Reconstruction of Functions From Generalized Hermite-Birkhoff Data

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Abstract. For a fixed conditionally positive definite function ψ we consider the linear space $\mathcal{R}$ containing all finite linear combinations of translates of ψ augmented by a polynomial part. We equip $\mathcal{R}$ with a topology and provide characterizations for the largest space $\mathcal{T}$ among all linear spaces whose elements are continuous on $\mathcal{R}$. Our approach serves to prove that all conditionally positive definite functions are available to provide solutions to generalized Hermite-Birkhoff interpolation problems, where the data $T_j f$ is generated by actions of finitely many compactly supported linear functionals $T_j \in \mathcal{T}$ on a specific function f.

§1. Introduction

Radial basis functions serve to provide a well-established tool for multivariate interpolation problems (see [1,2,3,10,11] for some surveys). Due to its interpolation scheme, for a fixed continuous radial basis function $\psi : \mathbb{R}^d \to \mathbb{R}$ the interpolant is assumed to be of the form

$$s_{\lambda,p}(x) = \lambda^y \psi(x-y) + p(x), \qquad \lambda \in \mathcal{L}_m^\perp, p \in \mathcal{P}_m^d. \tag{1}$$

Here $\mathcal{P}_m^d$ stands for the linear space of all d-variate polynomials of order m or less and

$$\mathcal{L}_m^\perp := \left\{ \lambda = \sum_{j=1}^N \lambda_j \delta_{x_j} : \lambda p = 0 \text{ for all } p \in \mathcal{P}_m^d, x_j \in \mathbb{R}^d \text{ pairwise distinct} \right\}$$

denotes the space of functionals containing all finite linear combinations of translates of the Dirac δ-functional which are orthogonal on all polynomials from $\mathcal{P}_m^d$.

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The set of all possible interpolants of the form (1) constitute a linear space

$$\mathcal{R} := \left\{ s_{\lambda,p} \,:\, \lambda \in \mathcal{L}_m^{\perp}, p \in \mathcal{P}_m^d \right\},$$

which we here refer to as *recovery space*. We will assume that ψ is *conditionally positive definite* of order m on $\mathbb{R}^d$, for it was established by Micchelli [9], that this ensures the well-posedness of the radial basis function interpolation method.

Definition 1.1. A continuous function $\psi : \mathbb{R}^d \to \mathbb{R}$ is in the class $\mathtt{cpd}(\mathcal{L}_m^{\perp})$ of all conditionally positive definite functions of order m on $\mathbb{R}^d$ if for all $\lambda \in \mathcal{L}_m^{\perp} \setminus \{0\}$ the quadratic form $\lambda^x \lambda^y \psi(x-y)$ is positive.

In order to keep our approach as general as possible, note that the definition of the function class $\mathtt{cpd}(\mathcal{L}_m^{\perp})$ is not restricted to radial functions and on this occasion we shall drop this requirement for ψ.

In this paper, under mild assumptions on ψ we shall provide characterizations of the largest space $\mathcal{T}$ among all spaces containing linear functionals which are continuous on $\mathcal{R}$ with respect to a specific topology to be introduced for $\mathcal{R}$. We identify the topological dual of $\mathcal{T}$ with the *native space* $\mathcal{F}$ that has already been extensively described for the radial basis function interpolation method in earlier papers of Madych & Nelson [6,7,8].

After some discussions on the native space we provide a reconstruction scheme that allows the recovery of a given function $f \in \mathcal{F}$ from finitely many scattered data $T_j f$ generated by actions of compactly supported linear functionals $T_j \in \mathcal{T}$ on f. This serves to provide a very general extension to the radial basis function interpolation method, including multivariate interpolation from scattered Hermite-Birkhoff data. It turns out that $\mathcal{F}$ is maximal among all spaces whose functions can be recovered by our method.

§2. Fundamental Assumptions

Note that the basis function $\psi \in \mathtt{cpd}(\mathcal{L}_m^{\perp})$ completely determines the properties of all of the aforementioned spaces $\mathcal{R}, \mathcal{T}$ and $\mathcal{F}$. Thus it is clear that this function shall play a central role throughout this paper. As a basic hypothesis we require ψ to possess a *generalized Fourier transform* in the distributional sense with respect to the space

$$\mathcal{D} := \left\{ \gamma \in C^{\infty}(\mathbb{R}^d, \mathbb{R}) : \operatorname{supp} \gamma \subset \mathbb{R}^d \text{ compact } \right\}$$

containing all real-valued compactly supported d-variate functions that possess derivatives up to an arbitrary order.

Definition 2.1. The function $\psi \in C(\mathbb{R}^d, \mathbb{R})$ is said to possess a generalized Fourier transform $\varphi \in C(\mathbb{R}^d \setminus \{0\})$ if there is a number $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, which is assumed to be minimal with the property $\|\omega\|^s \varphi(\omega) \in L_1(0)$, such that Parseval's identity generalized as

$$\int_{\mathbb{R}^d} \psi(x)\gamma(x)\,dx = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\gamma}(\omega)\varphi(\omega)\,d\omega \tag{2}$$

holds for all functions γ from

$$\mathcal{D}_s^\perp := \left\{ \gamma \in \mathcal{D} \,:\, \int_{\mathbb{R}^d} \gamma(x)p(x)\,dx = 0 \text{ for all } p \in \mathcal{P}_s^d \right\}.$$

For the rest of this paper we fix the function pair $\psi \in \mathtt{cpd}(\mathcal{L}_m^\perp)$ and φ related by the above definition. According to this, φ is allowed to possess an algebraic singularity of order s around zero, such that the right hand side of (2) is well-defined, since the (classical) Fourier transform

$$\hat{\gamma}(\omega) := \int_{\mathbb{R}^d} e^{-ix^T\omega}\gamma(x)\,dx$$

of a function $\gamma \in \mathcal{D}_s^\perp$ has a zero of order at least s at the origin.

As it is shown in [5], ψ is conditionally positive definite if and only if φ is nonnegative but positive on a nonempty open subset $U \subset \mathbb{R}^d \setminus \{0\}$:

$$\psi \in \mathtt{cpd}(\mathcal{L}_m^\perp) \qquad \Longleftrightarrow \qquad \varphi \geq 0, \varphi \not\equiv 0.$$

Additionally, it is proved in [5] that φ is integrable around infinity, such that there is a number $r \in \mathbb{N}_0$, here assumed to be maximal with the property $\|\omega\|^r \varphi(\omega) \in L_1(\infty)$. Note that the decay of φ at infinity determined by r corresponds to the order of differentiability of ψ.

§3. Structure of Basic Spaces

Our assumption on ψ to be conditionally positive definite of order m on $\mathbb{R}^d$ motivates us to introduce an inner product for the linear space $\mathcal{L}_m^\perp$ by

$$(\mu, \lambda)_\psi := \mu^x \lambda^y \psi(x - y) \qquad \text{for all } \lambda, \mu \in \mathcal{L}_m^\perp \tag{3}$$

which induces the norm $\|\cdot\|_\psi$. This suggests equipping the recovery space $\mathcal{R}$ with a topology given by the (semi-) inner product

$$\langle s_{\mu,q}, s_{\lambda,p} \rangle_\psi := (\mu, \lambda)_\psi \qquad \text{for all } \lambda, \mu \in \mathcal{L}_m^\perp,\ p, q \in \mathcal{P}_m^d.$$

As already mentioned in our introduction we shall characterize the linear space $\mathcal{T}$ containing all linear functionals that are bounded on $\mathcal{R}$. To this end, we consider the action of $\mu \in \mathcal{L}_m^\perp$ on a function $s_{\lambda,p} \in \mathcal{R}$ by

$$\mu s_{\lambda,p} = \mu^x \lambda^y \psi(x - y) = (\mu, \lambda)_\psi$$

which yields the desired estimate

$$|\mu s_{\lambda,p}| \leq \|\mu\|_\psi \|\lambda\|_\psi \qquad \text{for all } \lambda \in \mathcal{L}_m^\perp,\ p \in \mathcal{P}_m^d$$

for each $\mu \in \mathcal{L}_m^\perp$. Thus by continuity we obtain the following theorem.

Theorem 3.1. *Let $\mathcal{T}$ be the closure $\overline{\mathcal{L}_m^\perp}$ of $\mathcal{L}_m^\perp$ with respect to $\|\cdot\|_\psi$. Then, $\mathcal{T}$ is the largest among all spaces containing linear functionals such that for each $\tau \in \mathcal{T}$ the estimate*

$$|\tau s_{\lambda,p}| \le C_\tau \|\lambda\|_\psi \qquad \text{for all } \lambda \in \mathcal{L}_m^\perp,\ p \in \mathcal{P}_m^d$$

holds with a nonnegative constant C_τ. ∎

§4. Characterization of $\mathcal{T}$

Now we proceed to provide some practical characterizations of $\mathcal{T}$. For this purpose, let us recall our assumptions on ψ from section 2. As a consequence of (2), it follows that the identity

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x-y)\gamma(x)\gamma(y)\,dx\,dy = (2\pi)^{-d}\int_{\mathbb{R}^d} |\hat{\gamma}(\omega)|^2\varphi(\omega)\,d\omega \tag{4}$$

holds for all functions $\gamma \in \mathcal{D}_m^\perp$ with $m \ge s/2$. If we regard each function $\gamma \in \mathcal{D}_m^\perp$ as a linear functional whose action γf on a locally integrable function f is defined by

$$\gamma f := \int_{\mathbb{R}^d} \gamma(x) f(x)\,dx,$$

then we can rearrange (4) and topologize $\mathcal{D}_m^\perp$ by

$$(\beta,\gamma)_\psi := \beta^x\gamma^y\psi(x-y) \qquad \text{for all } \gamma,\beta \in \mathcal{D}_m^\perp.$$

This is an inner product for $\mathcal{D}_m^\perp$ with exactly the same definition that we have used in (3) for $\mathcal{L}_m^\perp$. In order to justify this notation, for a fixed functional $\lambda \in \mathcal{L}_m^\perp$ we consider the functions $\gamma_{n,\lambda}(x) := \lambda^y\gamma_{n,\delta}(x-y)$, where the sequence $\{\gamma_{n,\delta}\}_{n\in\mathbb{N}} \subset \mathcal{D}$ defines a specific *summability kernel*, generating Dirac's δ-distribution. A possible example for a summability kernel can be found in [12], Lemma 28.1. After verifying that each function $\gamma_{n,\lambda}$ is in $\mathcal{D}_m^\perp$, we obtain the identity

$$\gamma_{n,\lambda}^x\gamma_{n,\lambda}^y\psi(x-y) = (2\pi)^{-d}\int_{\mathbb{R}^d} |\hat{\gamma}_{n,\lambda}(\omega)|^2\varphi(\omega)\,d\omega$$

which tends, for $n \to \infty$, to

$$\lambda^x\lambda^y\psi(x-y) = (2\pi)^{-d}\int_{\mathbb{R}^d} |\sigma_\lambda(\omega)|^2\varphi(\omega)\,d\omega,$$

where the right hand side contains the *symbol function* $\sigma_\lambda(\omega) := \lambda^x e^{-ix^T\omega}$. This is the Fourier transform of the compactly supported functional λ. Note that this calculation generalizes to $\mathcal{T}$ by continuity so that we obtain

Theorem 4.1. $\mathcal{T} = \overline{\mathcal{D}_m^\perp}$. ■

For the purpose of our paper, it remains to provide a practical description of $\overline{\mathcal{D}_m^\perp}$. Therefore, note that the closure $\overline{\widehat{\mathcal{D}_m^\perp}}$ of $\widehat{\mathcal{D}_m^\perp} := \{\hat{\gamma} : \gamma \in \mathcal{D}_m^\perp\}$ is well-known to generate the weighted L_2 space

$$L_\varphi^2 := \left\{\hat{u} : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\} : |\hat{u}|_\varphi < \infty\right\}$$

being endowed with the (semi-) norm

$$|\hat{u}|_\varphi := \left(\int_{\mathbb{R}^d} |\hat{u}(\omega)|^2 \varphi(\omega)\, d\omega\right)^{1/2} \qquad \text{for all } \hat{u} \in L_\varphi^2.$$

Introducing the equivalence relation "$\sim$" which identifies two functions $\hat{u}, \hat{v} \in L_\varphi^2$ iff $|\hat{u} - \hat{v}|_\varphi = 0$, this leads to the Hilbert space $\mathcal{L}_\varphi^2 := L_\varphi^2/\sim$, which is isometric to $\overline{\widehat{\mathcal{D}_m^\perp}}$. Thus there is a unitary operator U with $U(\mathcal{L}_\varphi^2) = \overline{\widehat{\mathcal{D}_m^\perp}}$. If we understand the inverse Fourier transform $\mathtt{IF}$ of each element from $\overline{\widehat{\mathcal{D}_m^\perp}}$ in the sense of the Plancherel theorem we can provide a further characterization of $\mathcal{T}$ as follows.

Theorem 4.2. *$\mathcal{T}$ coincides with the space $(\mathtt{IF} \circ U)(\mathcal{L}_\varphi^2)$ containing all generalized inverse Fourier transforms of functions from the weighted L_2 space $\mathcal{L}_\varphi^2$.* ■

§5. Native Space

In earlier papers [6,7,8] Madych & Nelson provided practical characterizations of the native space

$$\mathcal{F} := \left\{f : \mathbb{R}^d \to \mathbb{R} : \exists C_f \geq 0 \text{ with } |\lambda f| \leq C_f \|\lambda\|_\psi \text{ for all } \lambda \in \mathcal{L}_m^\perp\right\},$$

which we identify by continuity with

$$\mathcal{F}_\mathcal{T} := \left\{f : \mathbb{R}^d \to \mathbb{R} : \exists C_f \geq 0 \text{ with } |\tau f| \leq C_f \|\tau\|_\psi \text{ for all } \tau \in \mathcal{T}\right\},$$

which is obviously the largest among all function spaces on which each functional from $\mathcal{T}$ is continuous. Madych & Nelson proved among many other things that the relation $\mathcal{F}/\mathcal{P}_m^d \cong \overline{\mathcal{L}_m^\perp}$ holds and since the native space has extensively been studied in [6,7,8] we close this section by merely stating the following theorem.

Theorem 5.1. $\mathcal{F}_\mathcal{T}/\mathcal{P}_m^d \cong \mathcal{T}$. ■

§6. Reconstruction of Functions From Scattered Data

Note that for $\lambda \in \mathcal{L}_m^\perp$ and $\gamma \in \mathcal{D}_m^\perp$ the analyticity of both the symbol function σ_λ and $\hat{\gamma} \in \widehat{\mathcal{D}_m^\perp}$ serves to show the definiteness of the corresponding inner product $(\cdot,\cdot)_\psi$. Adopting this observation, we note that due to the famous Paley-Wiener theorem (see [4]) each compactly supported functional possesses a generalized Fourier transform which is an analytic function.

Theorem 6.1. *Let τ be a compactly supported linear functional of order ℓ in the sense of distributions. Then, τ possesses a generalized Fourier transform which coincides with the function $\sigma_\tau(\omega) := \tau^x e^{-ix^T\omega}$ being an analytic function which is polynomially bounded such that*

$$|\sigma_\tau(\omega)| \leq C(1 + \|\omega\|)^\ell, \qquad \omega \in \mathbb{R}^d$$

holds. ■

Now, by

$$\Pi_\ell^\perp := \left\{\tau \,:\, \tau \text{ compactly supported }, \operatorname{ord} \tau \leq \ell,\, \tau(\mathcal{P}_m^d) = \{0\}\right\}$$

let us denote the linear space containing all $\mathcal{P}_m^d$-orthogonal functionals possessing an analytic generalized Fourier transform, whose growth at infinity can be bounded by a polynomial of order ℓ. Recall that due to our assumptions, $\|\omega\|^r \varphi(\omega) \in L_1(\infty)$ holds for a certain $r \in \mathbb{N}_0$. So for $\ell \leq r/2$ we obtain the identity

$$\tau^x \tau^y \psi(x-y) = (2\pi)^{-d} \int_{\mathbb{R}^d} |\sigma_\tau(\omega)|^2 \varphi(\omega)\, d\omega, \qquad \text{for all } \tau \in \Pi_\ell^\perp,$$

and therefore, $\Pi_\ell^\perp$ is an inner product space by

$$(\kappa, \tau)_\psi := \kappa^x \tau^y \psi(x-y), \qquad \text{for all } \tau, \kappa \in \Pi_\ell^\perp.$$

As a special case, for $\tau = \sum_{j=1}^N c_j T_j \in \Pi_\ell^\perp$ with linearly independent compactly supported linear functionals $T_1, \ldots, T_N$ of order less or equal to $\ell \leq r/2$ it follows that for the matrix $A_{\psi,\tau} := (T_j^x T_k^y \psi(x-y))_{1 \leq j,k \leq N}$ the quadratic form

$$c^T A_{\psi,T} c = \sum_{j,k=1}^N c_j c_k T_j^x T_k^y \psi(x-y) = \|\tau\|_\psi^2 \tag{5}$$

is nonnegative and vanishes if and only if $c = (c_1, \dots, c_N)^T = 0$. As a consequence, this provides a solution

$$s(x) = \sum_{j=1}^{N} c_j T_j^y \psi(x - y) + \sum_{n=1}^{Q} d_n p_n(x), \tag{6}$$

to the *reconstruction problem*

$$T_k f = T_k s, \qquad 1 \le k \le N, \tag{7}$$

under the constraints

$$\tau = \sum_{j=1}^{N} c_j T_j \in \Pi_\ell^\perp. \tag{8}$$

In the above, $p_1, \dots, p_Q$ denote a basis for $\mathcal{P}_m^d$ and f stands for a function for which $T_j f$ is well-defined for $1 \le j \le N$. Indeed, if we combine (7) and (8) this yields a linear system whose corresponding homogenous system has the form

$$\begin{aligned} A_{\psi,T} c + (TP) d &= 0, \\ (TP)^T c &= 0, \end{aligned}$$

where we set $TP := (T_j p_\ell)_{1 \le j \le N; 1 \le \ell \le Q}$ and $d := (d_1, \dots, d_Q) \in \mathbb{R}^Q$. Now, the equation

$$c^T A_{\psi,T} c + \underbrace{c^T (TP)}_{=0} d = 0$$

follows and this is true, according to (5), if and only if $c = 0$. Hence, the homogenous system only possesses the trivial solution if $(TP)d = 0$ implies $d = 0$, which is equivalent to the requirement

$$T_j p = 0, \quad 1 \le j \le N \qquad \Longrightarrow \qquad p \equiv 0 \text{ for all } p \in \mathcal{P}_m^d. \tag{9}$$

Theorem 6.2. *The reconstruction problem (7) has, under contraints (8), a solution of the form (6) which is unique provided that (9) holds.* ■

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