Sparse Approximation of Videos by Adaptive Linear Splines over Anisotropic Tetrahedralizations

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***Abstract*—We discuss theoretical aspects of sparse representations for videos by linear splines over anisotropic tetrahedralizations. In previous work, we have proposed a locally adaptive algorithm, *adaptive thinning*, for sparse approximation of images. Moreover, we have analyzed the asymptotic behaviour of N-term image approximations by linear splines over anisotropic Delaunay triangulations. In this paper, we generalize our previous results from image approximation to video approximation, i.e., from the approximation of bivariate to trivariate target functions.**

I. Introduction

During the last few years, there has been an increasing demand in sparse representations of signals. This, in particular, requires the construction of suitable dictionaries $\mathcal{A} = \{\varphi_j\}_{j\in\mathbb{N}}$ to obtain efficient representations of signals f by N-term approximations of the form

$$f_N = \sum_{j\in I_N} \alpha_j \varphi_j, \tag{1}$$

where $N = |I_N| \in \mathbb{N}$ is the size of the index set $I_N \subset \mathbb{N}$. The quality of an N-term approximation (1) is often measured by *rate-distortion curves*, reflecting the required amount of data (measured e.g. in file size of stored information) versus the approximation quality (measured e.g. in *peak signal-to-noise ratio* (PSNR) or in *structural similarity index* (SSIM)).

From a viewpoint of approximation theory, one important quality indicator is the decay rate of asymptotic N-term approximations $\{f_N\}_{N\in\mathbb{N}}$ in (1) that are obtained from the chosen dictionary $\mathcal{A}$. Popular methods for N-term image approximations can be found in [1], [6], [7], [11], [12].

In previous work [3], [10], we proposed N-term image approximations with optimal decay rates for relevant classes of target functions f, including bivariate horizon functions across α-Hölder smooth horizon boundaries. The decay rates in [3] were obtained from error estimates of the form

$$\|f - f_N\|^2_{L^2([0,1]^2)} = \mathcal{O}(N^{-\alpha}) \quad \text{for } N \to \infty, \tag{2}$$

where f_N is a (bivariate) linear spline over an anisotropic triangulation. In this case, the dictionary $\mathcal{A}$ is generated by all possible linear spline spaces over conformal triangulations that are covering the image domain. Therefore, the dictionary $\mathcal{A}$ is very large. But in [2], [4] we proposed an efficient image approximation algorithm of complexity $\mathcal{O}(N\log(N))$, termed *adaptive thinning* (AT), to compute a suitable sequence of spline spaces $\{\mathcal{S}_N\}_{N\in\mathbb{N}}$ over anisotropic Delaunay triangulations which are locally adapted to the geometry of the image. Our constructive approach in [2], [4] outputs a sequence of image approximations $f_N \in \mathcal{S}_N$ that are well-adapted to the local regularity of the target function f.

In this paper, we generalize the approximation method of [3], [10] from image to video approximation, i.e., from the approximation of *bivariate* functions to the approximation of *trivariate* functions. To this end, we introduce a class of piecewise affine-linear trivariate horizon functions, with singularities along α Hölder smooth surfaces. We approximate these prototypical test functions by linear splines over anisotropic tetrahedralizations. Moreover, we show how to maintain the decay rates of the asymptotic N-term approximations in (2).

The outline of this paper is as follows. In Section II, we briefly introduce linear splines over (conformal) tetrahedralizations. Then, in Section III, we turn to the approximation of trivariate horizon functions by trivariate linear splines. For the purpose of illustration, numerical simulations for a popular test case of video approximation are presented in Section IV.

II. Linear Splines over Tetrahedralizations

For a finite point set $Y \subset \mathbb{R}^3$, a (conformal) *tetrahedralization* $\mathcal{T} \equiv \mathcal{T}(Y)$ is a finite set $\mathcal{T} = \{T\}_{T\in\mathcal{T}}$ of tetrahedra satisfying the following properties.

(a) the vertex set of $\mathcal{T}$ is Y;
(b) two distinct tetrahedra in $\mathcal{T}$ intersect at most at one common vertex, common edge or common triangle;
(c) the convex hull $\mathrm{conv}(Y)$ of Y coincides with the area covered by the union of the tetrahedra in $\mathcal{T}$.

A tetrahedralization $\mathcal{T}$ of Y is called *Delaunay tetrahedralization* of Y, iff no circumsphere of a tetrahedron $T \in \mathcal{T}$ contains any point from Y in its interior. We recall that the Delaunay tetrahedralization $\mathcal{T}$ of Y is unique, provided that no five points in Y are co-spherical.

We remark that there are efficient algorithms for computing the Delaunay tetrahedralization (or its dual Voronoi diagram) for a given point set Y of size $N = |Y|$, where the expected combinatorial complexity is $\mathcal{O}(N)$ on average [8], although the worst case complexity is $\mathcal{O}(N^2)$ [9], [13].

In the following of this paper, we assume that $Y \subset \mathbb{R}^3$ is a set of pixel positions, where we require that the convex hull $\mathrm{conv}(Y)$ of Y coincides with the video domain, which, for simplicity, we assume to be the unit cube $[0,1]^3$, i.e., $\mathrm{conv}(Y) = [0,1]^3$. Moreover, we associate with any tetrahedralization $\mathcal{T}$ of Y the finite dimensional linear function space of *linear splines* over $\mathcal{T}$,

$$\mathcal{S}_{\mathcal{T}} = \left\{ g \in \mathscr{C}([0,1]^3) \ : g|_T \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T} \right\},$$

consisting of all continuous functions on $[0,1]^3$ whose restriction to any tetrahedron $T \in \mathcal{T}$ is a linear polynomial in $\mathcal{P}_1$.

Note that for any function $f \in \mathscr{C}([0,1]^3)$, there is a unique linear spline interpolant $s \in \mathcal{S}_{\mathcal{T}}$ to f over the vertices Y of $\mathcal{T}$ satisfying $s|_Y = f|_Y$. In particular, any linear spline $s \in \mathcal{S}_{\mathcal{T}}$ is uniquely determined by its values at the vertices Y of $\mathcal{T}$.

III. N-Term Approximation of Horizon Functions

In this section, we discuss asymptotic N-term approximations (1) by linear splines $f_N \in \mathcal{S}(\mathcal{T}_N)$ over anisotropic tetrahedralizations $\mathcal{T}_N$, for $N \in \mathbb{N}$. To this end, we explain how to construct sequences of tetrahedralizations $\{\mathcal{T}_N\}_{N \in \mathbb{N}}$, for vertex sets Y_N, such that there are constants $C, M > 0$ (independent of N) satisfying the following two properties.

(a) The size $|Y_N|$ of Y_N is bounded by $|Y_N| \leq M \times N$;
(b) the L^2-approximation error can be bounded above by

$$\|f - f_N\|^2_{L^2([0,1]^3)} \leq C N^{-\alpha},$$

where $f_N \in \mathcal{S}(\mathcal{T}_N)$ is the unique linear interpolant to f at Y_N, and where $\alpha > 0$ is related to the regularity of f.

Horizon functions [7] are popular prototypes for piecewise smooth images with discontinuities along Hölder smooth curves, exemplifying edges. In order to extend the model problem of horizon functions [7] from bivariate functions (i.e., images) to trivariate functions (i.e., videos), we first recall that a bivariate function $g : [0,1]^2 \to \mathbb{R}$ is said to be *Hölder continuous of order* $\beta \in (0,1]$, $g \in \mathscr{C}^\beta([0,1]^2)$, iff it satisfies

$$|g(x) - g(y)| \leq C\|x - y\|^\beta \qquad \text{for all } x, y \in [0,1]^2$$

for some $C > 0$. Moreover, for $\alpha = r + \beta$, with $r \in \mathbb{N}_0$ and $\beta \in (0,1]$, a function $g \in \mathscr{C}^r([0,1]^2)$ is said to be α-Hölder smooth, iff $\partial^\gamma g \in \mathscr{C}^\beta([0,1]^2)$ for all $\gamma \in \mathbb{N}_0^2$ with $|\gamma| = r$. Moreover, the linear space $\mathscr{C}^\alpha([0,1]^2)$ of all α-Hölder smooth functions over $[0,1]^2$ is equipped with the usual semi-norm

$$|g|_\alpha = \inf\{C : |\partial^\gamma g(x) - \partial^\gamma g(y)| \leq C\|x-y\|^\beta \ \forall x, y \in [0,1]^2\}.$$

In the following, we only require $\alpha \in (1,2]$, i.e., $\alpha = 1 + \beta$ for $\beta = \alpha - 1 \in (0,1]$. In this case, $\partial^\gamma g \in \mathscr{C}^{\alpha-1}([0,1]^2)$, for all $\gamma \in \mathbb{N}_0^2$ with $|\gamma| = 1$, where we let

$$|\partial^\gamma g|_{\alpha-1} = |g|_\alpha \qquad \text{for } g \in \mathscr{C}^\alpha([0,1]^2) \text{ and } |\gamma| = 1.$$

Now the class of α-horizon functions contains all piecewise affine-linear trivariate functions across α-Hölder smooth horizon surfaces, according to the following definition.

Definition 1: For any $\alpha \in (1,2]$, a function $f : [0,1]^3 \to \mathbb{R}$ is said to be an *α-horizon function*, iff it has the form

$$f(x,y,z) = \begin{cases} p(x,y,z) & \text{for } z \leq g(x,y), \\ q(x,y,z) & \text{otherwise,} \end{cases}$$

for affine-linear functions $p, q : \mathbb{R}^3 \to \mathbb{R}$ and $g \in \mathscr{C}^\alpha([0,1]^2)$ satisfying $g([0,1]^2) \subset (0,1)$. The α-Hölder smooth surface $g \in \mathscr{C}^\alpha([0,1]^2)$ is called *horizon boundary* of f. □

For the sake of brevity we decided to restrict ourselves to the approximation of horizon functions, although our approximation scheme can also be applied to piecewise smooth functions with one-dimensional singularities or with point singularities.

A. Approximation over Conformal Tetrahedralizations

We start with the approximation of horizon functions f over conformal tetrahedralizations $\mathcal{T}_N$. Our goal is to construct a sequence $\{\mathcal{T}_N\}_{N \in \mathbb{N}}$ of tetrahedralizations $\mathcal{T}_N$ in such a way, that the horizon boundary g is surrounded by an ε_N-corridor $K_{\varepsilon_N} \subset [0,1]^3$. To this end, we interpolate the horizon boundary g by a second order B-spline surface $P_N : [0,1]^2 \to \mathbb{R}$,

$$P_N(\mathbf{x}) = \sum_{i=0}^{n} \sum_{j=0}^{n} D_{i,j} N_{i,1}(x) N_{j,1}(y) \quad \text{ for } \mathbf{x} = (x,y), \quad (3)$$

where the samples $D_{i,j} = (\mathbf{x}_{i,j}, g(\mathbf{x}_{i,j}))$, $0 \leq i,j \leq n$ are taken over a regular grid in $[0,1]^2$ containing $n^2 = N$ cells.

We recall the following result from spline approximation.

Lemma 1: The L^∞-error between g and its interpolating surface $P_N(g)$ is bounded by

$$\|g - P_N(g)\|^2_{L^\infty([0,1]^2)} \leq C N^{-\alpha}.$$

□

In the following, it will be convenient to let $\varepsilon_N = N^{-\alpha}$, for $N \in \mathbb{N}$. Next we turn to the construction of an ε_N-corridor $K_{\varepsilon_N} \subset [0,1]^3$ containing the horizon boundary g. To this end, consider some $\mathbf{x}' \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$. Then, the spline interpolant P_N in (3) is locally given by a convex combination over the adjacent grid points. In this case, we have the inclusion

$$P_N(\mathbf{x}') \in \operatorname{conv}\{D_{i,j}, D_{i+1,j}, D_{i,j+1}, D_{i+1,j+1}\},$$

where the convex hull $\operatorname{conv}\{D_{i,j}, D_{i+1,j}, D_{i,j+1}, D_{i+1,j+1}\}$ is a non-degenerate tetrahedron, provided that the points $D_{i,j}, D_{i+1,j}, D_{i,j+1}, D_{i+1,j+1}$ are not co-planar.

Due to Lemma 1, the maximum distance between P_N and g is (up to a constant independent of N) less than ε_N. This allows us to construct an ε_N-corridor K_{ε_N} surrounding the surface g by offsetting each tetrahedron along the z-coordinate about offset ε_N. In this case, we have

$$g(\mathbf{x}') \in \operatorname{conv}\{D_{k,\ell} \pm (0,0,\varepsilon_N) : k \in \{i, i+1\}, \ell \in \{j, j+1\}\}.$$

In the following discussion, it will be convenient to let

$$A_{N,i,j} = \operatorname{conv}\{D_{k,\ell} \pm (0,0,\varepsilon_N) : k \in \{i, i+1\}, \ell \in \{j, j+1\}\}$$

for the convex hull of the (i,j)-th tetrahedron's offset. By the union of the convex pieces $A_{N,i,j} \subset [0,1]^3$ we obtain an ε_N-corridor $A_N = K_{\varepsilon_N}$ with the required properties, i.e.,

$$g\left([0,1]^2\right) \subset A_N := \bigcup_{0 \leq i,j \leq n-1} A_{N,i,j}.$$

Now the video domain $[0,1]^3$ is split into three parts, made up by A_N and the two subdomains $A_N^\pm \subset [0,1]^3$ lying above and below A_N, respectively. The construction of the point set

Y_N is now a rather straightforward task: For the grid points $\mathbf{x}_{i,j} \in [0,1]^2$, $0 \le i,j \le n$, we take for Y_N the union

$$Y_N = \bigcup_{0 \le i,j \le n} \left\{ (\mathbf{x}_{i,j}, 0), D_{i,j} \pm (0,0,\varepsilon_N), (\mathbf{x}_{i,j}, 1) \right\}.$$

Therefore, we have $|Y_N| \le 8 \times N$ as required. For the construction of the tetrahedralization $\mathcal{T}_{Y_N}$ of Y_N we split the grid cells individually, such that each tetrahedron in $\mathcal{T}_{Y_N}$ is either entirely contained in A_N or otherwise outside of A_N.

Now we are in a position where we can prove the following L^2-error estimate for conformal tetrahedralizations.

Proposition 1: For $\alpha \in (1,2]$, let $f : [0,1]^3 \to \mathbb{R}$ be an α-horizon function. Then there exist constants $C, M > 0$ (independent of N), such that for any $N \in \mathbb{N}$ there exists a tetrahedralization $\mathcal{T}_N$ with $|\mathcal{T}_N| \le M \times N$ vertices satisfying

$$\|f - f_N\|^2_{L^2([0,1]^3)} \le C N^{-\alpha},$$

where $f_N \in \mathcal{S}_{\mathcal{T}_N}$ interpolates f at the vertices in $\mathcal{T}_N$.

Proof: We approximate f by functions f_N of the form

$$f_N(\mathbf{x}) = \begin{cases} p(\mathbf{x}) & \text{for } \mathbf{x} \in A_N^- \\ q(\mathbf{x}) & \text{for } \mathbf{x} \in A_N^+ \\ g_N(\mathbf{x}) & \text{for } \mathbf{x} \in A_N \end{cases} \qquad \text{for } \mathbf{x} \in [0,1]^3,$$

where g_N is the interpolating linear spline to f at the vertices of the ε_N-corridor A_N. Note that f_N coincides with f outside of A_N, so that

$$\begin{aligned} \|f - f_N\|^2_{L^2([0,1]^3)} &= \|f - f_N\|^2_{L^2(A_N)} \qquad (4) \\ &= \sum_{0 \le i,j \le n-1} \|f - f_N\|^2_{L^2(A_{N,i,j})} \\ &= \sum_{0 \le i,j \le n-1} \sum_{T \in A_{N,i,j}} \|f - f_N\|^2_{L^2(T)}, \end{aligned}$$

where we tetrahedralized the corridor A_N. Hence, it remains to consider the L^2-error over each tetrahedron T in A_N.

Now note that the restriction $f_N|_T$ of f_N to a tetrahedron T interpolates f at the four vertices of T. Moreover, since both f and f_N are affine-linear functions outside the ε_N-corridor A_N, we can bound the L^2-error on tetrahedron T by

$$\|f - f_N\|^2_{L^2(T)} \le C \cdot |T| \cdot \|f\|^2_{L^2(T)},$$

where $|T|$ is the volume of T in $\mathbb{R}^3$, and where the constant C is independent of N. Therefore, the L^2-error in (4) can be bounded above by

$$\begin{aligned} \|f - f_N\|^2_{L^2([0,1]^3)} &\le \sum_{0 \le i,j \le n-1} \sum_{T \in A_{i,j,N}} C \cdot |T| \cdot \|f\|^2_{L^2(T)} \\ &\le C \sum_{0 \le i,j \le n-1} |A_{i,j,N}| \cdot \|f\|^2_{L^2(A_{i,j,N})}. \end{aligned}$$

By our construction of $A_{N,i,j}$, and since $g \in \mathscr{C}^\alpha([0,1]^2)$ with $\alpha \in (1,2]$, we can bound the volume of each $A_{N,i,j}$ by

$$|A_{N,i,j}| \le C N^{-\alpha}.$$

This allows us to refine our estimate on the L^2-error (4) by

$$\begin{aligned} \|f - f_N\|^2_{L^2([0,1]^3)} &\le C \sum_{0 \le i,j \le n-1} |A_{i,j,N}| \cdot \|f\|^2_{L^2(A_{i,j,N})} \\ &\le C \sum_{0 \le i,j \le n-1} N^{-\alpha} \cdot \|f\|^2_{L^2(A_{i,j,N})} \\ &\le C N^{-\alpha} \|f\|^2_{L^2(A_N)} \\ &\le C N^{-\alpha} \|f\|^2_{L^2([0,1]^3)}, \end{aligned}$$

which completes our proof. ■

B. Approximation over Delaunay Tetrahedralizations

Now we turn to the construction of Delaunay tetrahedralizations, where we wish to maintain the asymptotic L^2 error estimate of Proposition 1. To this end, recall that in our construction of conformal tetrahedralizations, any tetrahedron $T \in \mathcal{T}_N$ is either fully contained in A_N or outside of A_N, i.e.,

$$T \cap A_N \ne \emptyset \Longrightarrow T \subset A_N \quad \text{ for all } T \in \mathcal{T}_N. \qquad (5)$$

For the construction of the (more restrictive) Delaunay tetrahedralizations of Y_N, we can no longer maintain property (5).

Therefore, we consider refining the point set Y_N as follows.

Lemma 2: Let $f : [0,1]^3 \to \mathbb{R}$ be an α-horizon function, where $\alpha \in (1,2]$. Then there exists a refinement Z_N of the point set Y_N, where $|Z_N| \le M \times N$ (with M independent of N), such that any tetrahedron T in the Delaunay tetrahedralization $\mathcal{D}(Z_N)$ of Z_N satisfies property (5).

Proof: We can only give a short sketch of the proof. Our complete proof is contained in [14].

Recall that the Delaunay tetrahedralization of Y_N is the dual to its *Voronoi diagram* (see [13] for details). Therefore, any edge connecting $\mathbf{y}_i, \mathbf{y}_j \in Y_N$ satisfies the Delaunay property if there exists some $\mathbf{x} \in \mathbb{R}^3$ satisfying

(i) $\|\mathbf{x} - \mathbf{y}_i\| = \|\mathbf{x} - \mathbf{y}_j\|$
(ii) $\|\mathbf{x} - \mathbf{y}_i\| < \|\mathbf{x} - \mathbf{y}_k\|$ for all $k \ne i,j$.

Now let T be a tetrahedron which does not satisfy (5), so that $T \cap A_N \ne \emptyset$ and $T \cap ([0,1]^3 \setminus A_N) \ne \emptyset$. Then, there is at least one edge e in T passing through the boundary of A_N, and so there is one point $\mathbf{x} \in e$ satisfying properties (i), (ii).

To exclude edges e satisfying (i) and (ii), we refine the grid in such a way, that the new points are close enough to e and, moreover, split the edge e in two pieces.

Tetrahedra violating (5) are still possible, if the slope of g exhibits large local variations. In this case, we can refine Y_N accordingly to satisfy (5). To further explain this, we can select parameters $a, v, u \in \mathbb{N}$ (independent of N), where a depends on $|g|_\alpha$ and where v, u depend on $\max_{\mathbf{x} \in [0,1]^3} \|\partial_{x_i} g\|_{\infty,([0,1]^2)}$, $i \in \{1,2\}$, respectively.

Now we finally refine each cell of the regular grid by av cells along the x-coordinate and by au cells along the y-coordinate to construct Z_N over this new grid. Then, any tetrahedron in $\mathcal{D}(Z_N)$ satisfies (5) and, moreover, we have

$$|Z_N| \le av \times au \times |Y_N| \le av \times au \times 8 \times N.$$

■

We can summarize the discussion of this section as follows.

Theorem 1: Let $f : [0,1]^3 \to \mathbb{R}$ be an α-horizon function, where $\alpha \in (1,2]$. Then there exist constants $C, M > 0$ independent of N, such that there is a Delaunay tetrahedralization $\mathcal{D}_N = \mathcal{D}(Z_N)$, with $|Z_N| \leq M \times N$ for all $N \in \mathbb{N}$, satisfying

$$\|f - f_N\|^2_{L^2[0,1]^3} \leq CN^{-\alpha},$$

where $f_N \in S_{\mathcal{D}_N}$ is the spline interpolant to f over Z_N. $\square$

IV. Numerical Simulation by Adaptive Thinning

For the purpose of illustration, we present one numerical example, relying on the popular test video called `Suzie`, comprising 30 image frames. Figure 1 shows a sequence of five image frames, each of size 264×264, along with the approximation by adaptive thinning (AT), as obtained in [5].

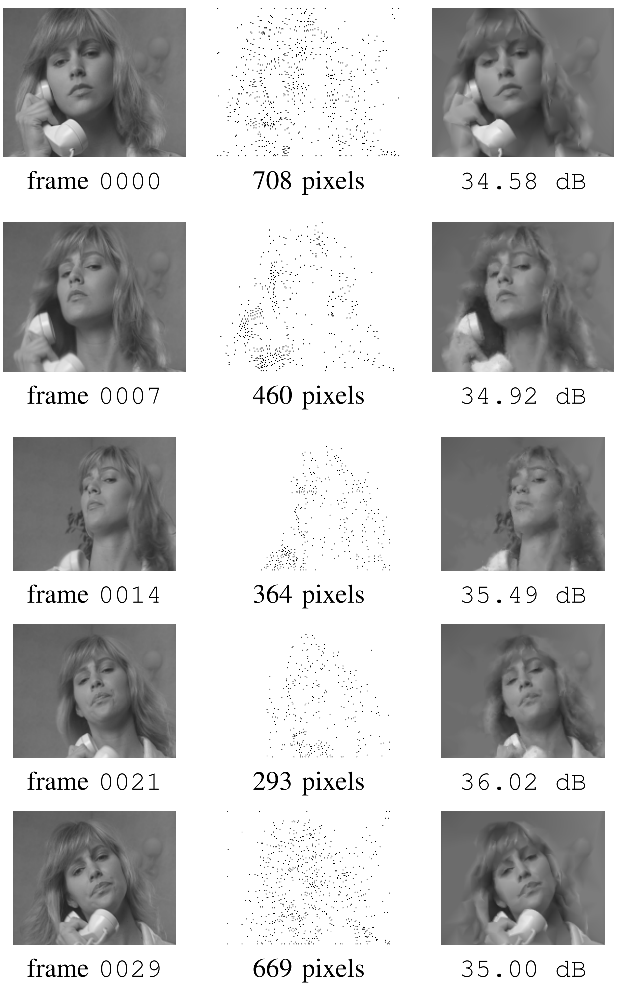

Fig. 1: Video approximation by adaptive thinning [5]. First column: original image frame; second column: significant pixels; third column: approximation by adaptive thinning, along with the PSNR value (measured in dB).

A more detailed description on the numerical results is contained in [5], where the entire set of image frames is shown. Note that adaptive thinning achieves to reconstruct the test data very well, especially the geometric features of the video. Moreover, note that the representation of the video data by the significant pixels is very sparse. This is due to a well-adapted distribution of the significant pixels, shown in Figs. 1-2. Their corresponding Delaunay tetrahedralization is in Fig. 2, where for three frames their intersecting tetrahedra are displayed.

We can conclude that adaptive thinning is quite competitive, as this is further supported by our numerical results in [5].

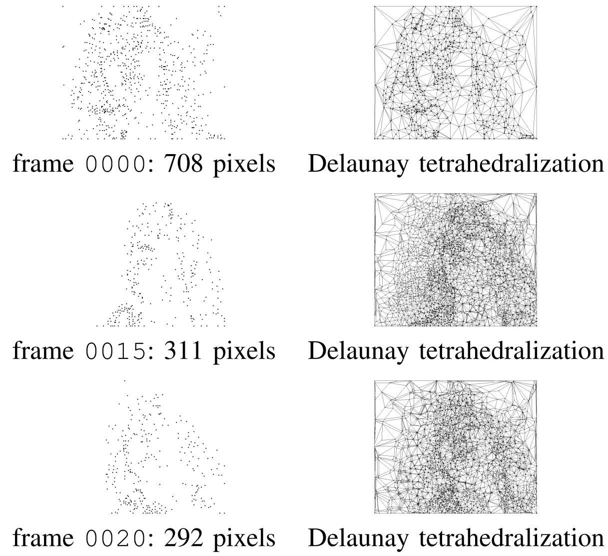

Fig. 2: Video approximation by adaptive thinning (AT) [5]. Significant pixels and Delaunay tetrahedralization.

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