$ of fractional order $\alpha > 0$, where we bound the FBP reconstruction error with respect to the weaker norms of the Sobolev spaces $\mathrm{H}^\sigma(\mathbb{R}^2)$, for $0 \leq \sigma \leq \alpha$. By only assuming Hölder continuity of the low-pass filter's window function, the results of this paper generalize previous of our findings in [2]–[4].**

I. Introduction

The method of *filtered back projection* (FBP) is a common reconstruction technique in computerized tomography (CT) and deals with recovering the interior structure of a scanned object from X-ray scans. This data can be interpreted as a set of line integrals of the object's *attenuation function* and, thus, the classical CT reconstruction problem reads as follows.

Problem 1 (Reconstruction problem): On domain $\Omega \subseteq \mathbb{R}^2$, reconstruct a bivariate function $f \in \mathrm{L}^1(\Omega)$ from given Radon data

$$\big\{\mathcal{R}f(t,\theta) \mid t \in \mathbb{R},\, \theta \in [0,\pi)\big\},$$

where the *Radon transform* $\mathcal{R}f$ of f is defined as

$$\mathcal{R}f(t,\theta) = \int_{\{x\cos(\theta)+y\sin(\theta)=t\}} f(x,y)\,\mathrm{d}(x,y)$$

for $(t,\theta) \in \mathbb{R} \times [0,\pi)$. ■

Thus, the CT reconstruction problem seeks for the inversion of the Radon transform $\mathcal{R}$. For a comprehensive mathematical treatment of $\mathcal{R}$ and its inversion, we refer to [6], [7], [12]. Due to the ill-posedness of the reconstruction problem, the inversion formula, see (1), cannot be used in practice. Instead, suitable low-pass filters of finite bandwidth L are employed leading to an *approximate* reconstruction of the target function.

In our previous work [2], [3], we derived L^2-error estimates and convergence rates (as $L \to \infty$) for target functions f from fractional Sobolev spaces $\mathrm{H}^\alpha(\mathbb{R}^2)$ with $\alpha > 0$. More recently, we also proved Sobolev error estimates and convergence rates in [1], [4]. In [2], [4] we considered low-pass filters whose window functions are continuously differentiable on $[-1,1]$. The proven rates of convergence saturate at *integer* order depending of the differentiability of the window. The primary goal of this paper is to generalize these previous results to Hölder continuous windows. This will allow us to predict saturation of the convergence rates at *fractional* order.

The outline of this paper is as follows. In Section II, we consider the inversion of the Radon transform by the classical FBP formula and show how the FBP can be stabilized by using low-pass filters of finite bandwidth. This standard approach leads us to an approximate reconstruction, whose approximation quality will be evaluated in this paper. To this end, in Section III, we discuss Sobolev error estimates for target functions from Sobolev spaces of fractional order. In Section IV, we finally derive asymptotic convergence rates for the special case of Hölder-windows as the bandwidth goes to infinity, where we will observe saturation at fractional order.

II. Filtered Back Projection

The inversion of the Radon transform $\mathcal{R}$ is well understood and given by the classical *filtered back projection formula*

$$f(x,y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}[|S|\mathcal{F}(\mathcal{R}f)(S,\theta)]\big)(x,y), \qquad (1)$$

which holds for $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathscr{C}(\mathbb{R}^2)$ (see [5, Theorem 6.2.]). Here, the *back projection* $\mathcal{B}h$ of $h \in \mathrm{L}^1(\mathbb{R} \times [0,\pi))$ is defined as

$$\mathcal{B}h(x,y) = \frac{1}{\pi}\int_0^\pi h(x\cos(\theta)+y\sin(\theta),\theta)\,\mathrm{d}\theta$$

for $(x,y) \in \mathbb{R}^2$. Note that, up to the constant $\frac{1}{\pi}$, the back projection $\mathcal{B}$ is the adjoint operator of the Radon transform $\mathcal{R}$.

We remark that the FBP formula is numerically *unstable*. Indeed, by applying the filter $|S|$ to the Fourier transform $\mathcal{F}(\mathcal{R}f)$ in (1), especially the high frequency components of $\mathcal{R}f$ are amplified by the magnitude of $|S|$ and, thus, the FBP formula is in particular highly sensitive with respect to noise.

In order to reduce the noise sensitivity, a standard approach is to replace the filter $|S|$ in (1) by a *low-pass filter* A_L of the form

$$A_L(S) = |S|\,W(S/L) \quad \text{for } S \in \mathbb{R}$$

with finite *bandwidth* $L > 0$ and an even *window function* $W \in \mathrm{L}^\infty(\mathbb{R})$ with compact support $\mathrm{supp}(W) \subseteq [-1,1]$.

Note that by replacing the filter $|S|$ in (1) by the low-pass filter A_L, the reconstruction of f is no longer exact and we only get an *approximate FBP reconstruction*, denoted by f_L.

For target functions $f \in \mathrm{L}^1(\mathbb{R}^2)$ the reconstruction f_L is defined almost everywhere on $\mathbb{R}^2$ (see [1, Proposition 3.1]) and the resulting *approximate FBP formula* can be simplified as

$$f_L = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L * \mathcal{R}f\big). \qquad (2)$$

Moreover, f_L belongs to $\mathrm{L}^2(\mathbb{R}^2)$ (see [1, Proposition 4.2]) and can be expressed in terms of the target function f via

$$f_L = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L * \mathcal{R}f\big) = f * K_L, \tag{3}$$

where the *convolution kernel* $K_L : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$K_L(x,y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L\big)(x,y) \quad \text{for } (x,y) \in \mathbb{R}^2.$$

For the sake of brevity, we call any application of the approximate FBP formula (2) an *FBP method*. Therefore, each FBP method provides one approximation f_L to f, $f_L \approx f$, whose quality depends on the choice of the low-pass filter A_L.

In the following, we analyse the intrinsic error of the FBP method which is incurred by the use of the low-pass filter A_L, i.e., we wish to analyse the reconstruction error

$$e_L = f - f_L$$

with respect to the filter's window W and bandwidth L.

We remark that pointwise and L^∞-error estimates on e_L were proven by Munshi et al. in [9]. Their theoretical results were further supported by numerical experiments in [10]. Error bounds for the L^p-norm of e_L, in terms of an L^p-modulus of continuity of f, were proven by Madych in [8].

In [1]–[4] we derived error estimates and convergence rates for target functions from fractional Sobolev spaces $\mathrm{H}^\alpha(\mathbb{R}^2)$. Let us recall that the *Sobolev space* $\mathrm{H}^\alpha(\mathbb{R}^2)$ of order $\alpha \in \mathbb{R}$ is defined as

$$\mathrm{H}^\alpha(\mathbb{R}^2) = \big\{f \in \mathcal{S}'(\mathbb{R}^2) \mid \|f\|_\alpha < \infty\big\},$$

where

$$\|f\|_\alpha^2 = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \big(1 + x^2 + y^2\big)^\alpha\, |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y)$$

and $\mathcal{S}'(\mathbb{R}^2)$ denotes the space of tempered distributions on $\mathbb{R}^2$.

We remark that in relevant applications of (medical) image processing, Sobolev spaces of compactly supported functions,

$$\mathrm{H}_0^\alpha(\Omega) = \big\{f \in \mathrm{H}^\alpha(\mathbb{R}^2) \mid \mathrm{supp}(f) \subseteq \overline{\Omega}\big\},$$

on an open and bounded domain $\Omega \subset \mathbb{R}^2$, and of fractional order $\alpha > 0$ play an important role (cf. [11]). In fact, we can consider the density of an image in $\Omega \subset \mathbb{R}^2$ as a function from the Sobolev space $\mathrm{H}_0^\alpha(\Omega)$ whose order α is close to $\frac{1}{2}$.

III. Error Analysis

In this section, we analyse certain Sobolev norms of the inherent FBP reconstruction error e_L for target functions f from the Sobolev space $\mathrm{H}^\alpha(\mathbb{R}^2)$ of fractional order $\alpha > 0$. To this end, we summarize our H^σ-error estimates in [1], [2] for $0 \leq \sigma \leq \alpha$. As we rely on these results in our discussion of Hölder-windows in Section IV, we recall some details for the reader's convenience.

Let us assume that $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$. We first show that the approximate FBP reconstruction f_L belongs to the Sobolev space $\mathrm{H}^\sigma(\mathbb{R}^2)$ for any $0 \leq \sigma \leq \alpha$.

Due to [1, Proposition 4.1], the convolution kernel K_L belongs to $\mathscr{C}_0(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$ and its Fourier transform satisfies

$$\mathcal{F}K_L(x,y) = W_L(x,y) \quad \text{for almost all } (x,y) \in \mathbb{R}^2.$$

Here, the bivariate window $W_L \in \mathrm{L}^\infty(\mathbb{R}^2)$ is defined as

$$W_L(x,y) = W\Big(\frac{r(x,y)}{L}\Big) \quad \text{for } (x,y) \in \mathbb{R}^2,$$

where

$$r(x,y) = \sqrt{x^2 + y^2} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

This in combination with representation (3) for f_L yields

$$\begin{aligned}\|f_L\|_\sigma^2 &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \big(1 + r(x,y)^2\big)^\sigma\, |(W_L \cdot \mathcal{F}f)(x,y)|^2 \,\mathrm{d}(x,y)\\ &\leq \Big(\sup_{r(x,y)\leq L} |W_L(x,y)|^2\Big)\, \|f\|_\alpha^2 = \|W\|_{\mathrm{L}^\infty(\mathbb{R})}^2\, \|f\|_\alpha^2.\end{aligned}$$

Thus, for $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ with $\alpha > 0$, the approximate FBP reconstruction f_L belongs to $\mathrm{H}^\sigma(\mathbb{R}^2)$ for all $0 \leq \sigma \leq \alpha$.

Let us now turn to the analysis of the FBP reconstruction error $e_L = f - f_L$ with respect to the H^σ-norm. For $\gamma \geq 0$, we define

$$r_\gamma(x,y) = \big(1 + x^2 + y^2\big)^\gamma \quad \text{for } (x,y) \in \mathbb{R}^2$$

so that the H^σ-norm of e_L can be expressed as

$$\begin{aligned}\|e_L\|_\sigma^2 &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} r_\sigma(x,y)\, |(\mathcal{F}f - W_L \cdot \mathcal{F}f)(x,y)|^2 \,\mathrm{d}(x,y)\\ &= I_1 + I_2,\end{aligned}$$

where

$$I_1 = \frac{1}{4\pi^2}\int_{B_L} r_\sigma(x,y)\, |1 - W_L(x,y)|^2\, |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y)$$

with

$$B_L = \big\{(x,y) \in \mathbb{R}^2 \mid r(x,y) \leq L\big\}$$

and

$$I_2 = \frac{1}{4\pi^2}\int_{\mathbb{R}^2\setminus B_L} r_\sigma(x,y)\, |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y).$$

For $\gamma \geq 0$, we define

$$\Phi_{\gamma,W}(L) = \sup_{S\in[-1,1]} \frac{(1 - W(S))^2}{(1 + L^2S^2)^\gamma} \quad \text{for } L > 0$$

so that we can bound I_1 from above by

$$I_1 \leq \Bigg(\sup_{(x,y)\in B_L} \frac{(1 - W_L(x,y))^2}{r_{\alpha-\sigma}(x,y)}\Bigg)\|f\|_\alpha^2 = \Phi_{\alpha-\sigma,W}(L)\|f\|_\alpha^2.$$

On the other hand, for $0 \leq \sigma \leq \alpha$, we can bound I_2 by

$$\begin{aligned}I_2 &\leq L^{2(\sigma-\alpha)}\, \frac{1}{4\pi^2}\int_{\mathbb{R}^2\setminus B_L} r_\alpha(x,y)\, |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y)\\ &\leq L^{2(\sigma-\alpha)}\, \|f\|_\alpha^2.\end{aligned}$$

Combining the estimates for I_1 and I_2, we finally obtain

$$\|e_L\|_\sigma^2 \leq \Big(\Phi_{\alpha-\sigma,W}(L) + L^{2(\sigma-\alpha)}\Big)\, \|f\|_\alpha^2.$$

We can summarize the discussion of this section as follows.

Theorem 1 (H^σ-error estimate, see [1, Theorem 5.2]): Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$ and let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even and compactly supported with $\mathrm{supp}(W) \subseteq [-1,1]$. Then, for $0 \leq \sigma \leq \alpha$, the H^σ-norm of the inherent FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_\sigma \leq \Big(\Phi^{1/2}_{\alpha-\sigma,W}(L) + L^{\sigma-\alpha}\Big) \, \|f\|_\alpha, \tag{4}$$

where

$$\Phi_{\alpha-\sigma,W}(L) = \sup_{S \in [-1,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^{\alpha-\sigma}} \quad \text{for } L > 0. \qquad \blacksquare$$

For the purpose of analysing the convergence behaviour of the error as the bandwidth goes to infinity, in [2], [4] we considered the special case of k-times continuously differentiable window functions whose first $k-1$ derivatives vanish at zero.

Theorem 2 (Estimate for $\mathscr{C}^k$-windows, see [4, Theorem 3]): Let the assumptions of Theorem 1 be satisfied. In addition, let $W \in \mathscr{C}^k([-1,1])$ for $k \geq 2$ with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k-1.$$

Then, for $0 \leq \sigma \leq \alpha$, the H^σ-norm of the inherent FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_\sigma \leq \Big(\frac{1}{k!} \, \|W^{(k)}\|_{\mathrm{L}^\infty([0,1])} + 1\Big) \, L^{\sigma-\alpha} \, \|f\|_\alpha$$

for $\alpha - \sigma \leq k$, and by

$$\|e_L\|_\sigma \leq \Big(\frac{c_{\alpha-\sigma,k}}{k!} \, \|W^{(k)}\|_{\mathrm{L}^\infty([0,1])} \, L^{-k} + L^{\sigma-\alpha}\Big) \, \|f\|_\alpha$$

for $\alpha - \sigma > k$ and sufficiently large $L > 0$, where the constant

$$c_{\gamma,k} = \Big(\frac{k}{\gamma-k}\Big)^{k/2} \Big(\frac{\gamma-k}{\gamma}\Big)^{\gamma/2}$$

is strictly monotonically decreasing in $\gamma > k$. In particular,

$$\|e_L\|_\sigma = \mathcal{O}\Big(L^{-\min\{k,\alpha-\sigma\}}\Big) \quad \text{for} \quad L \to \infty. \qquad \blacksquare$$

Note that for $\alpha - \sigma \leq k$ the decay rate of $\|e_L\|_\sigma$ is determined by the difference between the smoothness α of the target function f and the order σ of the considered Sobolev norm, whereas for $\alpha - \sigma > k$ the decay rate is predicted to saturate at *integer* order $\mathcal{O}(L^{-k})$. Here, k denotes the differentiability order of the window function W, whose first $k-1$ derivatives are required to vanish at zero.

IV. Convergence Rates for Hölder-Windows

In this section, we generalize our results in Theorem 2 for $\mathscr{C}^k$-windows by considering Hölder-windows. More precisely, we again consider even window functions $W \in \mathrm{L}^\infty(\mathbb{R})$ with $\mathrm{supp}(W) \subseteq [-1,1]$. Unlike in our previous work, we now assume that, for $k \in \mathbb{N}$ and $\nu \in (0,1]$, the window W satisfies $W \in \mathscr{C}^{k-1,\nu}([-1,1])$ with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k-1.$$

Thus, W is $(k-1)$-times continuously differentiable on $[-1,1]$ and $W^{(k-1)}$ is Hölder continuous on $[-1,1]$ with Hölder exponent $\nu \in (0,1]$ and Hölder constant $C_W > 0$ such that

$$|W^{(k-1)}(S) - W^{(k-1)}(t)| \leq C_W \, |S-t|^\nu \quad \forall \, S,t \in [0,1].$$

Under these assumptions, we will prove that the H^σ-norm of the FBP reconstruction error $e_L = f - f_L$ now behaves like

$$\|e_L\|_\sigma = \mathcal{O}\Big(L^{-\min\{k-1+\nu,\alpha-\sigma\}}\Big) \quad \text{for} \quad L \to \infty.$$

Theorem 3 (Estimate for Hölder-windows): Let the assumptions of Theorem 1 be satisfied. In addition, for $k \in \mathbb{N}$ and $0 < \nu \leq 1$ let $W \in \mathscr{C}^{k-1,\nu}([-1,1])$ with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k-1.$$

Then, for $0 \leq \sigma \leq \alpha$, the H^σ-norm of the inherent FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_\sigma \leq \Big(\frac{\Gamma(\nu+1)}{\Gamma(\nu+k)} \, C_W + 1\Big) \, L^{\sigma-\alpha} \, \|f\|_\alpha$$

for $\alpha - \sigma \leq k - 1 + \nu$, and by

$$\|e_L\|_\sigma \leq \Big(c_{\alpha-\sigma,k,\nu} \, \frac{\Gamma(\nu+1)}{\Gamma(\nu+k)} \, C_W \, L^{-k+1-\nu} + L^{\sigma-\alpha}\Big) \, \|f\|_\alpha$$

for $\alpha - \sigma > k - 1 + \nu$ and sufficiently large $L > 0$, where the constant

$$c_{\gamma,k,\nu} = \Big(\frac{k-1+\nu}{\gamma-k+1-\nu}\Big)^{(k-1+\nu)/2} \Big(\frac{\gamma-k+1-\nu}{\gamma}\Big)^{\gamma/2}$$

is strictly monotonically decreasing in $\gamma > k - 1 + \nu$. In particular,

$$\|e_L\|_\sigma = \mathcal{O}\Big(L^{-\min\{k-1+\nu,\alpha-\sigma\}}\Big) \quad \text{for} \quad L \to \infty.$$

Proof: Based on our assumptions and on the H^σ-error estimate (4) from Theorem 1, i.e.,

$$\|e_L\|_\sigma \leq \Big(\Phi^{1/2}_{\alpha-\sigma,W}(L) + L^{\sigma-\alpha}\Big) \, \|f\|_\alpha,$$

it is sufficient to analyse the error term

$$\Phi_{\gamma,W}(L) = \sup_{S \in [-1,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\gamma} = \max_{S \in [0,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\gamma}$$

for $\gamma \geq 0$. By assumption we have $W \in \mathscr{C}^{k-1,\nu}([-1,1])$ and

$$|W^{(k-1)}(S) - W^{(k-1)}(t)| \leq C_W \, |S-t|^\nu \quad \forall \, S,t \in [0,1].$$

Thus, if $k = 1$, the assumption $W(0) = 1$ gives

$$|1 - W(S)| \leq C_W \, S^\nu \quad \forall \, S \in [0,1].$$

If $k \geq 2$, the fundamental theorem of calculus gives

$$W^{(j)}(S) = W^{(j)}(0) + \int_0^S W^{(j+1)}(t) \, \mathrm{d}t \quad \forall \, 0 \leq j \leq k-2.$$

In particular,

$$W(S) = 1 + \int_0^S W'(t) \, \mathrm{d}t \quad \forall \, S \in [0,1]$$

and, for $k > 2$, iteratively applying integration by parts yields

$$W(S) = 1 + \frac{1}{(k-2)!} \int_0^S (S-t)^{k-2} \, W^{(k-1)}(t) \, \mathrm{d}t,$$

since

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall \, 1 \leq j \leq k-2.$$

Using $W^{(k-1)} \in \mathscr{C}^{0,\nu}([-1,1])$ and $W^{(k-1)}(0) = 0$, we have

$$|W^{(k-1)}(t)| \leq C_W\, t^\nu \;\; \forall\, t \in [0,1]$$

and, consequently, for all $S \in [0,1]$, it follows that

$$|1 - W(S)| \leq \frac{1}{(k-2)!}\, C_W \int_0^S (S-t)^{k-2}\, t^\nu \,\mathrm{d}t,$$

where

$$\int_0^S (S-t)^{k-2}\, t^\nu \,\mathrm{d}t = (k-2)!\, \frac{\Gamma(\nu+1)}{\Gamma(\nu+k)}\, S^{\nu+k-1}.$$

Hence, for any $k \in \mathbb{N}$ we have

$$|1 - W(S)| \leq \frac{\Gamma(\nu+1)}{\Gamma(\nu+k)}\, C_W\, S^{k-1+\nu} \;\; \forall\, S \in [0,1]$$

and the error term $\Phi_{\gamma,W}(L)$ is bounded above by

$$\Phi_{\gamma,W}(L) \leq \frac{\Gamma(\nu+1)^2}{\Gamma(\nu+k)^2}\, C_W^2 \max_{S \in [0,1]} \frac{S^{2(k-1+\nu)}}{(1+L^2S^2)^\gamma}.$$

It remains to analyse the function

$$\phi(S) = \frac{S^{2(k-1+\nu)}}{(1+L^2S^2)^\gamma} \quad \text{for } S \in [0,1].$$

Case 1: For $0 \leq \gamma \leq k-1+\nu$, the function ϕ is strictly monotonically increasing in $(0,1]$ so that

$$\max_{S \in [0,1]} \phi(S) = \phi(1) \leq L^{-2\gamma}.$$

Case 2: For $\gamma > k-1+\nu$, the first order necessary condition for a maximum of ϕ yields

$$\phi'(S) = 0 \quad \overset{S \neq 0}{\Longleftrightarrow} \quad (\gamma - k + 1 - \nu)\, L^2\, S^2 = k - 1 + \nu,$$

which has the unique positive solution

$$S^* = \frac{\sqrt{k-1+\nu}}{L\sqrt{\gamma-k+1-\nu}},$$

where

$$S^* \in (0,1] \quad \Longleftrightarrow \quad L \geq \frac{\sqrt{k-1+\nu}}{\sqrt{\gamma-k+1-\nu}} = L^*.$$

Furthermore, ϕ is strictly monotonically increasing in $(0,S^*)$ and strictly monotonically decreasing in (S^*,∞) so that

$$\arg\max_{S \in [0,1]} \phi(S) = \begin{cases} 1 & \text{for } L < L^* \\ S^* & \text{for } L \geq L^*. \end{cases}$$

With

$$\phi(S^*) = c_{\gamma,k,\nu}^2\, L^{-2(k-1+\nu)}$$

we finally get

$$\max_{S \in [0,1]} \phi(S) \leq \begin{cases} L^{-2\gamma} & \text{for } L < L^* \\ c_{\gamma,k,\nu}^2\, L^{-2(k-1+\nu)} & \text{for } L \geq L^*, \end{cases}$$

where the constant

$$c_{\gamma,k,\nu} = \Big(\frac{k-1+\nu}{\gamma-k+1-\nu}\Big)^{(k-1+\nu)/2} \Big(\frac{\gamma-k+1-\nu}{\gamma}\Big)^{\gamma/2}$$

is strictly monotonically decreasing in $\gamma > k-1+\nu$. ∎

Note that in Theorem 3 the convergence rate of $\|e_L\|_\sigma$ is determined by the difference between the smoothness α of the target function f and the order σ of the considered Sobolev norm, as long as $\alpha - \sigma \leq k-1+\nu$. For $\alpha - \sigma > k-1+\nu$ the order of convergence is predicted to saturate at *fractional* rate $\mathcal{O}(L^{-(k-1+\nu)})$. However, in this case the involved constant $c_{\alpha-\sigma,k,\nu}$ decreases at increasing α and, thus, a smoother target function still allows for a better approximation, as expected.

We remark that the results of Theorem 3 continue to hold if we assume $W \in \mathscr{C}^{k-1}([-1,1])$ and if $W^{(k-1)}$ satisfies a Hölder condition of order $0 < \nu \leq 1$ only at zero in the sense that there exists a constant $C_W > 0$ such that

$$|W^{(k-1)}(0) - W^{(k-1)}(S)| \leq C_W\, S^\nu \;\; \forall\, S \in [0,1].$$

Let us finally consider the special case of Lipschitz-windows $W \in \mathscr{C}^{k-1,1}([-1,1])$ with $k \in \mathbb{N}$. In this case, $W^{(k)}$ exists almost everywhere and we have $C_W = \|W^{(k)}\|_{\mathrm{L}^\infty([0,1])}$. Consequently, our error estimates in Theorem 3 reduce to

$$\|e_L\|_\sigma \leq \Big(\frac{1}{k!}\, \|W^{(k)}\|_{\mathrm{L}^\infty([0,1])} + 1\Big)\, L^{\sigma-\alpha}\, \|f\|_\alpha$$

for $\alpha - \sigma \leq k$ and

$$\|e_L\|_\sigma \leq \Big(\frac{c_{\alpha-\sigma,k}}{k!}\, \|W^{(k)}\|_{\mathrm{L}^\infty([0,1])}\, L^{-k} + L^{\sigma-\alpha}\Big)\, \|f\|_\alpha$$

for $\alpha - \sigma > k$, showing that the estimates in Theorem 2 remain valid under the weaker assumption $W \in \mathscr{C}^{k-1,1}([-1,1])$.

V. Conclusion

We conclude that the *flatness* of the window W at zero determines the convergence rate of the H^σ-error bounds for the inherent FBP reconstruction error. More precisely, if the first k derivatives vanish at zero and $|W^{(k)}(S)|$ grows like $|S|^\nu$ with $\nu \in (0,1]$, the decay rate of $\|f - f_L\|_\sigma$ is predicted to saturate at fractional order $\mathcal{O}(L^{-(k+\nu)})$ for $\alpha - \sigma > k + \nu$.

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