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Rational Homotopy Theory (Master)
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# Sheet 11

## Question 11.1

Consider a map of filtered complexes inducing quasi-isomorphisms on the associated graded complexes. Show that the map is a quasi-isomorphism if the filtrations are bounded.

## Question 11.2

Fix a field $k$ of coefficients. Assume that $E \to B$ is a fibration, $B$ is simply connected and the fiber $F$ is a homology $n$-sphere, i.e. $H_*(F) \cong H_*(S^n)$.

Show there is a long exact sequence

$$\to H^k(B) \xrightarrow{\gamma} H^{n+k+1}(B) \to H^{n+k+1}(E) \to H^{k+1}(B) \xrightarrow{\gamma} \cdots$$

Show moreover that there is a class $z \in H^{n+1}(B)$ such that $\gamma(u) = z \cup u$.

Apply this result to the fibration $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$.

## Question 11.3

Let $SU(n)$ denote the special unitary group, consisting of $n \times n$ complex matrices $A$ satisfying $A\bar{A}^T = \mathbf{1}_n$ and $\det(A) = 1$. This is a topological space with the subspace topology inherited from the natural embedding into $\mathbb{C}^{n^2}$.

It is not hard to check that $SU(2) \cong S^3$.

Letting $SU(3)$ act on any nonzero vector $v$ in $\mathbb{C}^3$ we obtain a map from $SU(3)$ to $S^5$ given by $A \mapsto Av$. This is a fibration. The fiber is given by $SU(2)$.

What are the cohomology groups of $SU(3)$ with coefficients in a field $k$?

What is the ring structure?

What is the cohomology of $SU(n)$ for $n \geq 4$?

**These questions will be discussed in the exercise class on 1.2.20.**

Questions with an asterisk are more challenging.