Sobolev Error Estimates for Filtered Back Projection Reconstructions

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Abstract—This paper concerns the approximation of bivariate functions using the filtered back projection (FBP) formula from computerized tomography. To this end, we prove error estimates and convergence rates for the FBP reconstruction method for target functions \( f \) from a Sobolev space \( H^\alpha(\mathbb{R}^2) \) of fractional order \( \alpha > 0 \), where we bound the FBP reconstruction error with respect to the (weaker) norms of the (rougher) Sobolev spaces \( H^\sigma(\mathbb{R}^2) \), for \( 0 \leq \sigma \leq \alpha \). The results of this paper generalize previous of our findings in [2]–[4] for \( L^2 \)-error estimates, i.e., for the case \( \sigma = 0 \), to Sobolev error estimates for all fractional orders \( \sigma \in [0, \alpha] \) and provide criteria to assess the performance of the utilized low-pass filter by means of its window function.

I. INTRODUCTION

The term filtered back projection (FBP) refers to a classical reconstruction technique in computerized tomography (CT), which deals with recovering the interior structure of a scanned object from X-ray scans. This X-ray data can be interpreted as a finite set of line integrals of the (unknown) attenuation function of the scanned object which describes the amount of energy that is absorbed by the medium.

We state the CT reconstruction problem as follows.

Problem 1 (Basic reconstruction problem): For \( \Omega \subset \mathbb{R}^2 \) reconstruct a bivariate function \( f \in L^1(\Omega) \) on its domain \( \Omega \) from given Radon data
\[
\{ Rf(t, \theta) \mid t \in \mathbb{R}, \ \theta \in [0, \pi) \},
\]
where the Radon transform \( Rf \) of \( f \) is defined as
\[
Rf(t, \theta) = \int_{\{ x \cos(\theta) + y \sin(\theta) = t \}} f(x, y) \, dx \, dy
\]
for \( (t, \theta) \in \mathbb{R} \times [0, \pi) \).

Thus, the CT reconstruction problem seeks for the inversion of the Radon transform \( R \). For a comprehensive mathematical treatment of \( R \) and its inversion, we refer to [6], [11].

In previous work [2]–[4] we derived \( L^2 \)-error estimates and convergence rates for target functions \( f \) from fractional Sobolev spaces \( H^\alpha(\mathbb{R}^2) \), where \( \alpha > 0 \). More recently, we also proved Sobolev error estimates and convergence rates [1]. Although we use some of the results from [1] here, the primary goal of this paper is to generalize our previous results in [2] from \( L^2 \)-error estimates to Sobolev error estimates in the rougher Sobolev spaces \( H^\sigma(\mathbb{R}^2) \), for \( \sigma \in [0, \alpha] \).

The outline of this paper is as follows. In Section IV we consider the inversion of the Radon transform by the classical FBP formula. Further, we describe how the FBP can be stabilized by using suitable low-pass filters of finite bandwidth and with a compactly supported window function. This standard approach leads us to an approximate reconstruction formula, whose approximation quality will be evaluated in this paper. To this end, in Section III we derive Sobolev error estimates for target functions from Sobolev spaces of fractional order. Additionally, we state asymptotic convergence rates as the bandwidth goes to infinity in Section IV. Asymptotic Sobolev error estimates with weaker assumptions are finally provided in Section V.

II. FILTERED BACK PROJECTION

The inversion of the Radon transform \( R \) is well understood and given by the classical filtered back projection formula
\[
f(x, y) = \frac{1}{2} B(F^{-1}||S|F(Rf)(S, \theta))(x, y),
\]
which holds for \( f \in L^1(\mathbb{R}^2) \cap C(\mathbb{R}^2) \) (see [5, Theorem 6.2.]). Here, the univariate Fourier transform \( F \) applies to variable \( S \) and the back projection \( Bh \) of \( h \in L^1(\mathbb{R} \times [0, \pi)) \) is defined as
\[
Bh(x, y) = \frac{1}{\pi} \int_0^\pi h(x \cos(\theta) + y \sin(\theta), \theta) \, d\theta
\]
for \( (x, y) \in \mathbb{R}^2 \). Note that, up to the constant \( \frac{1}{2} \), the back projection operator \( B \) is the adjoint operator of the Radon transform \( R \).

We remark that the FBP formula is numerically unstable. By applying the filter \( |S| \) to the Fourier transform \( F(Rf) \) in (1), especially the high frequency components of \( Rf \) are amplified by the magnitude of \( |S| \). Thus, the filtered back projection formula is in particular highly sensitive with respect to noise.

To reduce the noise sensitivity of the FBP formula, we follow a standard approach and replace the filter \( |S| \) in (1) by a low-pass filter \( A_L \) of the form
\[
A_L(S) = |S| W^{(S/L)} \quad \text{for } S \in \mathbb{R}
\]
with finite bandwidth \( L > 0 \) and an even window function \( W \in L^\infty(\mathbb{R}) \) with compact support \( \text{supp}(W) \subseteq [-1, 1] \).

When replacing the filter \( |S| \) in (1) by the low-pass filter \( A_L(S) \), the reconstruction of \( f \) is no longer exact and we only get an approximate FBP reconstruction, denoted by \( f_L \).

However, for target functions \( f \in L^1(\mathbb{R}^2) \) the reconstruction \( f_L \) is for any \( L > 0 \) defined almost everywhere on \( \mathbb{R}^2 \) (see [1, Proposition 3.1]) and, moreover, the resulting approximate FBP formula can be simplified as
\[
f_L = \frac{1}{2} B(F^{-1}A_L \ast Rf).
\]
Further, $f_L$ belongs to $L^2(\mathbb{R}^2)$ (see [1 Propostion 4.2]) and can be expressed in terms of the target function $f$ via

$$f_L = \frac{1}{2} B(\mathcal{F}^{-1} A_L * f) = f * K_L,$$  \hspace{1cm} (3)

where we define the convolution kernel $K_L : \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$K_L(x,y) = \frac{1}{2} B(\mathcal{F}^{-1} A_L)(x,y) \text{ for } (x,y) \in \mathbb{R}^2.$$

For the sake of brevity, we call any application of the approximate FBP formula (3) an FBP method. Therefore, each FBP method provides one approximation $f_L$ to $f$, $f_L \approx f$, whose quality depends on the choice of the low-pass filter $A_L$.

In the following, we analyse the intrinsic error of the FBP method which is incurred by the use of the low-pass filter $A_L$, i.e., we wish to analyse the reconstruction error

$$e_L = f - f_L$$

with respect to the filter's window $W$ and bandwidth $L$.

We remark that pointwise and $L_\infty$-error estimates on $e_L$ were proven by Munshi et al. in [8]. Their theoretical results were further supported by numerical experiments in [9]. Error bounds for the $L^p$-norm of $e_L$, in terms of an $L^p$-modulus of continuity of $f$, were proven by Madych in [7].

In [2]-[4] we derived $L^2$-error estimates and convergence rates for target functions from fractional Sobolev spaces $H^\alpha(\mathbb{R}^2)$. Let us recall that the Sobolev space $H^\alpha(\mathbb{R}^2)$ of order $\alpha \in \mathbb{R}$ is defined as

$$H^\alpha(\mathbb{R}^2) = \{ f \in \mathcal{S}'(\mathbb{R}^2) \mid \| f \|_\alpha < \infty \},$$

where

$$\| f \|_\alpha^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} (1 + x^2 + y^2)^\alpha |\mathcal{F}f(x,y)|^2 \, d(x,y),$$

and where $\mathcal{S}'(\mathbb{R}^2)$ denotes the Schwartz space of tempered distributions on $\mathbb{R}^2$.

We remark that in relevant applications of (medical) image processing, Sobolev spaces of compactly supported functions,

$$H^\alpha_0(\Omega) = \{ f \in H^\alpha(\mathbb{R}^2) \mid \text{supp}(f) \subseteq \overline{\Omega} \},$$

on an open and bounded domain $\Omega \subset \mathbb{R}^2$, and of fractional order $\alpha > 0$ play an important role (cf. [10]). In fact, we can consider the density of an image in $\Omega \subset \mathbb{R}^2$ as a function from the Sobolev space $H^\alpha_0(\Omega)$ whose order $\alpha$ is close to $\frac{1}{2}$.

III. Error Analysis

In this section, we analyse certain Sobolev norms of the inherent FBP reconstruction error $e_L$ for target functions $f$ from the Sobolev space $H^\alpha(\mathbb{R}^2)$ of fractional order $\alpha > 0$. To be more precise, we generalize our $L^2$-error estimates of [4] to $H^\sigma$-error estimates for $0 \leq \sigma \leq \alpha$. To this end, we partly rely on [1], as this is indicated in the following discussion.

Let us assume that $f \in L^1(\mathbb{R}^2) \cap H^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$. We first show that the approximate FBP reconstruction $f_L$ belongs to the Sobolev space $H^\sigma(\mathbb{R}^2)$ for $0 \leq \sigma \leq \alpha$.

In [1 Proposition 4.1] we have proven that the convolution kernel $K_L$ belongs to $C_0(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and, moreover, that its Fourier transform is given by

$$\mathcal{F}K_L(x,y) = W_L(x,y) \text{ for almost all } (x,y) \in \mathbb{R}^2.$$

Here, the compactly supported bivariate window function $W_L \in L^\infty(\mathbb{R}^2)$ is defined as

$$W_L(x,y) = W\left(\frac{r(x,y)}{L}\right) \text{ for } (x,y) \in \mathbb{R}^2,$$

where we let

$$r(x,y) = \sqrt{x^2 + y^2} \text{ for } (x,y) \in \mathbb{R}^2.$$

This in combination with representation (3) for $f_L$ yields

$$\| f_L \|_\sigma^2 = \| f * K_L \|_\sigma^2$$

$$= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} (1 + r(x,y)^2)^\sigma |(W_L \cdot \mathcal{F}f)(x,y)|^2 \, d(x,y)$$

$$\leq \left( \sup_{r(x,y) \leq L} \| W_L(x,y) \| \right)^2 \| f \|_\alpha^2 = \| W_L \|_{\infty,-1,1}^2 \| f \|_\alpha^2.$$

Thus, for $f \in L^1(\mathbb{R}^2) \cap H^\alpha(\mathbb{R}^2)$ with $\alpha > 0$, the approximate FBP reconstruction $f_L$ belongs to $H^\sigma(\mathbb{R}^2)$ for all $0 \leq \sigma \leq \alpha$.

Let us now turn to the analysis of the FBP reconstruction error $e_L = f - f_L$ with respect to the $H^\sigma$-norm. For $\gamma \geq 0$, we define

$$r_\gamma(x,y) = (1 + r(x,y)^2)^\gamma = (1 + x^2 + y^2)^\gamma \text{ for } (x,y) \in \mathbb{R}^2$$

so that the $H^\sigma$-norm of $e_L$ can be expressed as

$$\| e_L \|_\sigma^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} r_\gamma(x,y) |\mathcal{F}(f - f_L)(x,y)|^2 \, d(x,y)$$

$$= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} r_\gamma(x,y) |(\mathcal{F}f - W_L \cdot \mathcal{F}f)(x,y)|^2 \, d(x,y)$$

$$= I_1 + I_2,$$

where

$$I_1 = \frac{1}{4\pi^2} \int_{B_L} r_\gamma(x,y) |1 - W_L(x,y)|^2 |\mathcal{F}f(x,y)|^2 \, d(x,y)$$

with

$$B_L = \{(x,y) \in \mathbb{R}^2 \mid r(x,y) \leq L\}$$

and where

$$I_2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2 \setminus B_L} r_\gamma(x,y) |\mathcal{F}f(x,y)|^2 \, d(x,y).$$

For $\gamma \geq 0$, we define

$$\Phi_{\gamma,W}(L) = \sup_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\gamma} \text{ for } L > 0$$

so that we can bound $I_1$ from above by

$$I_1 \leq \left( \sup_{r(x,y) \in B_L} (1 - W_L(x,y))^2 \right) \| f \|_\alpha^2 = \Phi_{\alpha-W,L}(L) \| f \|_\alpha^2,$$

since

$$\sup_{r(x,y) \in B_L} \frac{(1 - W_L(x,y))^2}{r_{\alpha-\sigma}(x,y)} = \sup_{S \in [-L,L]} \frac{(1 - W(S))^2}{(1 + S^2)^{\alpha-\sigma}}.$$
For $0 \leq \sigma \leq \alpha$, we can bound $I_2$ by

$$I_2 \leq L^2(\sigma-\alpha) \frac{1}{4}\int_{F(x,y)} \left| F(x,y) \right|^2 d(x,y) \leq L^2(\sigma-\alpha) \| f \|^2 \alpha.$$

Combining the estimates for $I_1$ and $I_2$, we finally obtain

$$\| e_L \|_\sigma \leq \left( \Phi_{\alpha-\sigma,W}(L) + L^{2(\sigma-\alpha)} \right) \| f \|^2 \alpha,$$

where

$$\Phi_{\alpha-\sigma,W}(L) = \sup_{S \in [-1,1]} \frac{(1 - W(S))}{(1 + L^2 S^2)^{\alpha-\sigma}}$$

for $L > 0$.

We remark that for the special case $\sigma = 0$, the bound in (4) agrees with the $L^2$-error estimate of [11, Theorem 4.1].

Like the $L^2$-error bound in [2, Theorem 4.1], the $H^s$-error estimate (4) involves the error term $\Phi_{\gamma,W}(L)$, but now with $\gamma = \alpha - \sigma$ rather than $\gamma = \alpha$. Consequently, we can rely on the analysis in [2] concerning the properties of $\Phi_{\gamma,W}(L)$.

In [2, Theorem 2.4] we have proven that, if the window function $W$ is continuous on $[-1,1]$ and $W(0) = 1$, the error term $\Phi_{\gamma,W}(L)$ converges to 0 as $L$ goes to $\infty$ for all $\gamma > 0$. With this we get the following convergence result for the $H^s$-norm of the FBP reconstruction error $e_L = f - f_L$.

$$\| e_L \|_\sigma = o(1) \quad \text{for } L \to \infty.$$

We remark that the result in Corollary 1 is not covered by our previous paper [1]. Indeed, throughout [1], we only rely on the weaker assumption $W \in L^\infty(\mathbb{R})$ for the filter’s window function $W$ rather than on $W \in \mathcal{C}([-1,1])$.

### IV. RATE OF CONVERGENCE

In this section we analyse the convergence rate of the FBP reconstruction error $\| e_L \|_\sigma$ as $L$ goes to $\infty$. To this end, we partly rely on [11] as indicated in the following discussion.

Let $S_{\gamma,W,L} = \{0, 1\}$, for $\gamma \geq 0$, denote the smallest maximizer in $[0,1]$ of the even function

$$\Phi_{\gamma,W,L}(S) = \frac{(1 - W(S))}{(1 + L^2 S^2)^\gamma} \quad \text{for } S \in [-1,1].$$

To determine the rate of convergence for $\| e_L \|_\sigma$, we assume that $S_{\alpha-\sigma,W,L}$ is uniformly bounded away from 0, i.e., there exists a constant $c_{\alpha-\sigma,W} > 0$ satisfying

$$S_{\alpha-\sigma,W,L} \geq c_{\alpha-\sigma,W} \quad \text{for all } L > 0. \quad (5)$$

Then, the error term $\Phi_{\alpha-\sigma,W}(L)$ is bounded above by

$$\Phi_{\alpha-\sigma,W}(L) = \frac{(1 - W(S_{\alpha-\sigma,W,L}))}{(1 + L^2(S_{\alpha-\sigma,W,L})^\gamma)} \leq c_{\alpha-\sigma,W} L^{2(\sigma-\alpha)} \| f \|^2 \alpha.$$

Consequently, we obtain

$$\| e_L \|_\sigma \leq \left( c_{\alpha-\sigma,W} L^{2(\sigma-\alpha)} \| f \|^2 \alpha + 1 \right) L^{2(\sigma-\alpha)} \| f \|^2 \alpha,$$

so that

$$\| e_L \|_\sigma = O(L^{-(\alpha-\sigma)}) \quad \text{for } L \to \infty.$$

In particular, we obtain

$$\| e_L \|_\sigma = O(L^{-(\alpha-\sigma)}) \quad \text{for } L \to \infty.$$

Note that the decay rate $\alpha - \sigma$ is determined by the difference between the smoothness $\alpha$ of the target function $f$ and the order $\sigma$ of the Sobolev norm $\| \cdot \|_\sigma$ in which the reconstruction error $e_L$ is measured.

We remark that assumption (5) is satisfied for a large class of window functions. For example, let $W$ satisfy

$$W(S) = 1 \quad \text{for all } S \in (-\varepsilon, \varepsilon)$$

for some $\varepsilon \in (0,1)$. Then, assumption (5) is fulfilled with the constant $c_{\alpha-\sigma,W} = \varepsilon$ for all $0 \leq \sigma \leq \alpha$.

However, there are commonly used window functions $W$ which do not satisfy assumption (5). In fact, in [2] we investigated the behaviour of $S_{\gamma,W,L}$ and $\Phi_{\gamma,W,L}$ for $\gamma > 0$ numerically for the following window functions of the filter $A_L(S) = |S| W(S/L)$:

| Name   | $W(S)$ for $|S| \leq 1$ | Parameter |
|--------|-------------------------|-----------|
| Shepp–Logan | $\sin(\pi S/2)$ | $-$ |
| Cosine  | $\cos(\pi S/2)$ | $-$ |
| Hamming | $\beta(1 - \beta) \cos(\pi S)$ | $\beta \in [1/2, 1]$ |
| Gaussian | $\exp(-\pi S^2/\beta^2)$ | $\beta > 1$ |

We summarize our numerical results from [2] as follows. For $\gamma < 2$, we found that assumption (5) is fulfilled and

$$\Phi_{\gamma,W}(L) = O(L^{-2\gamma}) \quad \text{for } L \to \infty.$$

For $\gamma \geq 2$, assumption (5) is not fulfilled, since

$$S_{\gamma,W,L} \to 0 \quad \text{for } L \to \infty,$$

and the convergence rate of $\Phi_{\gamma,W}$ stagnates by

$$\Phi_{\gamma,W}(L) = O(L^{-1}) \quad \text{for } L \to \infty.$$

Note that all window functions in the above table are twice continuously differentiable on $[-1,1]$ with

$$W(0) = 1 \quad \text{and } W'(0) = 0.$$
This motivates us to analyse the convergence behaviour of the error term $\Phi_{\gamma, W}$ for the special case of $\mathcal{C}^k$-window functions whose first $k - 1$ derivatives vanish at zero (cf. [2]).

Consequently, to continue our analysis, we now consider even window functions $W$ with compact support in $[-1, 1]$ that additionally satisfy $W \in \mathcal{C}^k([-1, 1])$ for some $k \geq 2$ with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k - 1.$$  

Under this assumption, we have proven in [2] Theorem 6.1 that for $\gamma \leq k$ the error term $\Phi_{\gamma, W}(L)$ is bounded above by

$$\Phi_{\gamma, W}(L) \leq \frac{1}{(k!)^2} \|W^{(k)}\|_{\infty, [-1, 1]}^2 L^{-2\gamma} \quad \text{for all } L > 0$$

and for $\gamma > k$ by

$$\Phi_{\gamma, W}(L) \leq \begin{cases} \frac{1}{(k!)^2} \|W^{(k)}\|_{\infty, [-1, 1]}^2 L^{-2\gamma} & \text{for } L < \frac{\sqrt{\tau}}{\sqrt{\gamma - k}}, \\ \frac{c_{\gamma,k}}{(k!)^2} \|W^{(k)}\|_{\infty, [-1, 1]} L^{-2k} & \text{for } L \geq \frac{\sqrt{\tau}}{\sqrt{\gamma - k}}, \end{cases}$$

where the constant

$$c_{\gamma,k} = \left( \frac{k}{\gamma - k} \right)^{k/2} \left( \frac{\gamma - k}{\gamma} \right)^{\gamma/2}$$

is strictly monotonically decreasing in $\gamma > k$. In particular,

$$\Phi_{\gamma, W}(L) = O\left(L^{-2\min(k,\gamma)}\right) \quad \text{for } L \to \infty.$$

We remark that this theoretical result complies with our numerical experiments in [2]. In particular, the saturation of the convergence rate at order $O(L^{-2k})$ is observed in [2] and so the numerical experiments in [2] show that the stated convergence rate of $\Phi_{\gamma, W}(L)$ is optimal for the special case of $\mathcal{C}^k$-windows.

Using the above bound of $\Phi_{\gamma, W}(L)$ in Theorem 3 gives the following $H^\sigma$-error estimate for $\mathcal{C}^k$-windows.

**Theorem 3 ($H^\sigma$-error estimate for $\mathcal{C}^k$-windows):** Let the assumptions of Theorem 1 be satisfied. In addition, let $W \in \mathcal{C}^k([-1, 1])$ for $k \geq 2$ with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k - 1.$$  

Then, for $0 \leq \sigma \leq \alpha$, the $H^\sigma$-norm of the inherent FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_{H^\sigma} \leq \frac{1}{(k!)^2} \|W^{(k)}\|_{\infty, [-1, 1]} L^{\sigma - \alpha} \|f\|_{H^\alpha}$$

for $\alpha - \sigma \leq k$, and by

$$\|e_L\|_{H^\sigma} \leq \left( \frac{c_{\alpha - \sigma,k}}{k!} \|W^{(k)}\|_{\infty, [-1, 1]} L^{k - \alpha} + L^{\sigma - \alpha} \right) \|f\|_{H^\alpha}$$

for $\alpha - \sigma > k$ and sufficiently large $L > 0$. In particular,

$$\|e_L\|_{H^\sigma} = O\left(L^{-\min(k,\alpha - \sigma)}\right) \quad \text{for } L \to \infty.$$

Note that in Theorem 3 for $\alpha - \sigma \leq k$ the decay rate of $\|e_L\|_{H^\sigma}$ is determined by the difference between the smoothness $\alpha$ of the target function $f$ and the order $\sigma$ of the considered Sobolev norm, whereas for $\alpha - \sigma > k$ the decay rate saturates at $O(L^{-k})$. Here, $k$ denotes the differentiability order of the window function $W$, whose first $k - 1$ derivatives are required to vanish at zero. However, in this case the error bound still decreases at increasing $\alpha$, since the involved constant $c_{\alpha - \sigma,k}$ is strictly monotonically decreasing in $\alpha - \sigma > k$. Thus, a smoother target function still permits a better approximation, as expected. Nevertheless, the attainable convergence rate is limited by the differentiability order of the filter’s window function.

V. Asymptotic Error Estimates

In this section, we finally derive an asymptotic $H^\sigma$-error estimate for the FBP method under weaker assumptions.

For this purpose, we consider even window functions $W \in L^\infty(\mathbb{R})$ with compact support in $[-1, 1]$ that are $k$-times differentiable only at the origin for some $k \geq 2$ with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k - 1.$$  

As in the previous error estimates of this paper, we consider target functions $f \in L^1(\mathbb{R}^2) \cap H^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$ and analyse the $H^\sigma$-norm of the inherent FBP reconstruction error $e_L = f - f_L$ for $0 \leq \sigma \leq \alpha$.

We again start with splitting the $H^\sigma$-norm of $e_L$ into the sum of two integrals

$$\|e_L\|_{H^\sigma}^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} r_\sigma(x, y) |\mathcal{F}(f - f_L)(x, y)|^2 \, dx \, dy = I_1 + I_2,$$

where

$$I_1 = \frac{1}{4\pi^2} \int_{B_L} r_\sigma(x, y) \left|1 - W_L(x, y)\right|^2 |\mathcal{F}f(x, y)|^2 \, dx \, dy$$

with

$$B_L = \{(x, y) \in \mathbb{R}^2 \mid r(x, y) \leq L\}$$

and where

$$I_2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2 \setminus B_L} r_\sigma(x, y) |\mathcal{F}f(x, y)|^2 \, dx \, dy.$$  

As before, the integral $I_2$ can be bounded above by

$$I_2 \leq L^{2(\alpha - \sigma)} \|f\|_{H^\alpha}^2.$$

The integral $I_1$ can be expressed as

$$I_1 = \frac{1}{4\pi^2} \int_{B_L} r_\sigma(x, y) \left|1 - W\left(\frac{r(x, y)}{L}\right)\right|^2 |\mathcal{F}f(x, y)|^2 \, dx \, dy.$$  

Because $W \in L^\infty(\mathbb{R})$ is $k$-times differentiable at zero, we can apply Taylor’s theorem and, thus, there exists a function $h_k : \mathbb{R} \to \mathbb{R}$ satisfying

$$W(S) = \sum_{j=0}^k \frac{W^{(j)}(0)}{j!} S^j + h_k(S) S^k \quad \text{for all } S \in \mathbb{R}$$

and

$$\lim_{S \to 0} h_k(S) = 0.$$  

By assumption, the window $W$ satisfies

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k - 1.$$
Hence, for $(x, y) \in \mathbb{R}^2$ follows that

$$W\left(\frac{r(x, y)}{L}\right) = 1 + \left(\frac{W^{(k)}(0)}{k!} + h_k\left(\frac{r(x, y)}{L}\right)\right) \left(\frac{r(x, y)}{L}\right)^k$$

for all $L > 0$. If we now define, for $\gamma \geq 0$,

$$\phi^*_{\gamma,L,k} = \max_{(x,y)\in B_L} \frac{r(x,y)}{L}^{2k} \gamma_{\gamma}(x,y) = \max_{S \in [0,1]} \frac{S^{2k}}{(1 + L^2 S^2)^\gamma},$$

then integral $I_1$ can be bounded by

$$I_1 \leq 2 \phi^*_{\sigma-\gamma,L,k} (I_3 + I_4),$$

where we let

$$I_3 = \frac{1}{4\pi^2} \int_{B_L} \left(\frac{W^{(k)}(0)}{k!}\right)^2 r_{\alpha}(x,y) |Ff(x,y)|^2 \, dx \, dy$$

and

$$I_4 = \frac{1}{4\pi^2} \int_{B_L} h_k\left(\frac{r(x,y)}{L}\right)^2 r_{\alpha}(x,y) |Ff(x,y)|^2 \, dx \, dy.$$