Gödel's incompleteness theorems

The limits of the formal method

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Overview

1. Context and basics
   - History
   - Technical foundation
2. Gödel's first incompleteness theorem
   - The popular statement
   - Unraveling and preparing
   - Proving the first incompleteness theorem
3. Conclusion and preview
   - Relation to Hilbert's programme
   - The second incompleteness theorem
   - Why should we bother?

Logic and the axiomatic method I

- **ca 300 BCE:** The axiomatic method is used by Euclid of Alexandria in the context of geometry in his influential *Elements*.
- **1879-1903:** G. Frege attempts to found mathematics on pure logic; he introduces the first-order predicate logic.
- **1889:** Peano introduces the set of axioms known today as Peano's axioms in an attempt to formalize the natural numbers.
- **1903:** B. Russell detects Russell's Paradox in Frege's work. This sparks the foundational crisis.
- **1908-1922:** E. Zermelo, A. Fraenkel and Th. Skolem develop an axiomatic system for set theory, known today as ZFC.

Logic and the axiomatic method II

- **1910-1913:** B. Russell and A.N. Whitehead specify in their *Principia Mathematica* a formal system (axioms and rules of deduction), in which they establish parts of basic mathematics.
- **ca 1922:** D. Hilbert publicly announces his programme of proof theory – today known as Hilbert's programme.
- **1933:** K. Gödel publishes his two incompleteness theorems, proving the impossibility of carrying out Hilbert's programme.
- **1943-today:** Many examples in different branches of mathematics are found, which give a significance to Gödels first incompleteness theorem.

Russell's paradox

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Principle (Comprehension scheme)

For any (first-order) formula $\varphi(x)$ there is a set containing exactly all the sets x such that $\varphi(x)$ is true.

However this principle is inconsistent as was shown by B. Russell:

Proof of Russell's paradox.

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We fix a *signature* consisting of relation symbols, function symbols and constant symbols. This specifies our language.

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Given a signature, its language consists of all formulas build out of the symbols in the signature (used according to their kind) plus equality $=$ using logical connectives ($\vee, \wedge, \leftrightarrow, \neg$, etc.) and binding variables by quantifiers.

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1. $\forall x \forall y (x \cdot y = y \cdot x)$ is a formula in the language of groups.
2. $1 + x = y \cdot y$ is a formula in the language of rings.
3. $\forall x (\exists y (\mathbf{S}(y) = x) \vee x = 0)$ is a formula in the language of arithmetic.
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3. **Generalization:** If $\varphi(x)$ is established, where x is free, conclude $\forall x \varphi(x)$.

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Why does this formalization work?

We fix a certain finite* collection of rules. Let T be a theory, φ a formula. Then write $T \vdash \varphi$ iff there is a derivation from T s.t. φ is its final member.

Different perspective: We write $T \models \varphi$ iff in every mathematical structure, in which all formulas in T hold, also φ holds.

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Let T be the axioms of a group. Then $T \models \varphi$ means that φ is satisfied by any group. So, e.g., $T \not\models \forall x \forall y (x \cdot y = y \cdot x)$.

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The popular statement

We say that a theory T is *inconsistent* iff $T \vdash \exists x(x \neq x)$. Otherwise it is *consistent*.

We say that a theory T is *complete* iff for any sentence φ of the corresponding language either $T \vdash \varphi$ or $T \vdash \neg\varphi$. Otherwise it is *incomplete*.

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First things first...

Until further notice we work in the language of arithmetic.
Recall the (countably many Peano axioms):

1. $\forall x(\mathbf{S}(x) \neq 0)$;
2. $\forall x \forall y(\mathbf{S}(x) = \mathbf{S}(y) \rightarrow x = y)$;
3. $\forall x(x + 0 = x)$;
4. $\forall x \forall y(x + \mathbf{S}(y) = \mathbf{S}(x + y))$;
5. $\forall x(x \cdot 0 = 0)$;
6. $\forall x(x \cdot \mathbf{S}(y) = x \cdot y + x)$;
7. $\forall x \forall y(x \leq y \leftrightarrow \exists z(x + z = y))$;
8. $\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(\mathbf{S}(x))) \rightarrow \forall x \varphi(x)$,
   for any arithmetical formula $\varphi(x)$.

Let **PA** denote the set of Peano axioms.

Gödelization

We consider logical and arithmetic symbols as natural numbers via the following mapping:

ζ	$=$	$\neg$	$\wedge$	$\forall$	(	)	0	**S**	$+$	$\cdot$	x	y	...
$\#\zeta$	1	3	5	7	9	11	13	15	17	19	21	23	...

Let $\langle p_i \mid i \in \mathbb{N} \rangle$ be the enumeration of all prime numbers. Then we assign to a string of symbols $\xi = \zeta_0 \cdots \zeta_n$ the *Gödel number*

$$\dot{\xi} := p_0^{1+\#\zeta_0} \dots p_n^{1+\#\zeta_n}.$$

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Doing logic in $\mathbb{N}$

Now we can do logic in the structure $\mathbb{N}$ of the natural numbers and define the following relations on $\mathbb{N}$ for any arithmetic theory T:

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What does **PA** know about all this? (I)

In **PA** we can define any single $n \in \mathbb{N}$. We set $\underline{0} := 0$ and $\underline{n+1} := \mathbf{S}(\underline{n})$. Then in $\mathbb{N}$ a given term $\underline{n}$ gets interpreted as n.

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Now what does it mean to represent a relation on $\mathbb{N}$ in **PA**?

Definition

$P \subseteq \mathbb{N}^n$ is representable in a theory $T \supseteq$ **PA** if there is a formula $\alpha(\vec{x})$ such that for any $\vec{a} \in \mathbb{N}^n$:

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We say that a theory T is recursively axiomatizable iff there is a subset $T' \subseteq T$ such that for any $\varphi \in T$, $T' \vdash \varphi$ and T (informally) has the following property:
It is possible to write a computer program such that on any input $n \in \mathbb{N}$ it decides in finite time whether there is $\varphi \in T'$ such that $n = \ulcorner T \urcorner$.
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Let T be a recursively axiomatizable theory. Then the relations proof_T and prv_T are representable in **PA** *by formulas $\mathbf{proof}_T(x)$ and $\mathbf{prv}_T(x, y)$, respectively.*

Using this we get:

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For a theory T and two formulas φ, ψ we write $\varphi \equiv_T \psi$ to mean $T \vdash (\varphi \leftrightarrow \psi)$.

Lemma (Fixed point lemma)

Let $T \supseteq \mathbf{PA}$ be a theory. Then for every formula $\alpha(x)$ with exactly one free variable there is a sentence γ such that $\gamma \equiv_T \alpha(\ulcorner \gamma \urcorner)$.

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First we note that there is a formula $\mathbf{sb}(x_1, x_2, y)$ such that for any formula $\varphi = \varphi(x)$ we have $\mathbf{sb}(\ulcorner \varphi \urcorner, \underline{n}, y) \equiv_T y = \ulcorner \varphi(\underline{n}) \urcorner$. Then as a special case we have that

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Now we define $\beta(x) :\equiv \forall y(\mathbf{sb}(x, x, y) \to \alpha(y))$ and we define $\gamma :\equiv \beta(\ulcorner\beta\urcorner)$. Then:

$$\begin{aligned}
\gamma &\equiv \forall y(\mathbf{sb}(\ulcorner\beta\urcorner, \ulcorner\beta\urcorner, y) \to \alpha(y)) \\
&\equiv_T \forall y(y = \ulcorner\beta(\ulcorner\beta\urcorner)\urcorner \to \alpha(y)) \\
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Lemma (Non-representability lemma)

*Let $T \supseteq$ **PA** be a theory. Then $prvbl_T$ is not representable in T.*

Proof.

Let $\tau(x)$ be a formula representing prvbl_T. Then in particular we have that

$$T \nvdash \varphi \quad \Leftrightarrow \quad T \vdash \neg\tau(\ulcorner\varphi\urcorner). \qquad (*)$$

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Towards the incompleteness theorem (I)

We call a theory T *ω-consistent* iff whenever $T \vdash \exists x \varphi(x)$, then there is $n \in \mathbb{N}$ such that $T \nvdash \neg\varphi(\underline{n})$.

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Assume $T \vdash \neg\gamma$. Then $T \vdash \mathbf{prvble}_T(\ulcorner\gamma\urcorner)$, i.e., $T \vdash \exists y \mathbf{prv}_T(y, \ulcorner\gamma\urcorner)$. Then by ω-consistency we have that $T \nvdash \neg\mathbf{prvble}_T(\underline{n}, \ulcorner\gamma\urcorner)$ for some $n \in \mathbb{N}$,

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Formalize all mathematics in a formal language using a set of axioms that is easy to describe (like ZFC) and a finite set of inference rules to deduce theorems.

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Let T be a recursively axiomatizable theory and let $\mathbf{Con}_T :\equiv \forall x(\neg \mathbf{prov}(x, \ulcorner \exists x(x \neq x) \urcorner))$. Then $\mathbf{Con}_T$ is true in $\mathbb{N}$ iff T is consistent.

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