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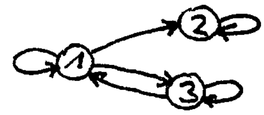
Übungen zu Stochastische Prozesse I

Lösungswege zu P 4.2.



Zu zeigen: $m_{32} = m_{31} + m_{12}$

Vorbetrachtungen: $m_{31} = 1 + p_{33} m_{31} \Rightarrow m_{31} = \frac{1}{1-p_{33}} < \infty$ (weil $1 - p_{33} = p_{31} > 0$)

ebenso $m_{12} = 1 + p_{11} m_{12} \Rightarrow m_{12} = 1/1-p_{11} < \infty$ (weil $1 - p_{11} \geq p_{12} > 0$)

Direkte Rechnung: $\underline{m_{32}} = 1 + p_{33} m_{32} + p_{31} m_{12} \Leftrightarrow m_{32} = \frac{1}{1-p_{33}} + \frac{p_{31}}{1-p_{33}} m_{12} = \underline{m_{31}} + 1 \cdot \underline{m_{12}}$.

Mit (elementarer) Markov-Eigenschaft: Beachte: $P(\tau_2 > \tau_1 | X_0 = 3) = 1$ (*)

$E(\tau_2 | X_0 = 3) = E(\tau_1 + (\tau_2 - \tau_1) | X_0 = 3) = \Sigma_k \Sigma_\ell (k+\ell) P(\tau_1 = k, \tau_2 - \tau_1 = \ell | X_0 = 3)$

$P(\tau_1 = k, \tau_2 - \tau_1 = \ell | X_0 = 3) = P(\tau_1 = k | X_0 = 3) P(\tau_2 - \tau_1 = \ell | X_0 = 3, \tau_1 = k, \ )$ $\{X_k = 1\}$ weil $\supset \{\tau_1 = k\}$

wg. (ME) $= P(\ \cdots\ ) P(X_{k+1} \neq 2, \ldots, X_{k+\ell} = 2 | X_0 = 3, \tau_1 = k, X_k = 1)$

wg. (ME): $= P(\quad) P(X_{k+1} \neq 2, \ldots, X_{k+\ell} = 2 | X_k = 1)$

wg. homogen: $= P(\quad) P(X_{0+1} \neq 2, \ldots, X_\ell = 2 | X_0 = 1)$

$= P(\tau_1 = k | X_0 = 3) P(\tau_2 = \ell | X_0 = 1)$

Eingesetzt:

$\underbrace{E(\tau_2 | X_0 = 3)}_{= m_{32}} = \sum_k \sum_\ell (k+\ell) P(\tau_1 = k | X_0 = 3) P(\tau_2 = \ell | X_0 = 1)$

$= \sum_k k \cdot P(\tau_1 = k | X_0 = 3) \sum_\ell P(\tau_2 = \ell | X_0 = 1)$

$+ \Sigma_k P(\tau_1 = k | X_0 = 3) \Sigma_\ell \ell \cdot P(\tau_2 = \ell | X_0 = 1)$

$= E(\tau_1 | X_0 = 3) + E(\tau_2 | X_0 = 1) = \underline{m_{31} + m_{12}}$.

Beachte:
$\Sigma_\ell P(\tau_2 = \ell | X_0 = 1) = 1$ wg. $m_{12} < \infty$
$\Sigma_k P(\tau_1 = k | X_0 = 3) = 1$ wg. $m_{31} < \infty$

Mit starker Markov-Eigenschaft:

$E(\tau_2 - \tau_1 | X_0 = 3) = \Sigma_{\ell=1}^{\infty} \ell \cdot P(\tau_2 - \tau_1 = \ell | X_0 = 3) =$

$= \Sigma_{\ell=1}^{\infty} \ell \cdot P(\tau_2 = \underline{\tau_1} + \ell | X_0 = 3, X_{\tau_1} = 1)$, weil $P(X_{\tau_1} = 1) = 1$

starke ME $= \Sigma_\ell \ \ell \cdot P(\tau_2 = \tau_1 + \ell | X_{\tau_1} = 1)$

$= \Sigma_\ell \ \ell \cdot P(\tau_2 = 0 + \ell | X_0 = 1) = E(\tau_2 | X_0 = 1) = m_{12}$

Zusammen: $\underline{m_{32}} = E(\tau_2 | X_0 = 3) = E(\tau_1 + (\tau_2 - \tau_1) | X_0 = 3) = \underline{m_{31} + m_{12}}$