Sheaves in Topology

Master's Course
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Contents

These are the lecture notes as of April 28, 2025.

Please contact me with any comments or corrections at `julian.holstein@uni-hamburg.de`.

An up to date version of these notes can be found at `http://www.math.uni-hamburg.de/home/holstein/lehre/STnotes.pdf`.

Here are some useful books:

[Wei95] C. Weibel, *An introduction to homological algebra*, CUP (1994). Very thorough general reference for homological algebra, occasionally a bit dated.

[GM03] S. Gelfand, Yu. Manin, *Methods of homological algebra*, Springer (2003). Another general reference for homological algebra, more modern but watch out for typos.

[KS90] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*, Springer (1990). Very dense and not always easy to use, but probably the most comprehensive book around.

[Ive86] B. Iversen, *Cohomology of sheaves*, Springer (1986). Good treatment of many of the topics of this course.

[Dim04] A. Dimca, *Sheaves in topology*, Springer (2004). Nice overview getting to some advanced topics, but light on details with very few proofs.

Many standard references in Algebraic Geometry or Topology have a useful perspective on some aspect of this course, e.g. Hartshorne *Algebraic Geometry*, Vakil *Foundations of Algebraic Geometry*, Voisin *Hodge Theory and Complex Algebraic Geometry I*, Bott & Tu *Differential forms in algebraic geometry* ...

1. Introduction

1.1. Overview

In this course we study the basic theory of sheaves with a view to applications in topology.

– presheaves and sheaves, stalks and sheafifiaciton, pushforward and pullback functors. , sheaf cohomology.

This will require some background in category theory and homological algebra, in particular the notion of derived functors, that I will review very very briefly.

Here is an outline of the course as it is planned at the moment. There may well be changes.

1. Basic definitions, examples and constructions. Presheaves, sheaves, stalks, sheafificaiton, pushforward, inverse image.

2. A very brief introduction to homological algebra. Derived functors, the derived category.

3. Cohomology as derived global sections. Injective, flasque and soft sheaves, de Rham and Cech cohomology.

4. Computations. Cohomology and pushforward with compact support; Mayer-Vietoris, base change; Projection formula.

5. Local systems. Cohomology with local coefficients, Riemann-Hilbert, constructible sheaves.

6. If time permits: Advanced topics.

This is an advanced graduate course, the main pre-requisites is a course and on advanced algebra (language of functors and homological algebra). A course on algebraic topology (including cohomology) is extremely useful, but can be taken at the same time.

The course is not complete in the sense that I reserve the right to leave out some details and use non-trivial results from the literature.

You can influence the pace and focus of the course somewhat by making requests, asking questions or telling me to slow down or speed up.

2. Basic theory of sheaves

2.1. Definitions and Examples

Let X be a topological space and $\mathrm{Op}(X)$ the category (poset) of open sets. The category has the open subsets of X as objects and a unique morphism $U \to V$, written $U \subset V$ if U is a subset of V and no other morphisms.

Definition 2.1. A *presheaf* on X with values in a category $\mathcal{C}$ is a functor $\mathcal{F} : \mathrm{Op}(X)^{\mathrm{op}} \to \mathcal{C}$

We call $\mathcal{F}(U)$ the *sections* of $\mathcal{F}$ on U.

A *morphism of presheaves* $\mathcal{F} \to \mathcal{G}$ is just a natural transformation.

We can unravel these abstract definitions: A presheaf on X provides an object $\mathcal{F}(U)$ of $\mathcal{C}$ for any open set in X and a restriction map $r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for any inclusion $V \to U$ that is compatible with compsition: $r_{UW} = r_{UV} \circ r_{VW}$. A morphism $f : \mathcal{F} \to \mathcal{G}$ is a map $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for every U such that $f_V \circ r^{\mathcal{F}}_{UV} = r^{\mathcal{G}}_{UV} \circ f_V$.

We will be mostly interested in the case that $\mathcal{C}$ is the category of abelian groups or more generally R-modules for some commutative ring R. We will always assume that $\mathcal{C}$ has all small limits and that it is a concrete category equipped with a forgetful functor to sets, i.e. we can characterise $\mathcal{F}(U)$ by its elements.

For a section $s \in \mathcal{F}(U)$ we also write $s|_V$ for $r_{UV}(s) \in \mathcal{F}(V)$.

Example 2.2.

1. On any X the functor sending any open set U to $\mathbb{Z}$ is a presheaf with values in abelian groups called the *constant presheaf*.

2. Ony any X the functor sending any open U to the set $\mathcal{C}^0(U, \mathbb{R})$ of continuous functions on U is a presheaf.

Definition 2.3. A collection $\{U_i\}_{i \in I}$ in $\mathrm{Op}(X)$ such that $\cup U_i = U$ is called a *cover*.

A presheaf $\mathcal{F}$ is called a *sheaf* if for any cover U_i of an open U and for any collection of sections $s_i \in \mathcal{F}(U_i)$ such that $\forall i, j \in I$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

The uniqueness of the section means that sections of a sheaf are determined by their restrictions, they are *locally determined*. A presheaf satisfying this condition is sometimes called *separated*.

The existence of the section means that sheaves can be *glued* from consistent local data.

We can write the sheaf condition somewhat compactly as a limit:

Lemma 2.4. *A presheaf $\mathcal{F}$ on X is a sheaf if and only if for any cover $\{U_i\}_{i\in I}$ of any open $U \subset X$ we have*

$$\mathcal{F}(U) = \mathrm{eq}\left(\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \cap U_j)\right)$$

Proof. Unravelling this limit returns the definition in words. □

From either definition we can read off two useful facts:

1. For any sheaf $\mathcal{F}(\amalg_i U_i) = \prod_i \mathcal{F}(U_i)$ as the U_i form a cover and all intersections are by definition empty.

2. For any sheaf $\mathcal{F}(\emptyset) = *$, the final object of the category $\mathcal{C}$. This is a special case of the previous point, we can cover the empty set by the empty set and read off that $\mathcal{F}(\emptyset)$ is the limit over the empty category, i.e. the final object!

Example 2.5. The constant presheaf on a topological space is typically *not* a sheaf. Assume X has two disjoint open subsets U, V and consider the constant sheaf with value $\mathbb{Z}$. Then for a sheaf $\mathcal{F}$ we have $\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V)$, but the constant sheaf takes value $\mathbb{Z} \neq \mathbb{Z} \times \mathbb{Z}$.

Example 2.6. Let Y be a topological space, for example $Y = \mathbb{R}$. Let X be an arbitrary toplogical space. Define $\mathcal{C}(U)$ to be the set of continuous maps $U \to Y$. Then $\mathcal{C}$ is a sheaf.

Let U_i be a cover of U. Then U is the colimit of the U_i, to be precise $U = \mathrm{coeq}(\amalg_i U_i \leftleftarrows U_i \cap U_j)$, which we write $\operatorname{colim} U_i$ by abuse of notation to simplify things. But then $\mathcal{C}$ is a sheaf because

$$\mathcal{C}(\operatorname{colim} U_i) := \mathrm{Hom}(\operatorname{colim} U_i, Y) = \lim \mathrm{Hom}(U_i, Y) = \lim \mathcal{C}(U_i)$$

by the fundamental property of limits and homs.

Alternatively, one can unravel the definitions.

In the case $Y = \mathbb{R}$ we call this the sheaf of real-valued (continuous) functons on X. I.e. the presheaf of real-valued continuous functions on X is a sheaf.

Example 2.7. In the previous example let Y have the discrete topology, for example $Y = \mathbb{Z}$. Then we have constructed the sheaf of locally constant functions on X with values in Y. We call it the *constant sheaf* and denote it by $\underline{Y}$. This is not to be confused with the constant presheaf. To be precise, the value on a set U is $\mathbb{Z}^{c(U)}$ where $c(U)$ is the number of connected components of U.

Example 2.8. Let E be a vector bundle of rank n on a topological space X, i.e. a space E with a surjection $p : E \to X$ such that X has a cover U_i and each $p^{-1}(U_i)$ is homeomorphic to $U_i \times \mathbb{R}^n$.

Then $\mathcal{E}$ defined by $\mathcal{E}(U) = \{s : U \to p^{-1}(U) \mid p \circ s = \mathbf{1}_U\}$ is a sheaf, the *sheaf of sections* of E. If $E = X \times \mathbb{R}$ is the trivial rank one vector bundle its sheaf of sections is the sheaf of $\mathbb{R}$-valued functions.

Example 2.9. More generally for any continuous map $p : Y \to X$ we may define the sheaf of sections $\mathcal{S}$ that sends any $U \subset X$ to the set of maps $s : U \to Y$ satisfying $ps = \mathbf{1}_U$. By definition $\mathcal{S}(U) = \mathcal{C}(U) \times_{\mathrm{Hom}(U,X)} \{\iota_U\}$ where ι_U is the inclusion $U \subset X$ and thus for a cover we have

$$\begin{aligned}\mathcal{S}(\operatorname*{colim}_i U_i) &\cong \mathcal{C}(\operatorname*{colim}_i U_i) \times_{\mathrm{Hom}(\mathrm{colim}_i U_i, X)} \{\iota_U\} \\ &\cong \left(\lim_i \mathcal{C}(U_i)\right) \times_{\lim \mathrm{Hom}(U_i,X)} \{\iota_{U_i}\} \\ &\cong \lim_i \left(\mathcal{C}(U_i) \times_{\mathrm{Hom}(U_i,X)} \{\iota_{U_i}\}\right) \cong \lim_i \mathcal{S}(U_i)\end{aligned}$$

as limits commute with limits, in particular the pullback commutes with the equalizer of products in the sheaf condition.

Example 2.10. As sheaves are defined locally we may make local modifications: If E is a smooth vector bundle on a smooth manifold the presheaf of smooth sections of E is a sheaf: As the presheaf of smooth sections is contained in the sheaf of continuous sections we can always glue compatible smooth sections to a unique continuous section. But this continuous section must be smooth as it restricts to a smooth section on each open in our cover.

Similarly we may define the sheaf of locally constant functions or holomorphic functions as a subsheaf of the sheaf of all continuous functions into $\mathbb{C}$.

Here and in future a *subsheaf* $\mathcal{F}$ of a sheaf $\mathcal{G}$ is just a sheaf on the same space such that $\mathcal{F}(U) \subset \mathcal{G}(U)$ for all U.

Example 2.11. Let $X = *$. Then a $\mathcal{C}$-valued sheaf on X is exactly an object of $\mathbb{C}$.

Let $*$ be a terminal object in $\mathcal{C}$. Then the constant presheaf with value $*$ is a sheaf.

Example 2.12. Let R be a commutative ring and M an R-module. We let Spec(R) be the set of all prime ideals of R and define a topology a follows. Let for each $f \in R$ $D_f \subset \operatorname{Spec} R$ be the set of prime ideals not containing f. This is a basis of open sets for a topology on $\operatorname{Spec} R$ called the *Zariski topology*. Define a presheaf $\tilde{M}$ as follows:

1. on the D_f by $\tilde{M}(D_f) = M_f$, the localisation of M at f, i.e. the R-module of formal quotients $\{\frac{m}{f^j} \mid m \in M, j \in \mathbb{N}\}$.
2. on an arbitrary $U = \cup_f D_f$ we define $\tilde{M}(U) = \lim \tilde{M}(D_f)$.

Then one can show with some commutative algebra that this is sheaf on $\operatorname{Spec} R$. In particular R itself gives rise to a sheaf on $\operatorname{Spec} R$ called the *structure sheaf* with the property that every $\tilde{M}(U)$ is a module over $\tilde{R}(U)$. We say $\tilde{M}$ is a *quasi-coherent sheaf* an the afine scheme $\operatorname{Spec} R$ and these (and their generalizations to general schemes) play a huge role in algebraic geometry, but our focus will lie elsewhere.

Definition 2.13. A topological space X equipped with a sheaf of rings $\mathcal{R}$ is called a *ringed space*. A *sheaf of $\mathcal{R}$-modules* is a sheaf $\mathcal{M}$ of abelian groups on X such that $\mathcal{M}(U)$ is a (left) $\mathcal{R}(U)$-module for every open set U in X. A morphism of sheaves of $\mathcal{R}$-modules is a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ such that each $\mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{R}(U)$-linear.

We will probably only look at sheaves of commutative rings, but there is no reason not to define things in general.

Definition 2.14. Given a topological space X and a category $\mathcal{C}$ we define the category $\mathsf{PSh}(X, \mathcal{C})$ as the category of presheaves on X.

We denote by $\mathsf{Sh}(X, \mathcal{C})$ the full subcategory of sheaves.

We will be particularly interested in sheaves with values in the category of R-modules for some commutative ring R.

We write $\mathsf{Sh}(X, R)$ for $\mathsf{Sh}(X, R\text{-Mod})$ for a commutative ring R and $\mathsf{Sh}(X)$ for $\mathsf{Sh}(X, \mathbb{Z}) = \mathsf{Sh}(X, \mathsf{Ab})$ for the category of sheaves of abelian groups. If $(X, \mathcal{R})$ is a ringed space we write $\mathsf{Sh}(X, \mathcal{R})$ for the category of sheaves of $\mathcal{R}$-modules.

2.2. Stalks and sheafification

As sheaves are local we may look at them at a point. We begin by looking at presheaves at points. To simplyif things we look at sheaves with values in an abelian category $\mathcal{A}$, for example abelian groups. Bt everything will be true in greater generality, for sheaves of sets one needs minor modifications of the proofs.

Definition 2.15. The *stalk* $\mathcal{F}_x$ of a presheaf $\mathcal{F}$ on X at a point $x \in X$ is defined as $\operatorname{colim}_{x \in U} \mathcal{F}(U)$ where the colimit is taken in the category $\mathcal{A}$ over all open sets containing x.

Given $s \in \mathcal{F}(U)$ we denote by $s|_x$ its image in $\mathcal{F}_x$, called the *germ* of s.

Explicitly, objects of $\mathcal{F}_x$ are pairs (U, s) with $x \in U \subset X$ open and $s \in \mathcal{F}(U)$ up to the equivalence $(U, s) \sim (W, t)$ if there is $V \subset U \cap W$ with $s|_V = t|_V$.

This is an example for a filtered colimit, which is sometimes (confusingly!) called a direct limit. See the section in the appendix if you are unfamiliar with these kinds of colimits.

Note that the stalk of a sheaf of rings is again a ring (whose underlying abelian group is the stalk of the underlying sheaf of abelian groups) by defining multiplication and addition of representatives in the obvious way: $[(U, s)] \cdot [(V, t)] = [(U \cap V, s|_{U \cap V} \cdot t|_{U \cap V})]$ etc.

Example 2.16. The constant presheaf with value R has stalk $R = \operatorname{colim} R$.

The constant sheaf $\underline{R}$ also has stalk R. The connected open neighbourhoods of a point P are final in all open neighbourhoods, thus we can compute the stalk on connected open sets, see Lemma A.35. But on a connected open set $\underline{R}(U) = R$.

Example 2.17. The presheaf of continuous functions $\mathcal{C}$ on a manifold M has as stalk at the point p the set (in fact, ring) of germs of functions at p.

Any morphism $f : \mathcal{F} \to \mathcal{G}$ induces a morphism of stalks $f_x : \mathcal{F}_x \to \mathcal{G}_x$ by sending the germ represented by (U, s) to the germ represented by $(U, f(s))$.

Lemma 2.18. *Two morphisms $f, g : \mathcal{F} \to \mathcal{G}$ of sheaves agree if they agree on stalks.*

Proof. For any U we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \big\downarrow & & \big\downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array} \tag{2.1}$$

and the vertical maps are injections: Assume given $s \in \mathcal{G}(U)$ with $s_x = 0$ for all $x \in U$. This means for any x there is some U_x on which s vanishes. But the $\{U_x\}$ form a cover of U and by the uniqueness part of the sheaf condition s must be 0.

As the maps induced by f, g in the bottom row agree, they must also agree in the top row. □

Lemma 2.19. *A morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves is an isomorphism if and only if all induced morphisms on stalks are isomorphisms.*

Proof. The only if direction is clear.

So let f be such that f_x is an isomorphism for all $x \in X$. We will show that for all U we have an isomorphism $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$, then $U \mapsto f_U^{-1}$ is an inverse morphism in the category of sheaves.

To show f is injective assume $f(s) = 0$ for all $s \in U$. In particular $f(s)_x = 0$ for all x, thus by injectivitiy $s_x = 0$, so there is some U_x with $s|_{U_x} = 0$. By the uniqueness property of sheaves this means $s|_U = 0$ as in Diagram 2.1.

To show surjectivity assume we have $t \in \mathcal{G}(U)$. By surjectivity on stalks at the point x there is some U_x and $s^x \in \mathcal{F}(U_x)$ such that $(f(s^x), U_x)$ represents t_x. Shrinking U_x if necessary we may even assume $f(s^x) = t|_{U_x}$.

We want to glue the s^x into a section of $\mathcal{F}(U)$. The U_x cover U, so we have to check overlaps. Let $U_{xy} = U_x \cap U_y$ be nonempty. Then $s^x|_{U_{xy}}$ and $s^y|_{U_{xy}}$ are sent to $t_{U_{xy}}$ by assumption. By the injectivity we have already established we have $s^x|_{U_{xy}} = s^y|_{U_{xy}}$. Thus by the sheaf property of $\mathcal{F}$ we can glue to obtain $s \in \mathcal{F}(U)$. As $f(s)$ agrees with t on all stalks we see that s maps to t by Diagram 2.1. □

The constant presheaf seemed like a reasonable construction and we did then construct something we called the constant sheaf. Could we have obtained the constant sheaf directly from the constant presheaf?

Definition 2.20. The *sheafification* of a presheaf $\mathcal{F}$ is defined as follows.

$$\mathcal{F}^{\mathsf{sh}}(U) := \{(f_p \in \mathcal{F}_p)_{p \in U} \mid f_p \text{ are compatible}\}$$

where compatibility means that for any $q \in U$ there is an open $q \in V \subset U$ and a section $s \in \mathcal{F}(V)$ with $f_p = s_p$ for $p \in V$. The restriction maps are the natural restriction maps.

Here the product is taken in the category $\mathcal{A}$ and the compatibility condition is expressible as an equaliser, so if $\mathcal{F}$ takes values in $\mathcal{A}$ so does $\mathcal{F}^{\mathsf{sh}}(U)$.

Theorem 2.21. *Given a presheaf $\mathcal{F}$ on X there is a natural map $u : \mathcal{F} \to \mathcal{F}^{\mathsf{sh}}$ such that any presheaf morphism $f : \mathcal{F} \to \mathcal{G}$ for a sheaf $\mathcal{G}$ factors uniquely through u.*

Proof. Let $\mathcal{F} \in \mathsf{PSh}(X)$. We first note that $\mathcal{F}^{\mathsf{sh}}$ is indeed a sheaf. Given any cover we have (U_i) and compatible sections $s_i \in \mathcal{F}^{\mathsf{sh}}(U_i)$ we define s by $((s_i)_x) \mid x \in U_i)$, i.e. we have to specify an element of the stalk $\mathcal{F}_x$ for any $x \in U$, and just choose any $x \in U_i$ in our cover and choose the germ $(s_i)_x$. By definition of the stalks this is well-defined. Thus we have existence of sections. But the construction is also unique as $s|_{U_i} = s_i$ implies $s_x = (s_i)_x$.

We now consider the map of presheaves $u : \mathcal{F} \to \mathcal{F}^{\mathsf{sh}}$ given on U by $s \in \mathcal{F}(U) \mapsto (s_x)_{x \in U} \in \mathcal{F}^{\mathsf{sh}}(U)$.

Let $\mathcal{G}$ be a sheaf and $f : \mathcal{F} \to \mathcal{G}$ a map of presheaves. We define $\mathcal{F}^{\mathsf{sh}}(U) \to \mathcal{G}(U)$ for any open U as follows. Take $s = (s_x)_{x \in U} \in \mathcal{F}^{\mathsf{sh}}(U)$. By definition there is a cover $\{U_i\}$ of U and sections $s_i \in \mathcal{F}(U_i)$ such that for all x we have $s_x = (s_i)_x$ for a suitable i. We consider $f(s_i) \in \mathcal{G}(U_i)$. By the sheaf property of $\mathcal{G}$ they glue to a section of $\mathcal{G}(U)$ that we call $f(s)$. (Note that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ as they agree on stalks.) This defines $f^{\#} : \mathcal{F}^{\mathsf{sh}} \to G$. This morphism is unique as morphisms of sheaves are determined on stalks by Lemma 2.18. □

Example 2.22. Let $\mathcal{F}$ be the constant presheaf with value R. Then $\mathcal{F}^{\mathsf{sh}}(U)$ is given by functions from U to R which locally come from a section of $\mathcal{F}(U) = R$, i.e. they are locally constant functions. Thus $\mathcal{F}^{\mathsf{sh}} = \underline{R}$, the constant sheaf is the sheafification of the constant presheaf.

Corollary 2.23. *We have $u_x : \mathcal{F}_x \cong (\mathcal{F}^{\mathsf{sh}})_x$ for any $x \in X$*

Proof. The morphism is from Theorem 2.21, the result follows by unravelling the definition of $(\mathcal{F}^{\mathsf{sh}})_x$. □

Corollary 2.24. *If $\mathcal{F}$ is a sheaf $\mathcal{F}$ is uniquely isomorphic to $\mathcal{F}^{\mathsf{sh}}$.*

Proof. We have a map $\mathcal{F} \to \mathcal{F}^{\mathsf{sh}}$ by Theorem 2.21. By Lemma 2.19 it suffices to compare stalks, so the result follows from Corollary 2.23. □

Corollary 2.25. *Sheafification provides a functor left adjoint to the inclusion $\iota : \mathsf{Sh}(X, \mathcal{A}) \to \mathsf{PSh}(X, \mathcal{A})$ of presheaves into sheaves, i.e. $\mathrm{Hom}_{\mathsf{Sh}(X,\mathcal{A})}(\mathcal{F}^{\mathsf{sh}}, \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{PSh}(X,\mathcal{A})}(\mathcal{F}, \mathcal{G})$ for a sheaf $\mathcal{G}$ and presheaf $\mathcal{F}$ on X.*

Proof. Given $f : \mathcal{F} \to \mathcal{G}$ a map of presheaves we obtain a map $f^{\mathsf{sh}} : \mathcal{F}^{\mathsf{sh}} \to \mathcal{G}^{\mathsf{sh}}$ by applying Theorem 2.21 to $\mathcal{F} \to \mathcal{G} \to \mathcal{G}^{\mathsf{sh}}$. Uniqueness ensures that this is functorial.

Theorem 2.21 provides the isomorphism of hom spaces for the adjunction. The map $u : \mathcal{F} \to \iota(\mathcal{F}^{\mathsf{sh}})$ is the unit and the identity map is the counit of this adjunction. □

Remark 2.26. There are different ways of considering sheafification. We may view the sheafification of a presheaf as the sheaf of sections of a certain space associated to the presehaf, the espace étalé, which is the union of all stalks of $\mathcal{F}$, equipped with a topology such that the natural projection map to X is a local homeomorphism.

This is just a different flavour of the construction we chose, but there are generally different constructions. Grothendieck's plus construction associates to any presheaf a separated presehaf and to any separated presheaf a sheaf, doing it twice is sheafification.

We could have of course also just defined sheafification as a left adjoint. We could have then shown existence by constructing it explicitly, or by some general machinery like an adjoint functor theorem. The main ingredient is checking that the inclusion of presheaves into sheaves preserves limits (see below for (co)limits of (pre)sheaves).

2.3. Limits and colimits

Recall that a category is called (co)complete if it has all (co)limits.

Theorem 2.27. *Let X be a topological spaces. If $\mathcal{C}$ is complete then so are $\mathsf{PSh}(X, \mathcal{C})$ and $\mathsf{Sh}(X, \mathcal{C})$. Limits of presheaves and sheaves are computed objectwise.*

If $\mathcal{C}$ is cocomplete then so are $\mathsf{PSh}(X, \mathcal{C})$ and $\mathsf{Sh}(X, \mathcal{C})$. Colimits of presheaves are computed objectwise while the colimit of a diagram of sheaves is the sheafifiaciton of the (objectwise) colimit of the underlying diagram of presheaves.

In particular the stalk of a colimit of sheaves is the colimit of the stalks.

Proof. We first observe that limits and colimits in the category of presheaves are determined objectwise. If you are less familiar with (co)limits it's a good exercise to check this for yourself.

By the adjunction $(-)^{\mathsf{sh}} \leftrightarrows \iota$ of Lemma 2.25 sheafification preserves colimits, thus with Corollary 2.24 we have

$$\operatorname*{colim}_j \mathcal{F}_j = \operatorname*{colim}_j (\iota\mathcal{F}_j)^{\mathsf{sh}} = (\operatorname*{colim}_j \iota\mathcal{F}_j)^{\mathsf{sh}}.$$

By Corollary 2.23 the statement about stalks follows.

To compute the limit of sheaves not that the objectwise limit of a diagram of sheaves is again a sheaf: The sheaf condition may be formulated as a limit and limits commute with limits. In other words, we may compute that for a cover $\{U_j\}$ of U and our diagram $\mathcal{F}_i$ of sheaves we have

$$\begin{aligned} \lim_i \mathcal{F}_i(U) &\cong \lim_i \lim_j \mathcal{F}_i(U_j) \\ &\cong \lim_j \lim_i \mathcal{F}_i(U_j) \end{aligned}$$

where we used that the $\mathcal{F}_i$ are sheaves and then that limits commute with limits (by what it means to be a limit). So the objectwise limit is a sheaf and satisfies the universal property of being a limit of presheaves, but then it also satisfies the weaker universal property of being a limit of sheaves.

Note that the fact that limits of sheaves exist and are given by the limit of presheaves also follows from the (non-trivial) category-theoretic statement that any inclusion with a left adjoint creates limits. □

We now consider sheaves with values in a fixed abelian category $\mathcal{A}$, for example R-modules for a fixed commutative ring R.

Then in particular a kernel of a map of sheaves is determined pointwise. We say that a map of sheaves is *injective* if its kernel is the 0 sheaf, i.e. it is injective on each open.

We say $f : \mathcal{F} \to \mathcal{G}$ is *surjective* if the cokernel is the 0 sheaf, which is the case if and only if all the maps $f_x : \mathcal{F}_x \to \mathcal{G}_x$ on stalks are surjective. In particular the map does not have to be surjective on each open. The condition is also called locally surjective to emphasize this point.

Remark 2.28. In fact these are precisely monomorphisms and epimorphisms in the category of sheaves and arguably these are the better terms to use. But enough people use the words injections and surjections.

Example 2.29. The need to sheafify the cokernel may look like a formal inconvenience, but it has a mathematical meaning. Let X be a complex manifold (like $\mathbb{C} \setminus \{0\}$) and $\mathcal{O}$ the sheaf of holomorphic functions.

Consider for example the inclusion of sheaves $\underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}$. This is the kernel of the exponential map from $\mathcal{O} \to \mathcal{O}^\times$ whose image as a presheaf we denote by $\mathcal{F}$. Then $\mathcal{F}$ is the presheaf of functions admitting a logarithm. We obtain a short exact sequence of presheaves

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{F} \to 0$$

which is just a compact way of saying $\mathcal{O} \to \mathcal{F}$ is an epimorphism with kernel $\mathbb{Z}$.

However, the presheaf cokernel $\mathcal{F}$ is not a sheaf. Having a logarithm is not a local property so if we try to glue locally defined functions which admit logarithms into a global function, the result will not in general have a logarithm.

The sheafification of $\mathcal{F}$ is $\mathcal{O}^\times$, the sheaf of invertible holomorphic functions. It is clear this is a sheaf so it suffices to check that $\mathcal{O}^\times$ is the stalkwise cokernel of the map $\underline{\mathbb{Z}} \to \mathcal{O}$. The sheaf of locally constant funnctions is the kernel of the exponentiation map, s we need to check surjectivity. Let (s, U) be a nonzero holomorphic function on some open U containing y. Shrinking U if necessary we may assume $s(y) \in B_{\frac{1}{2}|f(x)|}(f(x))$ and we have a well-defined logarithm.

The proof of the following lemma contains a brief reminder what an abelian category is.

Lemma 2.30. *The category* $\mathsf{Sh}(X, \mathcal{A})$ *of sheaves with values in the abelian category* $\mathcal{A}$ *is itself abelian.*

Proof. $\mathsf{Sh}(X)$ clearly has hom spaces which are abelian groups, it has a zero object given by the constant sheaf taking the value zero and we have seen it has finite limits and colimits in Theorem 2.27 as $\mathcal{A}$ has finite limits and colimits. We also observe that finite coproducts are equal to finite products. The presheaf finite product and coproduct agree, and this shows the finite coproduct is already a sheaf and thus equal to its own sheafifiaciton by Corollary 2.24 which is the coproduct of sheaves.

It remains to show that the natural map from the image of a map f (defined as $\ker \operatorname{coker}(f)$) to the coimage (defined as $\operatorname{coker} \ker(f)$) is an isomorphism. But this may be checked on stalks by Theorem 2.27 and Lemma 2.31 below, and on stalks it follows from the result in $\mathcal{A}$. □

Lemma 2.31. *Let $(\mathcal{F}_i)_{i \in I}$ be a finite diagram of sheaves on X. Then $(\lim \mathcal{F}_i)_x \cong \lim_i (\mathcal{F}_i)_x$ for all $x \in X$.*

Proof. By definition the stalk is a filtered colimit and colimits commute with finite limits in categories sufficiently like Set, see Theorem A.37.

But one can also prove this in a more elementary way. Every finite limit is an equalizer of maps between finite products by a variation of Lemma A.38. In an abelian category the finite products are finite coproducts and commute with stalks, and the equalizer of two maps f, g may be replaced by a kernel of $g - f$. Thus it suffices to show that given a map of sheaves $f : \mathcal{F} \to \mathcal{G}$ we have $\ker(f)_x = \ker(f_x : \mathcal{F}_x \to \mathcal{G}_x)$ and this follows by unravelling definitions: Elements of the left hand side are germs (U, s) with $f(s) = 0$ and elements of the right hand side are germs (V, t) with $f(t|_{V'}) = 0$ for some $x \in V' \subset V$. Up to equivalence of germs these sets agree. □

Note that infinite limits cannot usually be computed stalkwise.

2.4. Functors of sheaves

Given a continuous map $f : X \to Y$ of topological spaces we would like to transport sheaves along f.

Definition 2.32. Let $f : X \to Y$ be continuous and let $\mathcal{F}$ be a sheaf on X. Then we define the *pushforward sheaf* $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$ on Y.

Lemma 2.33. *The pushforward sheaf is indeed a sheaf.*

Proof. This follows as the preimage of a cover is a cover. □

Example 2.34. Let X be any topological space and $p : X \to *$ the only map to the one element space. Then for any $\mathcal{F}$ in $\mathsf{Sh}(X, \mathcal{C})$ the object $p_*\mathcal{F} = \mathcal{F}(X)$ in $\mathsf{Sh}(*, \mathcal{C}) = \mathcal{C}$ is also written as $\Gamma(X, \mathcal{F})$, the *global sections* of $\mathcal{F}$.

Example 2.35. Let $i : x \to X$ be an inclusion of a point and $M \in \mathcal{A}$. Then i_*M is the sheaf defined as $i_*M(U) = M$ if $x \in U$ and 0 otherwise. This is called the *skyscraper sheaf* at x.

Definition 2.36. Let $f : X \to Y$ be continuous and let $\mathcal{F}$ be a sheaf on X. Then we define the *pullback sheaf* $f^{-1}F$ as the sheafification of the presheaf $U \mapsto \operatorname{colim}_{f(U) \subset V} \mathcal{F}(V)$.

Example 2.37. Let X be any topological space and $p : X \to *$ the only map to the one element space. For U open in X and R a ring considered as a sheaf on $*$ then $p^{-1}R(U)$ is the sheaf associated to the presheaf $U \mapsto R$, using that the index category of all V with $p(U) \subset V$ only has the element $\{*\}$. Thus $p^{-1}R = \underline{R}$.

Example 2.38. Let $i : x \to X$ be the inclusion of a point and let $\mathcal{F}$ be a sheaf on X. Then $i^{-1}\mathcal{F}$ is by definition equal to the stalk $\mathcal{F}_x$.

Example 2.39. Let $j : U \to X$ be the inclusion of an open set and $\mathcal{F}$ a sheaf on X. Then $j^{-1}\mathcal{F}(V) = \mathcal{F}(V)$ with $V \subset U \subset X$. This is a sheaf (by the sheaf condition on X) and is also denoted $\mathcal{F}|_U$ and called the restricition of $\mathcal{F}$ to U. (This is not to be confused with the restriction maps of sections of a sheaf induced by an inclusion of open sets.)

Example 2.40. A sheaf $\mathcal{F}$ on X is called *locally constant* if there is a cover of X by open sets U_i such that each $\mathcal{F}|_{U_i}$ is isomorphic as a sheaf to the constant sheaf.

Let for example $X = S^1$ and M the open Möbius strip whith projection $p : M \to S^1$. Then the sheaf $\mathcal{S}$ of sections of M, defined as locally constant maps $s : U \to M \times S^1 U$ with $ps = \mathbf{1}_U$, forms a sheaf. (As being locally constant is a local condition this is a subsheaf of the sheaf of sections from Example 2.9) It is locally constant as we can cover X by two open intervals U_1, U_2 on which the Möbius band is homeomorphic to $U_i \times \mathbb{R}$. This identifies our sheaf of sections with the locally constant functions, which is the constant sheaf.

The definition of the sheaf pullback looks unwieldy, but it is well-behaved on stalks.

Lemma 2.41. *Let $f : X \to Y$ be continuous and let $\mathcal{F}$ be a sheaf on Y. We have $(f^{-1}\mathcal{F})_x = F_{f(x)}$. In particular let $i_y : * \to Y$ be the inclusion of a point. Then $(i_y)^{-1}\mathcal{F} = \mathcal{F}_y$.*

Proof. We may take the stalk $(f^{-1}\mathcal{F})_x$ as the stalk of the underlying presheaf, thus we compute $\operatorname{colim}_{x \in U} \operatorname{colim}_{f(U) \subset V} \mathcal{F}(V)$ which is exactly $\mathcal{F}_{f(x)} = \operatorname{colim}_{f(x) \in V} \mathcal{F}(V)$ by unravelling definitions. (Any V containing $f(x)$ also contains the image of an open containing x, namely $f^{-1}V$.) □

The following fact is extremely useful.

Theorem 2.42. *Given $f : X \to Y$ there is an adjunction $f^{-1} \dashv f_* : \mathsf{Sh}(Y) \leftrightarrows \mathsf{Sh}(X)$.*

Proof. We fix $\mathcal{F} \in \mathsf{Sh}(X)$ and $\mathcal{G} \in \mathsf{Sh}(Y)$. It is possible to write down natural maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ which are the unit and counit of the adjunction, or equivalently write down natural maps between $\operatorname{Hom}(\mathcal{G}, f_*\mathcal{F})$ and $\operatorname{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$. Checking the triangle equalities, respectively the fact the maps are indeed inverse is not pleasent (books like to skip this step). The following trick is from Vakil's *Foundations of Algebric Geometry*, Exercise 2.7.B.

Define the set $\operatorname{Hom}^C(\mathcal{G}, \mathcal{F})$ as the set of all collections of maps $\phi_{UV} : \mathcal{G}(V) \to \mathcal{F}(U)$ for $f(U) \subset V$ which are compatible with restrictions.

From the point of view of the open sets $U \subset X$ these maps are represented by maps $\operatorname{colim}_{f(U) \subset V} \mathcal{G}(V) \to \mathcal{F}(U)$. Compatibility with restriction means we have a map from the diagram of all V with $f(U) \subset V$, thus we obtain a map from the colimit.

From the point of view of the open sets $V \subset Y$ these maps are represented by maps $\mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$, as for a fixed V any U with $f(U) \subset V$ is a subset of $f^{-1}(V)$. □

For example we may compute for any sheaf $\mathcal{F}$ on X that

$$\mathrm{Hom}(\underline{\mathbb{Z}}, \mathcal{F}) \cong \mathrm{Hom}(p^{-1}\mathbb{Z}, \mathcal{F}) \cong \mathrm{Hom}(\mathbb{Z}, p_*\mathcal{F}) \cong \Gamma(X, \mathcal{F})$$

for $p : X \to *$.

Definition 2.43. Given $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}(X)$ we define the *sheaf of homomorphisms* $U \mapsto \mathcal{H}om_{\mathsf{Sh}(U,\mathcal{A}}(\mathcal{F}|_U, \mathcal{G}_U)$.

One can check that this is indeed a sheaf.
In particular the hom space $\mathrm{Hom}_{\mathsf{Sh}(X)}(\mathcal{F}, \mathcal{G})$ is nothing but $\Gamma(\mathcal{H}om(\mathcal{F}, \mathcal{G}))$.
Restricting Theorem 2.42 to open subset of Y shows the following:

Corollary 2.44. *For any $f : X \to Y$ and sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y we have $f_* \mathcal{H}om_X(f^*\mathcal{G}, \mathcal{F}) \cong \mathcal{H}om_Y(\mathcal{G}, f_*\mathcal{F})$.*

Proof. Let V be an open subset of Y and apply Theorem 2.42 to the restriciton $f' : f^{-1}V \to V$ to obtain $\mathrm{Hom}_{f^{-1}V}(f'^*\mathcal{G}, \mathcal{F}) \cong \mathcal{H}om_V(\mathcal{G}, f'_*\mathcal{F})$. This verifies the corollary on each open. □

Let now $\mathcal{R}$ be a sheaf of rings on X. We let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of left $\mathcal{R}$-modules on X, see Definition 2.13. We can define $\mathcal{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$ as the subsheaf of $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ that on each U consists of $\mathcal{R}(U)$-linear maps.

Similarly $\mathcal{R}$ be a sheaf of rings on X, $\mathcal{F}$ a sheaf of left $\mathcal{R}$-modules and $\mathcal{G}$ a sheaf of right $\mathcal{R}$-modules. (Equivalently $\mathcal{G}$ is a sheaf of left $\mathcal{R}^{\mathrm{op}}$-modules.) Then there is a presheaf of abelian groups $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{R}(U)} \mathcal{G}(U)$ which we may sheafify to obtain a tensor product of sheaves.

The tensor hom adjunction of modules directly gives us a tensor hom adjunction for sheaves:

Corollary 2.45. *Let $\mathcal{R}, \mathcal{S}$ be sheaves of rings on X. Let $\mathcal{F}$ be a $\mathcal{R} \otimes \mathcal{S}^{op}$-module sheaf, $\mathcal{G}$ a sheaf of $\mathcal{S}$-modules and $\mathcal{H}$ a sheaf of $\mathbb{R}$-modules. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{F} \otimes_{\mathcal{S}} \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}_{\mathcal{S}}(\mathcal{G}, \mathcal{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{H}))$$

where we used that $\mathrm{Hom}_{\mathcal{R}}(\mathcal{F}, \mathcal{H})$ has a natural $\mathcal{S}$-module structure.

Proof. We may check on each open, using Corollary 2.25. □

3. An introduction to homological algebra

3.1. Exactness

We now work in some general abelian category. This could be R-Mod for an arbitrary unital ring R or the category $\mathsf{Sh}(X, \mathcal{A})$ of sheaves on some space X with values in some other abelian category $\mathcal{A}$.

All our functors will be additive, i.e. they preserve finite sums (which are the same as finite products). It is a key question if they preserve kernels and/or cokernels.

Definition 3.1. A *(cochain) complex* in $\mathcal{A}$ is a sequence of objects $A^i \in \mathcal{A}$ where $i \in \mathbb{Z}$ with *differentials* $d_i : A^i \to A^{i+}$ satisfying $d_{i+1} \circ d_i = 0$.

A morphism of complexes $A \to B$ is called a *chain map*, it consists of maps $f^i : A^i \to B^i$ for every i which commute with the differential.

Complexes and the morphisms between them form the category $\mathsf{Ch}(\mathcal{A})$.

The i-th *cohomology* of a complex C is $\ker(d_i)/\operatorname{Im}(d_{i-1})$.

A cochain complex C whose cohomology group $H^i(C) = 0$ is called *exact at* C^i. And if all cohomology groups vanish it is called *exact* or *acyclic*. We also call an exact cochain complex an *exact sequence*.

It is often convenient to consider a *shifted* complex $A[1]$ defined by $A[1]^i = A^{i+1}$ and $d_i^{A[1]} = -d_{i+1}^A$.

This is *cohomological grading convention*. It is often convenient to instead use *homological grading convention* where the diferenital decreases degree.

We will identify objects of $\mathcal{A}$ with cochain complexes in $\mathsf{Ch}(\mathcal{A})$ concentrated in degree 0.

Definition 3.2. An exact chain complex $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ is called a *short exact sequence*.

We also say B is an *extension* of C by A.

Example 3.3. For any objects A, C in $\mathcal{A}$ there is a shorte exact sequence

$$0 \to A \to A \oplus C \to C \to 0$$

called a *split* short exact sequence. One can show an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split if and only if f has a left inverse or g has a right inverse.

In particular in a short exact sequence we have $\ker(f) = 0$, $\operatorname{coker}(g) = 0$ and $\ker(g) = \operatorname{Im}(f)$.

If you know homology from topology you know that the sequence of singular chains is exact at the object in degree n if there aren't any "holes" in degree n. In homological algebra you study this condition algebraically.

Lemma 3.4. *A sequence of sheaves* $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ *is a short exact sequence if and only if* $\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$ *is a short exact sequence at each* x

Proof. We have seen in the proof of Theorem 2.19 that f is injective if all f_x are injective. By definition we see that g is surjective if all g_x are surjective.

It remains to compare the image of f with the kernel of g. But as kernel and image (by definition the kernel of a cokernel) are computed stalkwise by Theorem 2.27 and Lemma 2.31 this follows. □

Example 3.5. Consider a point x_0 on the manifold $\mathbb{R}$. Then there is a short exact sequence of sheaves

$$0 \to \mathcal{C}^0 \xrightarrow{(x-x_0)\cdot} \mathcal{C}^0 \to \mathbb{R}_x \to 0$$

where $\mathbb{R}_x$ is the skyscraper sheaf at x.

3.2. Exact functors

Short exact sequences are thus a way to encode monomorphisms, epimorphisms and extensions. We now examine what functors do to them.

Definition 3.6. An additive functor that preserves short exact sequences is called *exact*. An additive functor that sends an exact sequence $0 \to A \to B \to C \to 0$ to an exact sequence $0 \to F(A) \to F(B) \to F(C)$ (not necessarily exact on the right!) is called *left exact*. Similarly for right exact functors.

Example 3.7. For any object M of $\mathcal{A}$ the functor $\mathrm{Hom}(M, -) : \mathcal{A} \to \mathsf{Ab}$ is left exact. The functor $\mathrm{Hom}(-, M) : \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ is also left exact.

Example 3.8. For any $f : X \to Y$ the functor f_* is left exact. By Theorem 2.42 we see that f_* is a right adjoint, thus it preserves all limits and in particular kernels. It follows that f_* is left exact. The lack of right exactness of f_* will occupy us for the rest of the semester.

Let us reiterate that by the argument in the example all left adjoints are right exact, and all right adjoints are left exact.

Remark 3.9. Arguing as in Lemma 2.31 a functor is left exact if and only if it preserves finite limits, and right exact if and only if it preserves finite colimits.

Example 3.10. The functor $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$ is not just right exact (as it's a left adjoint) but is exact. This follows as $f^{-1}\mathcal{F}_x = \mathcal{F}_{f(x)}$ by Lemma 2.41 and we can check exactness at stalks by Lemma 3.4.

Example 3.11. Let $i : Z \to X$ be a closed inclusion. Then i_* is exact. We use again that we can check exactness at stalks, so given an exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ we consider $(i_*\mathcal{F})_x \to (i_*\mathcal{G})_x \to (i_*\mathcal{H})_x$ at an arbitrary $x \in X$. As Z is closed we see that all stalks vanish for $x \notin Z$. If on the other hand $x \in Z$ we have $(i_*\mathcal{F})_x = \mathrm{colim}_{x \in V} \mathcal{F}(V \cap Z)$ which agrees with $\mathcal{F}_x$ as the $V \cap Z$ are exactly the open neighbourhoods of x in Z.

Example 3.12. The pushforward f_* is not exact in general. Consider the projection $p : X = C \setminus \{0\} \to *$ and the short exact sequence of sheaves $0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^\times \to 0$ from Example 2.29. Pushing forward along p is taking global sections, but the function $\exp : \mathcal{O}(X) \to \mathcal{O}^\times(X)$ is not surjective as there is no global logarithm of the identity function.

3.3. Derived functors

Consider the result of applying a left exact functor to a short exact sequence $A \to B \to C$. If F is not right exact then $F(B) \to F(C)$ is not an epimorphism. So there is a cokernel. Can we compute this cokernel in terms of F and the short exact sequence? Faling that, can we find something which contains the cokernel, and then try to determine the cokernel of the new map and so on. In other words, if we do not have a short exact sequence, can we get a long exact sequence?

One useful observation is that we know that any additive functor will preserve *split* exact sequences. We can relate being split to nice properties of modules:

Definition 3.13. An object M in an abelian category is *projective* if for any epi $q : A \to B$ and any map $f : M \to B$ there is a lift $g : M \to A$ such that $q \circ g = f$. An object N in an abelian category is *injective* if for any monomorphism $i : A \to B$ and any map $f : A \to N$ there is an extension $g : B \to N$ such that $g \circ i = f$.

It is easy to see that M is projective if only if $\mathrm{Hom}(M, -)$ is an exact functor and dually N is injective if and only if $\mathrm{Hom}(-, N)$ is an exact functor.

Example 3.14. In the category of R-modules any free module is projective. In fact projectives are exactly direct summands of free modules.

In the category of abelian groups the groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

Lemma 3.15. *If C is projective or A is injective then $A \to B \to C$ is split, i.e. $B \cong A \oplus C$.*

Sketch of proof. If C is projective use the identity map $C \to C$ to find a one-sided inverse of the map $B \to C$. Dually if A is injective. □

An object of $\mathcal{A}$ can be viewed as a complex concentrated in degree 0. We will now identify such objects with larger complexes consisting of nicer objects.

Definition 3.16. A *quasi-isomorphism* of complexes is a map of complexes $A \to B$ such that the induced map on cohomology is an isomorphism in every degree.

Definition 3.17. A *projective resolution* of A is a levelwise projective complex in nonpositive degrees $P^\bullet$ with a quasi-isomorphism to A.

An *injective resolution* of A is a levelwise injective complex $I^\bullet$ in nonnegative degrees with a quasi-isomorphism from A.

Definition 3.18. The *i-th left derived functor* of a right exact functor F is defined as $L_iF(A) := H_i(F(P))$ where P is a projective resolution of A.

The *i-th right derived functor* of a left exact functor G is defined as $R^iG(A) := H_i(F(I))$ where I is an injective resolution of A.

In the remainder of this section many results will have two versions, we one for left derived functors and one for right derived functors. I will only make statements for *right* derived functors, as these will be more interesting to us in this course, but it will be clear what the analogous statements for left derived functors are.

Lemma 3.19. *For any left exfact functor G we have $R^{<0}G(A) = 0$ and $R^0G(A) = G(A)$.*

Proof. The first statement follows from the definition. For the second statement we have by definition that $A = \ker(I^0 \to I^1)$ for an injective resolution. By left exactness of G and definition of R^0 we have $G(A) = \ker(GI^0 \to GI^1) = R^0G(A)$. □

Example 3.20. We define $\mathrm{Ext}^i_R(A, B)$ to be $R^i \operatorname{Hom}_R(-, B)(A)$. Consider the category of abelian groups, i.e. $R = \mathbb{Z}$. Note that an injective resolution in $\mathbb{Z}\text{-Mod}^{\mathrm{op}}$ is given by a projective resolution in $\mathbb{Z}$-Mod. So $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ is a suitable resolution of $\mathbb{Z}/p$ and we find $\mathrm{Ext}^*(\mathbb{Z}/p, B) = H^*(B \xrightarrow{p} B)$. So $\mathrm{Ext}^0(\mathbb{Z}/p, B) = {}_pB$, the submodule of p-torsion elements, and $\mathrm{Ext}^1(\mathbb{Z}/p, B) = B/pB$.

Definition 3.21. A category has *enough projectives* if for every object there is an epimorphism from a projective object. Dually a category has *enough injectives* if for every object there is a monomorphism to an injective object.

Example 3.22. R-Mod has enough projectives, there is always a surjection $F(M) \to M$, from the free module generated by the elements of M to M.

Lemma 3.23. *The category R-Mod has enough injectives.*

Proof. The proof is explained in [Wei95, Exercise 2.35] and the lead-up to that. □

We collect some fundamental facts for future reference which you hopefully know from previous exposure to homological algebra. Otherwise they are not hard to find, e.g. in the book of Weibel.

Theorem 3.24 (Comparison Theorem)**.** *Let $\epsilon : M \to I^\bullet$ and $\eta : N \to J^\bullet$ be injective resolutions and $f : M \to N$ a homomorphism. Then there is a lift $\tilde{f} : I^\bullet \to J^\bullet$ of f, i.e. we have $\eta \circ \tilde{f} = f \circ \epsilon$. Moreover, $\tilde{f}$ is unique up to chain homotopy equivalence.*

Proof. See [Wei95, Theorem 2.3.7] or do it as an exercise! □

Corollary 3.25. *Injective resolutions exist in $\mathcal{A}$ if there are enough injectives in $\mathcal{A}$. These resolutions are unique up to chain homotopy equivalence.*

Corollary 3.26. *The i-ith right derived functor is a well-defined functor.*

Lemma 3.27. *[Snake Lemma] Any short exact sequence of complexes* $0 \to A \to B \to C \to 0$ *induces a natural long exact sequence in cohomology groups:*

$$\cdots \to H^k(A) \to H^k(B) \to H^k(C) \to H^{k+1}(A) \to \cdots$$

Proof. See [Wei95, Theorem 1.3.1] □

We explain how these facts give the key result on derived functors:

Corollary 3.28. *Let* $F : \mathcal{A} \to \mathcal{B}$ *be a right exact functor between abelian categories. A s.e.s* $0 \to A \to B \to C \to 0$ *in* $\mathcal{A}$ *gives rise long exact sequence of derived functors*

$$0 \to FA \to FB \to FC \to R^1FA \to R^1FB \to R^1FC \to R^2FA \to \cdots$$

in $\mathcal{B}$*. The boundary maps are natural.*

Proof. We resolve all objects injectivly and use the lift from Theorem 3.24. This gives a short exact sequence of complexes which by Lemma 3.15 is split in every degree as all entries are injective. We apply F to obtain a new short exact sequence and taking cohomology we finish with the Snake Lemma 3.27. □

This definition of derived functors is the most direct one, it is not the only one. It is a bit ad hoc and we have to work to show what we have is well-defined.

Injective objects are not always easy to work with, so it is good to have other ways to compute. Let F be a left exact functor between two abelian categories.

Definition 3.29. An object A is *F-acyclic* if $R^iF(A) = 0$ for all $i > 0$.

Acyclic objects can be used to compute derived functors.

Proposition 3.30. *Let A be an object in an abelian category with enough injectives and let* $0 \to A \to S^0 \to S^1 \to \cdots \to S^m \to \cdots$ *be a resolution of A such that each* S^i *is F-acyclic. Then* $H^i(S^\bullet) \cong R^iF(A)$.

Proof. The proof technique here is called *dimension shifting*. We first consider $0 \to A \to S^0 \to Q_0 \to 0$ with Q the quotient object. Then by Lemma 3.28 we have $R^iF(A) \cong R^{i-1}FQ_0$ for $i \geq 2$ while $R^1F(A) \cong FQ_0/FS^0$. With $Q_0 = \ker(S^1 \to S^2)$ and F preserving kernels we get $R^1F(A) = H^1F(S^\bullet)$.

Now Q_0 has an F-acyclic resolution $S^1 \to S^2 \to \cdots$, thus by the same argument we see $R^1F(Q_0) = H^2F(S^\bullet)$. Together with the first part we get the result for $i = 2$.

Now we proceed by induction, letting $Q_i = S^i/S^{i-1} = S^i/Q_{-1} = \ker(S^{i+1} \to S^{i+2})$ we prove $R^{i+1}F(A) = FQ_i/FS^i = H^{i+1}F(S^\bullet)$. □

3.4. The derived category

To compute derived functors we replaced objects, considered as complexes concentrated in degree 0, by quasi-isomorphic complexes. After applying the functor we have a complex which is typically no longer quasi-isomorphic to a complex concentrated in degree 0. Hence it makes sense to consider all complexes, up to quasi-isomorphisms, and try to lift functors to this new category.

As complexes are now fundamental I will drop the $-^\bullet$ from the notation.

Remark 3.31. It is non-trivial to invert quasi-isomorphisms, mainly since it is unclear what happens to morphisms. We'd have to replace them by arbitrarily long zig-zags $* \to * \leftarrow * \to * \leftarrow \cdots \to *$ where all right-to-left maps are quasi-isomorphisms. But if we do not have a set of objects but a proper class then we quickly have a proper class of morphisms to consider, which is a problem.

We first note that there is a natural complex of morphism between two complexes.

Definition 3.32. Let $\mathcal{A}$ be an abelian category and $L, M \in \mathsf{Ch}(\mathcal{A})$. The *hom complex* $\underline{\mathrm{Hom}}(L, M)$is defined by $\underline{\mathrm{Hom}}^i(L, M) = \{f^\bullet : L^\bullet \to M^{\bullet+i}\}$ and $df : a \mapsto d(fa) - (-1)^{|f|} f(da)$ where $|f|$ denotes the degree of f.

In particular a chain map is a cocycle in degree 0.

Definition 3.33. Two chain maps $f, g : L \to M$ in $\mathsf{Ch}(\mathcal{A})$ are *homotopic* if there is a map $h : L \to M[1]$ such that $dh = g - f$. In other words, they agree in $H^0(\underline{\mathrm{Hom}}(L, M))$.

Given an abelian category $\mathcal{A}$ we define the *homotopy category* $K(\mathcal{A})$ to be the category with the same objects as $\mathsf{Ch}(\mathcal{A})$ but with morphisms equal to the *homotopy classes* of chain maps.

There are different boundedness conditions we can put on chain complexes, and hence on $\mathsf{Ch}(\mathcal{A})$ and $K(\mathcal{A})$. Let $\mathsf{Ch}^b(\mathcal{A})$ to be the category of *bounded cochain complexes*, i.e. those A_* such that $A_n = 0$ for all but finitely many n. We also define $\mathsf{Ch}^+(\mathcal{A})$, resp. $\mathsf{Ch}^-(\mathcal{A})$, to be the categories of chain complexes that are bounded below, resp. above. $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ etc. are defined similarly.

Definition 3.34. Given a category $\mathcal{A}$ and a class of morphisms S we define the *localization* of $\mathcal{A}$ at S to be a category $\mathcal{B}$ with a functor $Q : \mathcal{A} \to \mathcal{B}$ such that $Q(s)$ is an isomorphism for any $s \in S$ and which is universal with this property: Any $\mathcal{A} \to \mathcal{C}$ that sends all $s \in S$ to isomorphisms factors through Q.

Definition 3.35. We define the *derived category* $D(\mathcal{A})$ as the localization of $K(\mathcal{A})$ at the class of quasi-isomorphisms. Write $Q_{\mathcal{A}} : K(\mathcal{A}) \to D(\mathcal{A})$ for the natural functor. $D^b(\mathcal{A})$ is defined similarly from $K^b(\mathcal{A})$, etc.

Theorem 3.36. *$D(\mathcal{A})$ exists (as a locally small category).*

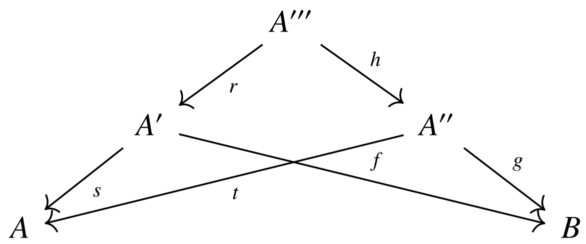
Not a proof. See [Wei95, Sections 10.3 and 10.4] or [Huy06, Section 2.1] for a detailed proof.

I'll just give some comments about the shape of the proof: Localization means that we throw in an inverse f^{-1} for every quasi-isomorphism f. This is a lot like Ore localization for (noncommutative) rings, if you've met that. The content is in working out suitable conditions for the existence of a localization, and checking that they are satisfied in our case. A class of morphisms is called *localising* if

- S contains the identities and is closed under composition.
- Given morphisms $A \xleftarrow{s} A' \to B$ with $s \in S$ there are morphisms $A \to B' \xleftarrow{t} B$ with $t \in S$ making the obvious diagram commute. Dually given the second pair of morphisms there exists the first one.
- Given any morphism f, g the existence of $s \in S$ with $sf = sg$ is equivalent to the existence of $t \in S$ with $ft = gt$.

These conditions hold for quasi-isomorphisms in the homotopy category, but not in the category of chain complexes. □

The second condition in the proof allows us to write any morphism in the derived category as a 2-term zig-zag or "roof" $(s, f) := A \xleftarrow{s} A' \xrightarrow{f} B$ with $s \in S$. Two morphisms (s, f) and (t, g) represent the same map if there is a common roof (r, h) with $sr = th$ and $hg = fr$.



This is sometimes called a *calculus of fractions*.

Remark 3.37. Any complex with homology bounded above has a natural quasi-isomorphism from a bounded above complex. Any complex with homology bounded below has a quasi-isomorphism to a bounded below complex. Together this gives a chain of quasi-isomorphisms between a complex with bounded cohomology and a bounded complex. There will not be chain homotopies in general.

Example 3.38. Let R be a ring. We define the *derived category* of R, $D(R)$, as the derived category of $\mathsf{Ch}(R\text{-Mod})$.

Because of its definition $D(\mathcal{A})$ is a bit hard to work with. For example, it's an additive category, but that is not obvious from the definition!

A morphism $A \xleftarrow{s} A' \to\xrightarrow{f} B$ in $D(\mathcal{A})$ is 0 if it is equivalent to a morphism homotopy equivalent to 0, and unravelling definitions this means there must exist a quasi-isomoprhism r with $f \circ r$ chain homotopic to 0.

Remark 3.39. $D(\mathscr{A})$ is not abelian but is an example of a *triangulated category*. Identifying complexes up to quasi-isomorphism is a good middle ground between the homotopy category of all complexes (which is very large) and the category $\mathscr{A}$ (which is not well-suited to cohomology and derived phenomena). One issue with the derived category is that while it admits derived functors (as we will see) it does not keep track of higher coherences. This makes it ill-suited for some applications. For example it is not possible to glue locally defined derived categories.

To keep track of this extra structure one may replace the derived category by a certain *stable ∞-category* which is obtained as the ∞-categorical localisation of $K(\mathscr{A})$ at all quasi-isomorphisms. One can do this explicitily to build a simplicial derived category using the so-called "hammock localization" or by abstract properties of ∞-categories, as explained e.g. in [Cis19].

For the purposes of this course we will need neither the details of triangulated categories nor of stable ∞-categories.

The name triangulated category refers to the following very useful construction:

Definition 3.40. Given a chain map $f : A \to B$ in $K(\mathscr{A})$ its *cone* is defined as the complex C with $C^n = A^{n+1} \oplus B^n$ and $d_n : (a, b) \mapsto (-da, db - fa)$.

By construction there are natural maps $B \to \mathrm{cone}(f)$ and $\mathrm{cone}(f) \to A[1]$, and we can build a sequence of morphisms $A \xrightarrow{f} B' \to \mathrm{cone}(f) \to A[1]$ (which may of course be continued to the left and to the right).

We call any sequence $A \to B \to C \to A[1]$ in $K(\mathscr{A})$ or $D(\mathscr{A})$ that is isomorphic (in $K(\mathscr{A})$, respectively $D(\mathscr{A})$) to a sequence of the form $D \to \mathrm{cone}\,(g) \to E \xrightarrow{g}\to D[1]$ an *exact triangle*. Here an isomorphism of sequences means there are isomorphisms $A \to D$, $B \to \mathrm{cone}(g)$ and $C \to E$ making the obvious diagraom commute.

One can show that if $A \to B \to C \to A[1]$ is an exact triangle so is $B \to C \to A[1] \to B[1]$ and so forth, see [Wei95, Example 10.1.6]. Thus while exact triangles are related to short exact sequences in an abelian category there is no object singled out. (This also shows our slightly non-standard definition is equivalent to the usual ones.)

Proposition 3.41. *A chain map $f : A \to B$ is a quasi-isomorphism if and only if* $\mathrm{cone}(f)$ *is acyclic, i.e. it has no homology.*

Proof. By definition we have a short exact sequence of complexes $B \to \mathrm{cone}(f) \to A[1]$. In the associated long exact sequence of homology groups (Lemma 3.27) the boundary maps are the maps induced by f on cohomology, thus the result follows. □

To get a more concrete representation of hom spaces in the derived category we have the following very useful result:

Theorem 3.42. *Given a complex of injectives $I \in K^+(\mathscr{A})$ and any cochain complex A we have* $\mathrm{Hom}_{K(A)}(A, I) \cong \mathrm{Hom}_{D(\mathscr{A})}(A, I)$.

Sketch of proof. We first show that $\underline{\mathrm{Hom}}_{K(\mathcal{A})}(-, I)$ sends quasi-isomorphisms to quasi-isomorphisms. We know by Proposition 3.41 that $f : A \to B$ is a quasi-isomorphism if its cone $C := \mathrm{cone}(f)$ is acyclic. So we let $B \to C \to A[1]$ be the short exact sequence in $\mathsf{Ch}(\mathcal{A})$ associated to the map $f : A \to B$. We apply $\underline{\mathrm{Hom}}(-, I)$ to obtain a sequence $\underline{\mathrm{Hom}}(A[1], I) \to \underline{\mathrm{Hom}}(C, I) \to \underline{\mathrm{Hom}}(B, I)$. Since our short exact sequence is levelwise split this is a level-wise split short exact sequence. (Unravelling definitions we have $\prod_n \mathrm{Hom}(A^{n+1}, I^{n+i})) \to \prod_n \mathrm{Hom}(C^n, I^{n+i}) \to \prod_n \mathrm{Hom}(B^n, I^{n+1})$ in each degree, which is split exact.)

By construction $\underline{\mathrm{Hom}}(C, I)$ is the cone of $\underline{\mathrm{Hom}}(f, I)$, thus if $\underline{\mathrm{Hom}}(C, I) \simeq 0$ we have that the natural map $\underline{\mathrm{Hom}}(B, I) \to \underline{\mathrm{Hom}}(A, I)$ is a quasi-isomorphism.

To show that $\underline{\mathrm{Hom}}(C, I)$ is indeed acyclic if C is we build a homotopy to 0 for an arbitrary chain map $g : C \to I[i]$. This shows that $\underline{\mathrm{Hom}}(C, I)$ has no cohomology and the desired result follows.

We build our homotopy $h : C \to I[i-1]$. As I is bounded below we may use induction and start with $h^{k_0} = 0$ for some small enough k_0. Assuming we found $h^{\leq k}$ with $g^{k-1} = dh^{k-1} - (-1)^{i-1} i h^k d : C^{k-1} \to I^{k-1+i}$ we consider $g^k - dh^k : C^k \to I^{k+i}$. Using that g is a chain map (i.e. $dg = (-1)^i gd$) this factors through C^k/C^{k-1}:

$$\begin{aligned}(g^k - dh^k)d &= (-1)^i dg^{k-1} - dh^k d \\ &= (-1)^i d(dh^{k-1} - (-1)^{i-1} h^k) - dh^k d \\ &= dh^k d - dh^k d = 0\end{aligned}$$

Since C^k/C^{k-1} injects into C^{k+1}, by injectivity of I we may extend to a map $(-1)^k h^k$ which satisfies precisely $g^k = dh^k - (-1)^k h^{k+1} d$.

We now consider the map $f \mapsto (\mathbf{1}, f)$ from $\mathrm{Hom}_{K(A)}(A, I)$ to $\mathrm{Hom}_{D(\mathcal{A})}(A, I)$. Let (s, f) be morphism $A \to B$ in $D(\mathcal{A})$, to show the theorem we have to show it is equivalent to a unique $(\mathbf{1}, g)$. By our first claim there is a unique g with $gs = f$ and then $(\mathbf{1}, g)$ is equivalent to (s, f). To show uniqueness we look at the equivalence criterion for fractions. It suffices to show that $(\mathbf{1}, g)$ and $(\mathbf{1}, g')$ are only equivalent if g and g' are homotopic. From the diagram we read off that $gr \simeq g'r$ for some quasi-isomorphism r. Applying H^0 to the first claim this implies $g \simeq g'$. □

The following corollary allows us to compute hom-sets in the derived category.

Corollary 3.43. *Assume $\mathcal{A}$ has enough injectives. Then for objects $A, B \in \mathcal{A}$ considered as complexes concentrated in degree 0 we have* $\mathrm{Hom}_{D(\mathcal{A})}(A, B[i]) = \mathrm{Ext}^i(A, B)$.

Proof. The two sides may be identified with the two sides of $\mathrm{Hom}_{D(\mathcal{A})}(A, I[i]) \cong \mathrm{Hom}_{K(\mathcal{A})}(A, I[i]))$ where I is an injective resolution of B. □

Remark 3.44. This result remains true with the same proof for A, B any bounded below complex as long as we define $\mathrm{Ext}^i(A, B)$ suitably, i.e. via level-wise injective resolution of B.

Corollary 3.45. *Assume $\mathcal{A}$ has enough injectives. Consider the subcategory $K^+(Inj(\mathcal{A}))$ of $K^+(\mathcal{A})$ that consists of levelwise injective complexes. Then the natural quotient map $Q_{\mathcal{A}} : K^+(Inj(\mathcal{A})) \to D^+(\mathcal{A})$ is an equivalence of categories.*

Sketch of proof. Full faithfulness follows from Theorem 3.42. To show inclusion is essentially surjective we have to injectively resolve complexes that are bounded below. There is a natural but technical proof proceding by induction and using the existence of enough injectives, details are in [GM03, p. III.5.25]. □

Remark 3.46. Total derived functors may be constructed even in the absence of injective or projective resolutions. Say a class of objects $\mathcal{R} \subset K^+(\mathcal{A})$ is *adapted* to a left exact functor $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ if it preserves acyclic complexes and any $A \in K^+(\mathcal{A})$ is quasi-isomorphic to an object R_A in $\mathcal{R}$.

Then $RF(A)$ is defined as $F(R_A)$ and has all the desirable properties, see [GM03].

A. Basic category theory

I will give a rapid fire overview of category theory. The focus is on definitions and examples, with a few results thrown in, but no proofs (those can be found in any standard reference, e.g. Mac Lane's "Categories for the working mathematician").

If you have met a few concepts here and there this should be nice refresher putting everything we need together in a systematic way

If you are comfortable with categories up to limits and adjunctions you can skip this. The least standard part is probably Section A.2.3 on filtered colimits.

A.1. Basics

A.1.1. Categories and Functors

Definition A.1. A *category* $\mathcal{C}$ consists of the following data:

- a class of *objects* $\mathrm{Ob}(\mathcal{C})$,
- for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a class of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (also called arrows),
- for every object X a distinguished morphism $\mathbf{1}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, the *identity*
- for every three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a *composition* $\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$,

such that

- compositon is associative: $(f \circ g) \circ h = f \circ (g \circ h)$,
- the identity is an identity for composition: $\mathbf{1}_Y \circ f = f = f \circ \mathbf{1}_X$ for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Given f in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ we call X the *source* and Y the *target* of f.

Example A.2.

1. Sets and functions form a category we denote by Set. (Since we want to consider the category of all sets and want to avoid paradoxa we referred to a class of objects in our definition.)
2. Topological spaces and continuous maps form a category Top. It is easy to consider the subcategory of CW complexes or path connected spaces etc.

3. There is also a category Top_* whose objects are pointed topological spaces (X, x_0) and whose morphisms are base-point preserving maps, i.e. $f : (X, x_0) \to (Y, y_0)$ is given by $f : X \to Y$ with $f(x_0) = y_0$.

   This is an example of an *undercategory*: Given any category $\mathcal{C}$ with an object C there is a category whose objects are arrows $f : C \to D$ in $\mathcal{C}$, and whose morphisms are maps $g : D \to D'$ making the obvious triangle commute: $g \circ f = f' : C \to D1$. Top_* is the category of topological spaces under the one point space.

4. In algebra we find many further categories: Groups and homomorphisms form the category Group, vector spaces over k and linear maps form Vect_k, abelian groups, rings, fields, etc. all form categories

5. There is a category with one object and one morphism (the identity of the object). In general a category is called *discrete* if the identities are the only morphisms. Every set I can be considered as a discrete category $\mathbf{I}$ with $\mathrm{Ob}(\mathbf{I}) = I$.

6. For every category $\mathcal{C}$ there is an *opposite category* $\mathcal{C}^{op}$ with the same objects, $\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$ and $f \circ_{\mathcal{C}^{op}} g := g \circ_{\mathcal{C}} f$. Thus we obtain the opposite category $\mathcal{C}^{op}$ from $\mathcal{C}$ by turning around all arrows.

We will often abuse notation and write $C \in \mathcal{C}$ as a shortcut for "C is an object of $\mathcal{C}$".

Definition A.3. A morphism $f : C \to D$ is called *isomorphism*, if there is $g : D \to C$ such that $g \circ f = \mathbf{1}_C$ and $f \circ g = \mathbf{1}_D$.

Homeomorphisms and (group/ring/vector space) isomorphisms are examples.
In all categories we consider isomorphic object as equivalent and (almost) interchangeable.

Remark A.4. If the objects and morphisms of a category form sets we call it a *small category*. If there may be a class of objects but the morphisms between any two pair of objects form a set we say the category is *locally small*.

Many categories we are interested in, like Top, Set and Group are not small, but locally small.

Example A.5. A small category in which there is at most one morphism between any two objects and in which any isomorphism is an identity is called a *partial order*. Then the composition is uniquely determined by the morphisms (as there is only one function into a set with one element).

An example is the category $\mathbb{N}$ whose objects are the natural numbers and where there is a morphism $i \to j$ if and only if $i \leq j$.

An important motivation for the study of category theory is the observation that mathematical objects are often better understood through the morphisms between them. The same principle holds for categories.

Definition A.6. A *functor* F between two categories $\mathcal{C}$ and $\mathcal{D}$ consists of the following data:

- a map that associates to any $X \in \mathrm{Ob}(\mathcal{C})$ an object $F(X) \in \mathrm{Ob}(\mathcal{D})$.
- for each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a map from $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ which we write as $f \mapsto F(f)$,

such that

- F is compatible with composition: $F(f \circ g) = F(f) \circ F(g)$,
- F preserves the identities: $F(\mathbf{1}_X) = \mathbf{1}_{F(X)}$.

Example A.7.

1. For every category $\mathcal{C}$ there is an identity functor $\mathbf{1}_{\mathcal{C}}$ that does nothing on objects and morphisms.
2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and D an object of $\mathcal{D}$. Then there is a constant functor $c_D : \mathcal{C} \to \mathcal{D}$ that sends every object of $\mathcal{C}$ to D and any morphism of $\mathcal{C}$ to $\mathbf{1}_D$.
3. A family of topological spaces $(X_i)_{i \in I}$ is nothing but a functor from I, considered as a discrete category, to Top.
4. From every category whose objects have an underlying set e.g. $\mathsf{Top}, \mathsf{Group}, \mathsf{Vect}_k$) there is a *forgetful functor* to Set, that forgets all additional structure.
5. Algebraic Topology is in no small part the study of functors from topological spaces to algebraic categories.

   The homotopy groups are functors $\pi_n : \mathsf{Top}_* \to \mathsf{Group}$ associating to any pointed topological space (X, x_0) the homotopy group $\pi_n(X, x_0)$ and to any map $f : X \to Y$ the induced map f_*.

   Similary homology groups are functors $H_n : \mathsf{Top} \to \mathsf{Ab}$.

   Cohomology groups are functors $H^n : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Ab}$. Note that these functors turns around the direction of arrows, which is why we write it as a functor from the opposite category. We also call such functors *contravariant*.

It is easy to see that functors can be composed, so there is a *category of categories* whose objects are (small) categories and whose morphisms are functors.

A.1.2. Natural Transformations

Remarkably, there are not just maps between categories (the functors) but also maps between maps betwen categories.

Definition A.8. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* α from F to G consists of maps $\alpha_C : FC \to GC$ for every $C \in \mathcal{C}$ such that for every map $f : C \to C'$ in $\mathcal{C}$ there is a commutative diagram:

$$\begin{CD} FC @>{Ff}>> FC' \\ @V{\alpha_C}VV @VV{\alpha_{C'}}V \\ GC @>{Gf}>> GC' \end{CD}$$

Remark A.9. You might think that it is easier to write $\alpha_{C'} \circ Ff = Gf \circ \alpha_C$ instead of drawing the commutative diagram.

The commutative diagram has the advantage that it keeps track of all the objects as well as the morphisms between them. More importantly, in category theory, algebraic topology and homological algebra there is often a plethora of maps whose compositions we want to compare, and it is much easier to keep track if one arrange them all in a beautiful diagram.

Example A.10.

1. There is a functor $D : \mathsf{Vect}_k \to \mathsf{Vect}_k$ that takes every vector space to its double dual $V \mapsto (V^*)^*$. Then for every vector space there is a map $\iota : V \to DV$ that sends $v \in V$ to the functional $\alpha \mapsto \alpha(v)$. This map is natural, meaning it is compatible with linear maps. In other words, ι is a natural transformation from the identity functor $\mathbf{1}_{\mathsf{Vect}}$ to the double dual D.

2. For any functor $F : \mathcal{C} \to \mathcal{D}$ there is the identity natural transformation $\mathbf{1}_F$ defined by $(\mathbf{1}_F)_C = \mathbf{1}_{FC}$ for every $C \in \mathcal{C}$.

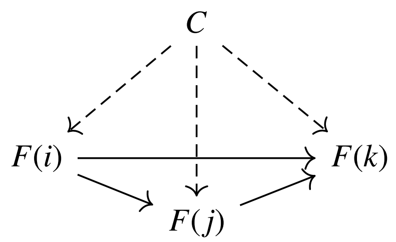
3. Fix two categories I and $\mathcal{C}$, where we may think of I as being somehow small.

   We will consider a functor $F : I \to \mathcal{C}$ as a *diagram* in $\mathcal{C}$, given by objects $F(i)$ together with arrows $F(f) : F(i) \to F(j)$ for every morphism $f : i \to j$ in I.

   Any object C of $\mathcal{C}$ determines a constant functor $c_C : I \to \mathcal{C}$ that sends any i to C and any $f : i \to j$ to $\mathbf{1}_C$.

   Then natural transformation from c to another functor $F : I \to \mathcal{C}$ is given by maps $\alpha_i : C \to F(i)$ for every $i \in I$ such that $F(f) \circ \alpha_i = \alpha_j$ for every $f : i \to j$.

   We call a natural transformation from a constant diagram to F a *cone* over F. We think of C as the tip of the cone, and there are arrows going to all the vertices of the diagram, making all the triangles commute.



4. For every $n \geq 1$ the Hurewicz homomorphism $h_n : \pi_n(X, *) \to H_n(X, \mathbb{Z})$ from homotopy to homologoy of path connected spaces is a natural transformation. (To be precise it is a natural transformation from π_n to the composition of homology with the functor forgetting basepoints. If $n = 1$ we also have to compose with the inclusion functor from abelian groups to all groups.)

5. For every topological space X we have a functor which takes the underlying set of X and equips it with the discrete topology, write this as X^δ. Then the identity map from X^δ to X is continuous. In fact it is a natural transformation from the discretization functor to the identity functor $X^\delta \mapsto X$.

Natural transformations may be composed and form the morphism in the *category of functors* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ between two categories.

Definition A.11. A natural tranformation α such that all α_C are isomorphisms is an isomorphism in the category of functors and is called a *natural isomorphism*.

A.1.3. Equivalences

Definition A.12. Two categories are *equivalent* if there are functor $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ is naturally isomorphic to $\mathbf{1}_{\mathcal{D}}$ and $G \circ F$ is naturally isomorphic to $\mathbf{1}_{\mathcal{C}}$.

We can give a more concrete description, for which we need some definitions.

Definition A.13. functor $F : \mathcal{C} \to \mathcal{D}$ is *full* if it induces surjections on all hom sets, i.e. every $g : FC \to FC'$ in $\mathcal{D}$ is $F(f)$ for some $f : C \to C'$.

The functor F is *faithful* if it induces injections on all hom sets, i.e. $F(f) = F(f')$ only if $f = f'$.

F is *fully faithful* if it is both full and faithful.

F is *essentially surjective* if every object in $\mathcal{D}$ is isomorphic to some object FC in the image of F.

Then one can prove that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective. (The "if" direction needs the axiom of choice.)

Example A.14. 1. Let k be a field. There is an equivalenc of categories from finite-dimensional k-vector spaces to its opposite category, given by $V \mapsto V^*$ on objects.

2. Let Mat be the category whose objects are non-negative integers and whose morphisms from m to n are $(m \times n)$-matrices. Composition is given by matrix multiplication.

   Then there is a natural functor from Mat to the category of finite-dimensional $\mathbb{R}$-vector spaces, given by $n \mapsto \mathbb{R}^n$ on objects. This is an equivalence of categories.

A.1.4. Opposite categories

We recall the following Example A.2.6:

Definition A.15. Let $\mathcal{C}$ be any category. Then its *opposite category* $\mathcal{C}^{\mathrm{op}}$ is defined to have the same objects as $\mathcal{C}$ but $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(C, D) := \mathrm{Hom}_{\mathcal{C}}(D, C)$ and $f \circ_{\mathcal{C}^{\mathrm{op}}} g := g \circ_{\mathcal{C}} f$.

In words $\mathcal{C}$ is obtained by turning around all the arrows in $\mathcal{C}$.
Clearly any functor $F : \mathcal{C} \to \mathcal{D}$ induces an opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.
Many natural functors, like cohomology, turn around the order of arrows, i.e. cohomology is a functor $\mathsf{Top}^{\mathrm{op}} \to \mathsf{Ab}$.

Definition A.16. We call a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ a *contravariant functor* from $\mathcal{C} \to \mathcal{D}$.

By using the opposite of categories and functors, we can dualize all the definitions and results in category theory.
Moreover, whenever we prove a statement about a category $\mathcal{C}$ then the *dual statement* holds for its opposite category.
This is a very powerful idea, which we will come back to soon.

A.1.5. The hom functor

Forming the hom sets in a category is actually functorial. Let us explain what this means.
Let $\mathcal{C}$ be a locally small category, i.e. the morphisms between any two objects form a set (rather than a proper class). Let C be an object of $\mathcal{C}$.

Definition A.17. The *hom-functor*, denoted $h_C : \mathcal{C} \to \mathsf{Set}$, sends any object D to $\mathrm{Hom}_{\mathcal{C}}(C, D)$ and any morphism $f : D \to D'$ to the map $f_* : \mathrm{Hom}_{\mathcal{C}}(C, D)$ to $\mathrm{Hom}_{\mathcal{C}}(C, D')$ defined by $g \mapsto f \circ g$.

We can of course also put the object C in the second place of Hom. Then our functor will be contravariant and turn around the order of arrows. We obtain $h^C : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ which is defined by $D \mapsto \mathrm{Hom}_{\mathcal{C}}(D, C)$ and $f \mapsto f^*$, where $f^*(g) = g \circ f$.
For another level of abstraction, $h_{(-)}$ defines a functor from $\mathcal{C}^{\mathrm{op}}$ to the category of functors $\mathrm{Fun}(\mathcal{C}, \mathsf{Set})$. This is a fully faithful functor that is called the *Yoneda embedding*. Any functor naturally isomorphic to h_C is called *representable*.

Example A.18. The forgetful funtor U : Group $\to$ Set is representable by the group of integers.

Unravelling our definition this means that there for every group G there is an isomorphism $\mathrm{Hom}_{\mathsf{Group}}(\mathbb{Z}, G) \cong U(G)$, and these isomorphisms are compatible with group homomorphisms.

But this just says that the set of morphisms from $\mathbb{Z}$ to G is exactly the set of elements of G, the isomorphism is given by sending $f : \mathbb{Z} \to G$ to $f(1) \in G$.

Remark A.19. A key result in category theory is the *Yoneda lemma*. It states that natural transformations from h^C to some other functor $F : \mathcal{C} \to$ Set are in natural bijection with $F(C)$. It's not hard, but very consequential. (Although we won't need it.)

A.2. Universal constructions

A.2.1. Limits

Category theory allows us to unify many constructions in mathematics, in particular those characterised by *universal properties*.

Definition A.20. Let I be a small category and $\mathcal{C}$ any category. A *diagram of shape I* in $\mathcal{C}$ is just a functor $D : I \to \mathcal{C}$.

A *cone over D* is an object C in $\mathcal{C}$ together a natural transformation from the constant diagram C to D.

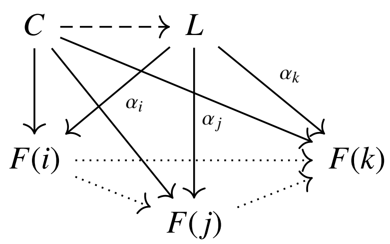
Explicitly a cone consists of C with maps $\gamma_i : C \to D(i)$ for all objects i in I such that for any $a : i \to j$ we have $D(a) \circ \gamma_i = \gamma_j$.

A map of cones $(C, \gamma) \to (E, \epsilon)$ is a map $f : C \to E$ compatible with the maps, i.e. $\epsilon_i \circ f = \gamma_i$.

We will often write F_i for the objects $F(i)$ for $i \in I$.

Definition A.21. A *limit* of the diagram $F : I \to \mathcal{C}$ is a cone (L, α_i) over F that is universal in the sense that any cone (C, γ_i) maps uniquely to (L, α_i).

In other words, L and α have the property that whenever we have C in the following diagram there is exactly one dashed arrow $C \to L$ making the diagram commute.



This universal property (like all universal property) ensures that if there are two limits L and L' there is a unique isomorphism between them: As L is a limit there is a unique map

$g : L' \to L$ and as L' is a limit there is a unique map $g' : L \to L'$. As $g'g$ and $\mathbf{1}_{L'}$ are both maps of cones from L' to itself they must agree and g' and g are inverse.

We thus also speak of *the limit* and denote it by $\lim_I F$ or $\lim F_i$.

Remark A.22. Note that the limit need not exist! If we can form arbitrary (small) limits in a category $\mathcal{C}$ we say that $\mathcal{C}$ *has all small limits*.

Let us make this more concrete.

Definition A.23. Let I a set considered as a discrete category. The limit of $F : I \to \mathcal{C}$ is called the product of the $F(i)$, often written $\prod_{i\in I} F_i$.

Thus $\prod_i F_i$ has the property that there are natural maps $\pi_j : \prod_i F_i \to F_j$ for all j (called *projection*) and whenever we are given maps $\beta_j : C \to F_j$ for all j we obtain a map $\beta : C \to \prod_i F_i$ such that $\beta_j = \pi_j \circ \beta$.

This recovers the familiar product of sets, topological spaces, abelian groups etc.

We consider a special case:

Definition A.24. Let I be the empty set considered as a discrete category without objects! The limit of the unique functor $I \to \mathcal{C}$ is called the *terminal* object of $\mathcal{C}$, often written $*$. It has the property that for every $C \in \mathcal{C}$ there is a unique morphism $C \to *$.

The terminal object in Set is the set with 1 Element.

Definition A.25. Let I be the category with two objects and two arrows in the same direction $\bullet \rightrightarrows \bullet$. The limit of $F : I \to \mathcal{C}$ is called *equalizer*.

Definition A.26. Let I be the category with three objects $\bullet \to \bullet \leftarrow \bullet$. The limit of $F : I \to \mathcal{C}$ is called *pullback*.

Example A.27.

1. The terminal object in Groups is the group with 1 element.
2. The terminal object in Top is the topological space with 1 point.
3. In the diagram $\bullet \to \bullet \leftarrow \bullet$ that defines pull-backs the middle object is terminal.
4. If a pull-back diagram in Set or Top takes the form $* \to Y \xleftarrow{f} X$ then the pull-back is the fiber of f (equipped with the subspace topology in the case of Top).
5. If a pull-back diagram takes the form $X \to * \leftarrow Y$, i.e. the middle object goes to the terminal object of $\mathcal{C}$, then the limit is the product $X \times Y$.
6. In the category Groups there is a unique map from $*$ to any group H and the pullback of the diagram $* \to H \xleftarrow{f} G$ is nothing but the kernel of f.
7. The equalizer of two maps $f, g : A \to B$ in Set is exactly the subset of A given by all elements a with $f(a) = g(a)$, this explains the name.

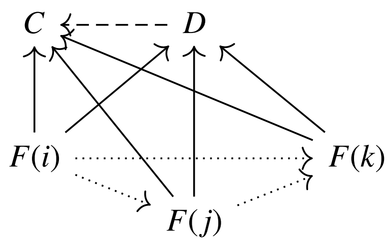
A.2.2. Colimits

We now apply the idea of *dualizing* categorical notions by turning around all the arrows to the previous section.

So we change the orientation of all the arrows in the definition of a limit. This gives the dual notion of a limit, called the colimit.

Definition A.28. A *colimit* of the diagram $F : I \to \mathcal{C}$, denoted by $\mathrm{colim}_I F$, is an object D of $\mathcal{C}$ together with a natural transformation $\alpha : F \Rightarrow c_D$ that is *universal*, in the sense that any natural transformation from F to a constant functor c_C factors uniquely through c_D.

The corresponding diagram looks like this:



Remark A.29. To make the duality of limit and colimit more precise we can observe that (D, α) is a colimit of the diagram $F : I \to \mathcal{C}$ exactly if $(D, \alpha^{\mathrm{op}})$ is a limit of the diagram $F^{\mathrm{op}} : I^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. Here $\alpha^{\mathrm{op}} : c_D^{\mathrm{op}} \Rightarrow F^{\mathrm{op}}$ is the natural transformation corresponding to $\alpha : F \Rightarrow c_D$ under the correspondence of morphisms in $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$.

Definition A.30. The colimit over a discrete category is called the *coproduct* or sum.

The colimit of the empty diagram is called the *initial object*.

The colimit of the diagram $\bullet \leftarrow \bullet \to \bullet$ is called *pushout*.

The colimit of a diagram of shape $\bullet \leftleftarrows \bullet$ is called *coequalizer*.

Example A.31.

1. In Set and Top ithe coproduct is given by the disjoint union.
2. In Group the coproduct is given by the free product of groups.
3. In Vect the product and coproduct of two vector spaces V and W agree, both are given by $V \oplus W$. (This holds for all finite products and coproducts in Vect, but it is no longer true for infinite products and coproducts!)
4. The initial object in Set is given by the empty set.
5. The group with one object is both initial and terminal.
6. The pushout of the diagram $0 \leftarrow V \to W$ of vector spaces is the quotient space W/V.
7. The coequalizer of two maps $f, g : A \to B$ in Set is given by the quotient of B by the relation generated by $f(a) \sim g(a)$ for all $a \in A$.

From the definition of limit and colimits it is not hard to obtain the following extremely useful result:

Lemma A.32. *Let $F : I \to \mathcal{C}$ and $G : J \to \mathcal{C}$ be diagrams. Then we have natural isomorphisms*

$$\operatorname{Hom}_{\mathcal{C}}(C, \lim_I F_i) \cong \lim_I \operatorname{Hom}_{\mathcal{C}}(C, F_i)$$

and

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim}_J G_i, C) \cong \lim_J \operatorname{Hom}_{\mathcal{C}}(G_j, C)$$

A.2.3. Filtered colimits

A special kind of colimit is given by the following.

A category I is *filtered* if any finite diagram in I has a cone. Equivalently I is filtered when it is not empty, for every two objects i, i' there exists an object k with two arrows $i \to k$ and $i' \to k$; for any two parallel arrows $u, v : i \rightrightarrows j$ there is an object k and morphism $f : j \to k$ with $fu = fv$.

A *filtered diagram* is a diagram $I \to \mathcal{C}$ with I filtered.

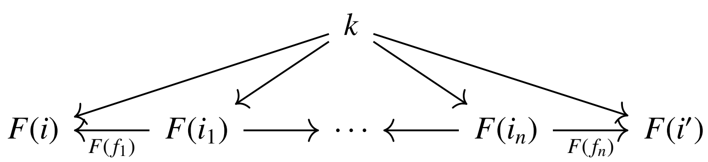
Definition A.33. A colimit over a filtered diagram is a *filtered colimit*

Example A.34.
1. The category $(\mathbb{N}, \leq)$ with objects the natural numbers and a single morphism $a \to b$ whenevere $a \leq b$ is filtered. A colimit indexed by $(\mathbb{N}, \leq)$ is also called a sequential colimit. Increasing unions are a typical example: $\mathbb{R} = \operatorname{colim}_{a \in \mathbb{N}}(-a, a)$ as sets or topological spaces.
2. The set of all neighbourhoods of a point x in a topological space X is a filtered category under inclusion.

Such examples where there is at most one morphism between two objects are also called posets.

A functor $F : I \to J$ is called *cofinal* if

1. For any object j in J there is i in I with a morphism $j \to F(i)$
2. For any two arrows $j \to F(i)$ and $j \to F(i')$ there is a zig-zag of arrows $i \overset{f_1}{\longleftarrow} \cdots \overset{f_n}{\longrightarrow} i1$ making the natural diagram commute:



Note that the second condition is automatic if J is filtered.

Lemma A.35. *Let $F : I \to J$ be a final functor and $G : J \to \mathcal{C}$ a diagram. Then if* $\operatorname{colim}_I GF$ *exists then* $\operatorname{colim}_J G$ *also exists and agrees with* $\operatorname{colim}_I GF$.

Example A.36. The inclusion of all prime numbers into $(\mathbb{N}, \leq)$ is final.

The inclusion of connected open neighbourhoods in all neighbourhoods of a point in a topological set is final.

The key result about filtered colimits is the following:

Theorem A.37. *In the category* Set *and A-Mod for any ring A finite limits commute with filtered colimits.*

A.2.4. Existence of (co)limits

We say a category $\mathcal{C}$ *has all small limits* or *is complete* if every diagram $I \to \mathcal{C}$ has a limit. Similarly we say $\mathcal{C}$ has all small colimits or is cocomplete if every diagram $I \to \mathcal{C}$ has a colimit.

This may seem extremely difficult to check, but in fact one can build any limit from just two types of limit:

Recall that an equalizer is a limit for a diagram of the shape $\bullet \rightrightarrows \bullet$ and a product is a diagram whose shape is a discrete category.

We say a category $\mathcal{C}$ has all equalizers if any equalizer diagram has a limit, and similarly for products (and other shapes of diagrams).

Lemma A.38. *A category $\mathcal{C}$ has all limits if and only if it has all products and equalizers. It has all colimits if and only if it has all coproducts and coequalizers.*

A.2.5. Adjunctions

It is rare that categories are equivalent, but a weaker notion is extremely fruitful.

Definition A.39. We say $F : \mathcal{C} \to \mathcal{D}$ is *left adjoint* to $G : \mathcal{D} \to \mathcal{C}$, in symbols $F \dashv G$ if for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$ there are natural isomorphisms

$$\phi_{C,D} : \operatorname{Hom}_{\mathcal{C}}(C, GD) \cong \operatorname{Hom}_{\mathcal{D}}(FC, D)$$

Here naturality means that for every map $C \to C'$ in $\mathcal{C}$ the natural diagram commutes:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{C}}(C', GD) & \xrightarrow{\phi_{C',D}} & \operatorname{Hom}_{\mathcal{D}}(FC', D) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle Ff^*} \\
\operatorname{Hom}_{\mathcal{C}}(C, GD) & \xrightarrow{\phi_{C,D}} & \operatorname{Hom}_{\mathcal{D}}(FC, D)
\end{array}$$

and a similar diagram commutes for $g : D \to D'$ in $\mathcal{D}$.

If $\mathcal{C}$ and $\mathcal{D}$ are locally small we can also phrase naturality as saying that the two functors $\operatorname{Hom}_{\mathcal{C}}(-, G(-))$ and $\operatorname{Hom}_{\mathcal{D}}(F(-), -)$ from $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ to Set are naturally isomorphic.

Example A.40. 1. Throughout algebra there are adjunctions between free and forgetful functors. For example the forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ has a left adjoint given by taking a set X to the free group with set of X as set of generators.

2. The forgetful functor $\mathsf{Top} \to \mathsf{Set}$ has a left adjoint given by equipping any set with the discrete topology. It also has a right adjoint given by equipping any set with the indiscrete topology.

Left and right adjoints are naturally dual: If $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to G, then $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ is right adjoint to G^{op}.

Let $F \dashv G : \mathcal{C} \rightleftarrows \mathcal{D}$ and $C \in \mathcal{C}$. By the adjunction the identity map $\mathbf{1}_{FC} : FC \to FC$ corresponds to a map $\epsilon_C : C \to GFC$. By naturality in the definition of an adjunction the ϵ assemble into a natural transformation $\epsilon : \mathbf{1}_{\mathcal{C}} \Rightarrow GF$. This is called the *unit* of the adjunction.

Simlarly there is a natural transformation $\eta : FG \Rightarrow \mathbf{1}_{\mathcal{D}}$, called the *counit* of the adjunction.

Lemma A.41. *Let $F \dashv G$. Then unit and counit satisfy the following identities of natural transformations: For every $C \in \mathcal{C}$ we have*

$$\eta_{FC} \circ F(\epsilon_C) = \mathbf{1}_{FC}$$

and for every $D \in \mathcal{D}$ we have

$$G(\eta_C) \circ \epsilon_{GD} = \mathbf{1}_{GD}.$$

Put a little differently, we have the following identities of natural transformations: $G\eta \circ \epsilon_G = \mathbf{1}_G$ and $\eta_F \circ F\epsilon = \mathbf{1}_F$.

In fact, adjoints may be equivalently characterized by the existence of unit and counit.

Remark A.42. An adjunction induces an equivalence of categories if and only if unit and counit are natural isomorphisms.

One can also show that adjoints are given by a universal property and are thus unique up to unique natural isomorphism.

Adjoints are closely related to limts:

Lemma A.43. *Let F be a left adjoint. Then F preserves colimits, i.e. whenever (D, α) is a colimit of a diagram $G : I \to \mathcal{C}$ then $(FD, F\alpha)$ is a colimit for $F \circ G : I \to \mathcal{D}$.*

Dually, if G is a right adjoint then G preserves limits.

Remark A.44. Under some assumption on the categories $\mathcal{C}$ and $\mathcal{D}$ there is even a converse to the lemma: Any functor preserving all colimits has a left adjoint. There are different theorems, depending on the precise assumptions made, but they are all called *adjoint functor theorems*.

We can even characterize limits using adjoints.

Lemma A.45. *Consider the category* $\text{Fun}(I, \mathcal{C})$ *of I-shaped diagrams in $\mathcal{C}$. There is a diagonal functor $\Delta : \mathcal{C} \to$* $\text{Fun}(I, \mathcal{C})$ *sending any object C to the constant functor c_C. Then taking the limit of a diagram is right adjoint to Δ, and taking the colimit is left adjoint.*

Bibliography

[Cis19] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2019. DOI: 10.1017/9781108588737.

[Dim04] Alexandru Dimca. *Sheaves in topology*. Universitext. Berlin: Springer, 2004.

[GM03] Gelfand, S.I. and Y. Manin. *Methods of Homological Algebra: Springer monographs in mathematics*. Springer Monographs in Mathematics. Springer, 2003.

[Huy06] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. Oxford: Oxford Univ. Press, 2006.

[Ive86] Birger Iversen. *Cohomology of sheaves*. Universitext. Springer, 1986.

[KS90] M. Kashiwara and P. Schapira. *Sheaves on Manifolds:* Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer, 1990.

[Wei95] C.A. Weibel. *An introduction to homological algebra*. Cambridge University Press, 1995.