Sheet 10

Question 10.1

(a) Compute $H^*_{dR}(S^1)$ for $X = S^1$, exhibiting explicit generators for all cohomology classes.

(b) Consider the map $f_n : \theta \mapsto n\theta$ from S^1 to S^1. Compute the induced map on cohomology.

(c) Assuming the Künneth theorem give explicit generators of $H^*_{dR}(T^n)$ where $T^n = (S^1)^n$ is the n-torus. Describe the product structure.

Question 10.2

Fix a cover $\mathfrak{U}$ of a manifold X sucht that all U_i and their intersections are contractible. Define a product on the $C = \check{C}^*(\mathfrak{U}, \underline{\mathbb{R}})$ by considering

$$f \cup g|_{U_{i_0 \cdots i_{p+q}}} = f|_{U_{i_0 \cdots i_p}} \cdot g|_{U_{i_p} \cdots U_{i_{p+q}}}$$

for $f \in \check{C}^p(\mathfrak{U}, \underline{\mathbb{R}})$ and $g \in \check{C}^q(\mathfrak{U}, \underline{\mathbb{R}})$. Check it is compatible with the Čech differential.

By considering a suitable double complex show that $H_{dR}(X, \mathbb{R})$ and $\check{H}(\mathfrak{U}, \underline{\mathbb{R}})$ are isomorphic as graded algebras.

Question 10.3

Let X be a simply connected space and L a locally constant sheaf on L. Let $x \in X$.

(a) Show that the support of any section of a locally constant sheaf on an arbitrary space is open.

(b) Show the restriction map $r : \Gamma(X, L) \to L_x$ is injective.

(c) * Show that r is also surjective.

(d) Deduce that L is locally constant.

These questions will be discussed in the exercise class on 27 June 2025.

Questions with an asterisk are more challenging.