Homological and Homotopical Algebra

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1.1. Preamble

This course is an introduction to homological and homotopical algebra. By the end of it you should

1. see homology groups everywhere,
2. be able to compute lots of them, and
3. not be scared of the word derived.

I have tried to keep the required background to a minimum, and I have probably failed. Note the following:

- Algebraic topology will often provide motivational and understandable examples.
- We will need some category theory. In particular we will freely use limits and universal properties. There will be appeals to dual statements. A nice reference are Julia Goedecke's notes for the Cambridge Category Theory course, available at www.dpmms.cam.ac.uk/ jg352/teaching.html.

Some proofs will be relegated to example sheets or left out. Not because they are hard, but because they are long (or rather because the course is short).

Some sections of these notes were not lectured in class, they are marked as such in the margins. (They are obviously not examinable.)

Brief annotated bibliography:

[W] Weibel, *An introduction to homological algebra*, CUP 1995. Highly recommended as a general reference, full of interesting examples, source of a lot of the material in these notes.

[GM] Gelfand-Manin, *Methods of homological algebra*, Springer 2003. A bit more modern on the theory, lighter on applications.

[DS] Dwyer-Spalinski, *Homotopy theories and model categories*. In Handbook of Algebraic Topology, 1995. Very readable introduction to the theory and applications of model categories. (Freely available at hopf.math.purdue.edu/Dwyer-Spalinski/theories.pdf.)

[Hov] Hovey, *Model categories*, AMS 1999. More comprehensive introduction to model category theory.

[Huy] Huybrechts:, *Fourier-Mukai transforms in algebraic geometry*, OUP 2006. An introduction to the beautiful theory of derived categories in algebraic geometry that is conspicuously absent from this course

Comments and corrections are very welcome, please email `jvsh2@cam.ac.uk`.

1.2. Motivating example

We begin with the graded ring $R = k[x_0, \ldots, x_n]$ with $\deg(x_i) = 1$. (A grading on a ring is a direct sum decomposition $R = \oplus R_d$ such that $R_d.R_e \subset R_{d+e}$, an element $r \in R_d$ is said to have degree d. Hilbert was very interested in graded modules over R. A graded module is just an R-module that can be written as $M = \oplus_d M_d$ such that $R_d M_n \subset M_{d+n}$. Then we consider the function $d \mapsto H_M(d) = \dim_k M_d$. Hilbert tried to compute this function in great generality, and proved that it is equal to a polynomial function for large enough d. It's now called the *Hilbert polynomial*.

One reason to care is that R is the homogeneous coordinate ring of projective space $\mathbb{P}^n_k$, and projective varieties correspond to homogenous ideals I and the quotient $M = R/I$ is the homogeneous coordinate ring of the projective variety $V(I)$.

The Hilbert polynomial is extremely important in algebraic geometry, for example when considering moduli spaces one often fixes the Hilbert polynomial. If $S = R/I$ is the coordinate ring of a curve C in $\mathbb{P}^n$ then the Hilbert polynomial is

$$H(d) = \deg(C) \cdot d + (1 - g(C))$$

To compute we would like to replace M by free modules for which the computation becomes straightforward. We can begin by writing down a surjection $F_0 \twoheadrightarrow M$, this will exist as M has generators. Then we consider the kernel K_0, given by all the relations between generators (finitely generated by Hilbert's basis theorem). Hilbert (following Sylvester) called the elements of K_0 *syzygies*. If K_0 is free, we can stop. Otherwise there are relations between relations, forming what's called the *2nd module of syzygies*. We continue by writing $K_0 = F_1/K_1$, and so on.

We write the end result $\ldots \twoheadrightarrow F_2 \twoheadrightarrow F_1 \twoheadrightarrow F_0 \twoheadrightarrow M$ and call it a *free resolution*.

Theorem (Hilbert 1890)**.** *M has a free resolution of length at most $n+1$, i.e. s.t. $F_{n+2} = 0$.*

Now observe that $H_M(d) = \sum_i (-1)^i H_{F_i}(d)$ by applying the rank nullity theorem a few times. Next we compute H_{F_i}. We can write $F_i = \oplus_j R(-a_{ij})$, where $R(b)$ stands for the graded module R shifted by b, so $R(b)_c = R_{b+c}$. This is the free R-module with a single generator in degree b. It suffices to compute $H_{R(-a)}$. But it is an easy exercise in combinatorics to show that $H_R(d) = \binom{n+d}{n}$. Then $H_{R(-a_{ij})} = \binom{n+d-a_{ij}}{n}$. Putting it all together we find

$$H_M(d) = \sum_i (-1)^i \sum_j \binom{n+d-a_{ij}}{n}$$

As soon as $d \geq \max_{i,j}(a_{ij}) - r$ this expression becomes a polynomial.

1.3. Abelian categories

We will begin with abelian categories. But historically and logically we should begin with an example:

Example 1. You are welcome not to worry about the definition of an abelian category and only ever think about R-Mod, the category of left modules over an associative unital ring R. We denote by **Ab** the category of $\mathbb{Z}$-modules, also known as abelian groups.

Example 2. Unless you are interested in geometry or topology, in which case you also want to remember the abelian category of sheaves (or presheaves) of abelian groups on a topological space, or $\mathscr{O}$-modules or quasi-coherent sheaves or coherent sheaves on a scheme, or some other categories of sheaves on sites, for example $\mathscr{D}$-modules on a smooth variety.

So you might want to read the definition of an abelian category as a lemma (exercise) about the category of R-modules.

Definition. An *additive category* is a category such that

1. every hom-set has the structure of an abelian group and composition distributes over addition.
2. there is a 0-object, i.e. an object that is both initial and terminal, i.e. an object that admits a unique morphism to and from every object.
3. For any objects A, B the product $A \times B$ exists. It is automatically equal to the coproduct $A \amalg B$ and we write it as $A \oplus B$.

A functor between additive categories is *additive* if it induces homomorphisms on hom-sets. (We'll see it then preserves finite direct sums.)

An *abelian category* $\mathscr{A}$ is an additive category such that

1. every map in $\mathscr{A}$ has a kernel and a cokernel
2. For every map f we have $\ker(\operatorname{coker}(f)) \cong \operatorname{coker}(\ker(f))$, image equals coimage.

Recall that the kernel of $f : A \longrightarrow B$ is the equalizer of f and the 0-map $A \longrightarrow 0 \longrightarrow B$, i.e. the universal map to A that is killed by composition with f. The dual is the cokernel, which for rings is given by the usual quotient.

A word about the image: Given a map $f : A \longrightarrow B$ there is a map $\ker(\operatorname{coker} f) : I \longrightarrow B$. We call I the image of f. There is dually the *coimage* $\operatorname{coker}(\ker f)$ and there is always a natural map from the image to the coimage, and in an abelian category this map is an isomorphism.

Example 3. The last condition can be a bit confusing. Consider the category whose objects are inclusions of vector spaces $V \subset W$, and whose morphisms are compatible pairs of linear maps. Consider the natural inclusion map i from $0 \subset V$ to $V \subset V$. It is easy to see this map is monic and epic. So the kernel and cokernel are trivial and $\operatorname{coker}(\ker(i)) = (0 \subset V) \neq (V \subset V) = \ker(\operatorname{coker}(i))$. (This argument shows that the category of filtered vector spaces is not abelian.)

Convention. From now on $\mathscr{A}$ will always be an arbitrary abelian category.

In general, objects of abelian categories need not have underlying sets, and need not have elements. However, it is often incredibly useful to be able to pick an element and see what happens to it (this proof-technique is called *diagram-chasing*). In R-Mod this is fine. In categories of sheaves we can argue with sections which behave very much like elements.

Remark. In greater generality one appeals to the *Freyd-Mitchell embedding theorem*: Let $\mathscr{A}$ be a small abelian category. Then there is a ring R and a fully faithful exact functor (i.e. preserving kernels and cokernels) $\mathscr{A} \longrightarrow R$-Mod. Hence most results that are proved using diagram chases (in finite diagrams) apply to all abelian categories. There are some subtleties, however. (For example the embedding need not preserve infinite products.)

1.4. Exactness

Definition. Consider a sequence of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$. If $g \circ f = 0$ that says $\mathrm{Im}(f) \subset \ker(g)$. If now also $\ker(g) \subset \mathrm{Im}(f)$ we say the sequence is *exact* at B. An exact sequence is a sequence of morphisms that is exact at every object.

If you know homology from topology you know that the sequence of singular chains is exact at the object in degree n if there aren't any "holes" in degree n. It makes sense to study this condition algebraically.

Remark. Recall that a map is *monic* or a *mono(morphism)* if it is left cancellable, i.e. if $fg = fh$ implies $g = h$, and is *epic* or an *epi(morphism)* if it is right cancellable. In R-modules a morphism is epic iff it is surjective iff it is a quotient iff it is the cokernel of some map, and it is monic iff it is injective iff it is the kernel of some map. This follows from the isomorphism theorems. But note that $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epi of rings.

If moreover f is monic and g is epic we call $A \longrightarrow B \longrightarrow C$ a *short exact sequence*. We can also write this as an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$. One way of looking at this is to say that B has quotient C and the kernel is A, so B is made up of C and A, we say it as an *extension* of C by A. For example, we could have $B = C \oplus A$, in this case we say the short exact sequence is *split*. We will return to this viewpoint.

Example 4. In the category of abelian groups we have the short exact sequences $\mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$ and $\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2$. There are also exact sequences $\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$ and $\mathbb{Z} \xrightarrow{3} \mathbb{Z} \longrightarrow \mathbb{Z}/3$.

2. Chain complexes

Chain complexes and cochain complexes, homotopies and quasi-isomorphisms, exact functors.

2.1. Chain and cochain complexes

Definition. Let us consider a collection $(C_i)_{i\in\mathbb{Z}}$ of objects of $\mathscr{A}$ and a sequence of maps $d_i : C_i \longrightarrow C_{i-1}$ satisfying $d_{i-1} \circ d_i = 0$, but not necessarily exact. We often just write $d^2 = 0$. We call $(C_\bullet, d)$ a *chain complex* and the d are called *differentials* or sometimes *boundary operators*.

Definition. We also define $Z_i = \ker(d_i)$ and $B_i = \mathrm{Im}(d_{i+1})$, the *cycles* and *boundaries*. Then we can measure the failure of C to be exact by considering $\mathrm{coker}(B_i) \longrightarrow Z_i)$, which we call the i-th *homology object*, $H_i(C)$.

Remark. Homology began as counting holes, or rather: determining higher connectivity. Riemann, Betti and Poincare developed Betti numbers and torsion coefficients. Emmy Noether noticed that the numbers are ranks of groups. One can see this as an instance of "groupification", which is a precursor of *categorification*, interpreting a mathematical object as the shadow of some other, richer object. For example seeing a set as the equivalence classes of objects of a category.

Why is the condition $d^2 = 0$ the right one? It seems quite a deep definition. Historically d is a boundary operator, and the boundary of a boundary should be empty. (Of course that's not in general true for topological spaces.)

A complex all of whose homology groups are 0 is also called *acyclic* or (as before) exact. Note we can also say the complex is exact everywhere. You can think of it like a contractible topological space. It may be large, but there is nothing interesting going on. (Unless there is extra structure, like a group action.)

It is often convenient to consider differentials going the other way and increasing degree. Then by convention we write $d^i : C^i \longrightarrow C^{i+1}$ and call $(C^\bullet, d)$ a cochain complex and we define cohomology objects $H^i(C)$. One can write $C^i = C_{-i}$.

Example 5. Singular chains and cochains on a topological space. Chains an a simplicial complex. The de Rham complex. The Bar complex. A flabby resolution of a sheaf. A free or projective resolution of a module. The Čech complex. The Koszul complex. You name it. (Don't worry if you haven't seen most of these before.)

Definition. A *chain map* is a level-wise morphisms that commute with the differential.

The category of chain complexes over an abelian category $\mathscr{A}$, denoted $\mathbf{Ch}(\mathscr{A})$ has as objects the chain complexes over $\mathscr{A}$ and as morphisms the chain maps.

$\mathbf{Ch}(\mathscr{A})$ is additive and for every n there is an additive functor $H_n : \mathbf{Ch}(\mathscr{A}) \longrightarrow \mathscr{A}$. That means given a map $f : B \longrightarrow C$ there is a group homomorphism $f_* : H_n(B) \longrightarrow H_n(C)$. You just define the map in the only way you can and check it is well-defined.

It's easy to see that $\mathbf{Ch}(\mathscr{A})$ is an abelian category, kernel and cokernel are defined level-wise. So it makes sense to talk about a short exact sequence of chain complexes.

Lemma 1 (Snake Lemma)**.** *Given a short exact sequence $A \longrightarrow B \longrightarrow C$ of chain complexes there are boundary maps ∂ fitting into an exact sequence*

$$\ldots \longrightarrow H_i(B) \longrightarrow H_i(C) \xrightarrow{\partial} H_{i-1}(A) \longrightarrow H_{i-1}(B) \longrightarrow \ldots$$

of homology objects.

Proof. Exercise! □

Revisiting our definition for chain maps we see that $f \mapsto df - fd$ looks like a differential. In fact, it can be made into a differential on a suitable complex.

Definition. Define the hom-complex $\underline{\text{Hom}}(A, B)$ between two complexes as follows: In degree n we have all collections of maps $\{f_i : A_i \longrightarrow B_{i+n}\}$. (These are not chain maps). The differential is $f \mapsto fd - (-1)^n df$.

Indeed $d^2(f) = d(df - (-1)^n fd) - (-1)^1(df - (-1)^n fd)d = 0$.

Remark. There is a sign rule that says whenever an element a of degree $|a|$ moves past an element b of degree $|b|$ a sign $(-1)^{|a||b|}$ should appear.

Remark. Instead of a hom-set we have now defined a hom-complex and in this way $\mathbf{Ch}(\mathscr{A})$ can actually be *enriched* over chain complexes $\mathbf{Ch}(\mathbf{Ab})$.

Note that the hom set is the Z_0 of the hom complex. That begs the question if we should not be talking about $H_0(\underline{\text{Hom}})$ instead.

Definition. A *chain homotopy* between to chain maps f and g from A to B is a collection of map $s : A_n \longrightarrow B_{n+1}$ with $ds + sd = f - g$. A map $f : A \longrightarrow B$ is a *chain homotopy equivalence* if there is $g : B \longrightarrow A$ such that gf and fg are homotopic to the identity.

This is equivalent to saying there is a map $s \in \mathrm{Hom}_1(A, B)$ such that $d(s) = f - g$. (Note there is a change of sign, which does not matter.)

The next definition is very convenient:

Definition. We define the *shifted complex* by $C[n]_i = C_{i+n}$ (and correspondingly $C[n]^i = C^{i+n}$). We also change the differential of $C[n]$ by a factor of $(-1)^n$. Note that $H_i(C) = H_0(C[i])$.

One often writes $H_*(C)$ for the graded object $\oplus_i H_i(C)[-i]$. (A graded object is a complex without differentials.)

One confusing thing about this definition is that a chain complex equal to A concentrated in degree n is $A[-n]$.

Remark. You will also find the definition $C[n]^i = C^{i-n}$, e.g. in [W].

Note that shift is a functor, we just change indices of chain maps.

With this definition a chain map $A \longrightarrow B[n]$ is precisely a cycle in $\mathrm{Hom}_n(A, B)$.

Lemma 2. *Chain homotopic maps induce the same maps on homology .*

Proof. Just check it. □

Definition. A chain map inducing isomorphisms on homology is called a *quasi-isomorphism*. A and B are quasi-isomorphic if there exists a quasi-isomorphism between them (in either direction!)

We often write $A \simeq B$ if A and B are quasi-isomorphic, which is bad notation since it is an asymmetric definition.

Example 6. For example the chain complex $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ in degrees 1 and 0 is quasi-isomorphic to $\mathbb{Z}/p$. But note there is no morphism, never mind a quasi-isomorphism, from $\mathbb{Z}/p$ to $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$.

It follows from the lemma that chain homotopy equivalences are quasi-isomorphisms. The converse is not true.

Note that chain homotopies and chain homotopy equivalences are preserved by additive functors as they are defined in terms of composition and addition. On the other hand quasi-isomorphisms are not necessarily preserved by additive functors.

Definition. Given a chain map $f : A \longrightarrow B$ we define the *cone*, $\operatorname{cone}(f)$ with underlying graded module $A[-1] \oplus B$ and differential $\begin{pmatrix} -d_A & \\ -f & d_B \end{pmatrix}$. If A and B are cochain complexes we have to take the underlying graded module to be $A[1] \oplus B$.

Lemma 3. *The map f is a quasi-isomorphism if and only if* $\operatorname{cone}(f)$ *is exact.*

Proof. There is a short exact sequence $B \longrightarrow \operatorname{cone}(f) \longrightarrow A[-1]$. The diagram chase of the snake lemma shows that the boundary in the associated long exact sequence is $f_* : H_*(A) \longrightarrow H_*(B)$. Then the result follows. □

Definition. A non-negative chain complex C_* with homology equal to M in degree 0 and zero elsewhere is quasi-isomorphic to M by the map that is 0 in degrees other than zero and projection $C_0 = Z_0(C) \longrightarrow H_0(C)$ in degree 0. We call such a map a *resolution* of M by C and write $M_* \longrightarrow C$.

2.2. Exact functors

Short exact sequences are a way to encode injections, surjections and extensions. We now examine what functors do to them.

Definition. An additive functor that preserves short exact sequences is called *exact*. An additive functor that sends an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ to an exact sequence $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ (not necessarily exact on the right!) is called *left exact*. Similarly for right exact functors.

Example 7. For any object M of $\mathscr{A}$ the functor $\operatorname{Hom}(M, -) : \mathscr{A} \longrightarrow \mathbf{Ab}$ is left exact. The functor $\operatorname{Hom}(-, M) : \mathscr{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ is also left exact.

Let M be an arbitrary R-module. The functor $- \otimes_R M : R\text{-Mod} \longrightarrow \mathbf{Ab}$ is right exact.

All of this is easy to check. On the other hand, none of these functors is exact in general.

Note that left adjoints are right exact, right adjoints are left exact. This follows since a functor is right exact if it preserves cokernels. Cokernels are colimits and left adjoints preserve colimits.

Example 8. Let M be an (R, S)-bimodule, i.e. it is a left R-module and a right S-module (equivalently, it is a left $R \otimes S^{\mathrm{op}}$-module). Then $\mathrm{Hom}_R(M, -)$: R-Mod $\longrightarrow$ -Mod is a right adjoint of $M \otimes_S -$: S-Mod $\longrightarrow$ R-Mod. I.e. given an R-module A and an S-module B we have $\mathrm{Hom}_S(B, \mathrm{Hom}_R(M, A)) \cong \mathrm{Hom}_R(M \otimes_S B, A)$ and the isomorphism is natural in A and B.

Our next goal is to approximate functors by exact functors. First note that every additive functor preserves, i.e. is exact on split exact sequences.

This is thanks to the 3rd characterisation in the following lemma.

Lemma (Splitting Lemma)**.** *For a short sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in an additive category the following are equivalent:*

1. *The morphism g is a cokernel of f and f has a left inverse r.*
2. *The morphism f is a kernel of g and g has a right inverse s.*
3. *There are morphisms $A \overset{r}{\leftarrow} B \overset{s}{\leftarrow} C$ such that the identities $rf = \mathbf{1}_A$, $gf = 0$, $rs = 0$, $gs = \mathbf{1}_C$ and $\mathbf{1}_B = fr + sg$ hold.*
4. *The sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is isomorphic to the sequence $A \xrightarrow{i} A \oplus C \xrightarrow{p} C$, i.e. there is an isomorphism $B \cong A \oplus C$ making the obvious diagram commute.*

Proof. The last two items are equivalent by definition (the isomorphism $A \oplus C \cong B$ is given by maps $f + s : A \oplus C \longrightarrow B$ and $(r, g) : B \longrightarrow A \oplus C$). Clearly these imply the first two items. So let us show 2. implies 3. (1. implies 3. will be similar). By assumption we have $gs = \mathbf{1}_C$ and $gf = 0$.

Next we need to produce r. Consider $h = \mathbf{1}_B - sg$. As $gh = 0$ we know h factors through f uniquely, say $h = fr$. This gives $\mathbf{1}_B = fr + sg$. Since $frf = hf = f$ and f is monic we have $rf = \mathbf{1}_A$. Moreover $hs = 0$ gives $rs = 0$. □

3. Derived functors

Derived functors via projective resolutions. The long exact sequence. Examples.

3.1. Introduction

Consider the result of applying a right exact functor to a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$. If F is not left exact then $F(A) \longrightarrow F(B)$ is not monic. So there is a cokernel. Can we compute this cokernel in terms of F and the short exact sequence? Maybe not the cokernel, but something which contains the cokernel, and then we try to determine the cokernel of the new map ... In other words, if we can't get a short exact sequence, can we get a long exact sequence?

If, for example, F was exact on some short exact sequence of complexes, that would give rise to a long exact sequence of homology groups. We are not given complexes, we are given objects in $\mathscr{A}$. But recall that complexes with no homology are like contractible spaces. So we can identify complexes with a single homology group in degree 0 with their homology group.

Next we need to force F to be exact on a sequence of complexes. We know that any additive functor will preserve split exact sequences. We can relate being split to nice properties of modules:

Definition. An object M in an abelian category is *projective* if for any epi $q : A \twoheadrightarrow B$ and any map $f : M \longrightarrow B$ there is a lift $g : M \longrightarrow A$ such that $q \circ g = f$. The dual notion is called *injective*.

It is easy to see that M is projective if only if $\mathrm{Hom}(M, -)$ is an exact functor and dually N is injective if and only if $\mathrm{Hom}(-, N)$ is an exact functor.

Example 9. In R-Mod it is easy to show that M is projective if and only if it is a summand of a free module. An example of a non-free projective R-module is given for $R = Mat_n(S)$ by the module of column vectors S^n.

Injective modules are more unwieldy. An example of an injective $\mathbb{Z}$-module is $\mathbb{Q}$.

Lemma 4. *If C is projective then $A \rightarrowtail B \twoheadrightarrow C$ is split. Similarly if A is injective.*

Proof. Use the identity map $C \longrightarrow C$ and the splitting lemma. □

Definition. A *projective resolution* of A is a levelwise projective complex quasi-isomorphic to A.

Definition. The *i-th left derived functor* of a right exact functor F is defined as $L_iF(A) := H_i(F(P))$ where P is a projective resolution of A.

Note that by right exactness of F we always have $L_0F(A) = F(A)$ and $L_{<0}F(A) = 0$.

Example 10. We define $\mathrm{Tor}_i^R(A, B)$ to be $L_i(- \otimes_R B)(A)$. Consider the category of abelian groups, i.e. $R = \mathbb{Z}$. We have a projective resolution $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$. We find that $\mathrm{Tor}(\mathbb{Z}/p, B) = H(B \xrightarrow{p} B)$. Hence $\mathrm{Tor}_0(\mathbb{Z}/p, B) = B/pB$ and $\mathrm{Tor}_1(\mathbb{Z}/p, B) = {}_pB$

Example 11. We define $\mathrm{Ext}_R^i(A, B)$ to be $R^i\mathrm{Hom}_R(-, B)(A)$. Consider the category of abelian groups, i.e. $R = \mathbb{Z}$. Note that an injective resolution in $\mathbb{Z}\text{-Mod}^{\mathrm{op}}$ is given by a projective resolution in $\mathbb{Z}$-Mod. So $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ is a suitable resolution of $\mathbb{Z}/p$ again and we find $\mathrm{Ext}^*(\mathbb{Z}/p, B) = H^*(B \xrightarrow{p} B)$. So $\mathrm{Ext}^0(\mathbb{Z}/p, B) = {}_pB$, the submodule of p-torsion elements, and $\mathrm{Ext}^1(\mathbb{Z}/p, B) = B/pB$.

Remark. You may wonder what happens if instead we consider $L_i(A \otimes_R -)(B)$ or $R^i\mathrm{Hom}_R(A, -)(B)$. We'll prove in Proposition 32 that we get the same answers as before, this is called *balancing*.

3.2. Proofs

Definition. A category has *enough projectives* if for every object there is an epimorphism from a projective object. Dually a category has *enough injectives* if for every object there is a monomorphism to an injective object.

Example 12. R-Mod has enough projectives, there is always a surjection $F(M) \longrightarrow M$, from the free module generated by the elements of M to M. It also turns out that R-Mod has enough injectives. Details are on the example sheet.

The abelian category of sheaves on X has enough injectives but does not have enough projectives.

Lemma 5. *Projective resolutions exist in $\mathscr{A}$ if there are enough projectives in $\mathscr{A}$.*

Proof. Let A be an object of $\mathscr{A}$. There is a projective object P_0 and a short exact sequence $0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$. Inductively there are short exact sequences $K_{i+1} \longrightarrow P_i \longrightarrow K_i$. Since the kernel of one such sequence is the cokernel of the next these short exact sequences assemble to give a long exact sequence $\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$. (This is called *splicing*.) □

Theorem 6 (Comparison Theorem)**.** *Let $\epsilon : P_* \longrightarrow M$ and $\eta : Q_* \longrightarrow N$ be projective resolutions and $f : M \longrightarrow Q$ a homomorphism. Then there is a lift $\tilde{f} : P_* \longrightarrow Q_*$ of f, i.e. there we have $\eta \circ \tilde{f} = f \circ \epsilon$. Moreover, $\tilde{f}$ is unique up to chain homotopy equivalence.*

Proof. Exercise! (See example sheet.) □

Corollary 7. *Projective resolutions are well-defined up to a chain homotopy and hence the derived functors are well-defined.*

Proof. Lift the identity to two comparison maps between the two resolutions. Their compositions lift the identity, so the maps are homotopy equivalences. □

The theorem also allows us to define $L_i f$ as $H_i(\tilde{f})$.

Corollary 8. *The i-ith left derived functor is a functor.*

Lemma 9 (Horseshoe lemma)**.** *Given a short exact sequence $A^1 \longrightarrow A^2 \longrightarrow A^3$ with projective resolution $P^1 \longrightarrow A^1$ and $P^3 \longrightarrow A^3$ there exists a projective resolution $P^2 \longrightarrow A^2$ such that $P_i^2 = P_i^1 \oplus P_i^3$ and the inclusion and projection maps lift the maps $A^1 \longrightarrow A^2 \longrightarrow A^3$.*

Proof. We proceed by induction. Let $P_0^2 = P_0^1 \oplus P_0^3$. To define the map ϵ to A^2 we lift $\epsilon^3 : P_0^3 \longrightarrow A^3$ to map to A^2 and we compose ϵ^1 with the inclusion $A^1 \longrightarrow A^2$. Now we see $\mathrm{coker}(\epsilon^2) = 0$ by the snake lemma, and $\ker(\epsilon^1) \longrightarrow \ker(\epsilon^2) \longrightarrow \ker(\epsilon^3)$ is exact, also by the snake lemma. This completes the induction step. □

Corollary 10. *A s.e.s $A \longrightarrow B \longrightarrow C$ in $\mathscr{A}$ gives a long exact sequence of derived functors*

$$\ldots LF_2(C) \longrightarrow L_1F(A) \longrightarrow L_1F(B) \longrightarrow L_1F(C) \longrightarrow FA \longrightarrow FB \longrightarrow FC \longrightarrow 0$$

Proof. By the horseshoe lemma we know we can compatibly resolve and get a short exact sequence of projective complexes that is levelwise split. As any additive functor is exact on split exact sequences we are done by the snake lemma. □

Proposition 11. *The boundary map is natural, i.e. given two short exact sequences with maps f_i between them we have $\partial \circ L_i f_3 = L_{i-1} f_1 \circ \partial$.*

Not a proof. This requires a bit of work, see [W] Theorem 2.4.6. □

To compute right derived functors we do everything with injective resolutions. We do not need to do any more work, we just argue in the opposite category!

Similarly, note that $\mathrm{Hom}(-, M)$ is contravariant, so it's a functor $\mathscr{A}^{\mathrm{op}} \longrightarrow R\text{-Mod}$. Now an injective resolution in $\mathscr{A}^{\mathrm{op}}$ is a resolution by projectives in $\mathscr{A}$.

We would like some universal property to make our derived functors canonical. Note that what we have here is enough to work with most derived functors. Moreover, we will look at universality later in the context of derived categories and model categories to get a better foundational understanding. For completeness here is the definition: not lectured

Definition. A *homological δ-functor* is a a collection of additive functors T_i, together with boundary maps $\partial : T_n(C) \longrightarrow T_{n-1}(A)$ for every short exact sequence, that form a long exact sequence for every short exact sequence, and which are natural with respect to maps of short exact sequences. A *universal homological δ-functor* is a δ-functor T_n that is universal, i.e. given any δ-functor S and map $f_0 : S_0 \longrightarrow T_0$ there is a unique morphism of δ-functors $\{S_n \longrightarrow T_n\}$ extending f_0.

As it's defined via a universal property the universal δ-functor is unique. One can show that our left derived functors are universal δ-functors, see [W] 2.4.7.

3.3. F-acyclics

not lectured

Do we really need injective and projective objects to compute derived functors?

Definition. Let F be a left exact functor. We say an object A is F-acyclic if $R^{\neq 0}(A) = 0$. Similarly for right exact functors. An F-acyclic resolution is defined as a resolution by F-acyclic objects.

Example. In the category $\mathbf{Ch}_R$ flat modules are $-\otimes_R M$-acyclic for any module M. Recall a module is flat if $-\otimes_R M$ preserves injections, hence if $-\otimes_R M$ is exact and all $\mathrm{Tor}_{>0}(A, M) = 0$. (We are using balancing.)

It turns out that a resolution of F-acyclics suffices to compute derived functors.

Proposition. *Let F be left exact and $A \longrightarrow J$ an F-acyclic resolution. Then $H^*(FJ) \cong R^*FA$.*

Proof. See first example sheet. □

Remark. This is useful, but clearly not enough. It is well known that the category of quasi-coherent sheaves on a variety does not have enough projectives. Still we would like to derive the tensor product. We say a class of objects $\mathscr{D}$ is *adapted* to a left exact functor F if F maps any acyclic bounded below cochain complex of objects in $\mathscr{D}$ into an acyclic complex, and if any object injects into an object of $\mathscr{D}$. Dually for right exact functors. (In particular restrict to bounded below chain complexes.)

In the category of $\mathscr{O}_X$-modules on a variety X on sees that locally free sheaves are flat (a sheaf is flat if its stalks are) and quasi-coherent sheaves have flat resolutions. Flat sheaves are adapted to the tensor product and one can compute the derived functors of $-\otimes \mathscr{F}$ even though there are not enough projectives.

To see how to develop the theory of derived functors from adapted objects see [GM] III.6.

4. Derived categories

Homotopy category and the derived category. Hom-sets and the subcategory of injectives. Total derived functors.

4.1. The derived category

To compute derived functors we replaced objects, considered as complexes concentrated in degree 0, by quasi-isomorphic complexes. After applying the functor we have a complex which is typically no longer quasi-isomorphic to a complex concentrated in degree 0. Hence it makes sense to consider all complexes, up to quasi-isomorphisms, and try to lift functors to this new category.

Remark. It is non-trivial to invert quasi-isomorphisms, mainly since it is unclear what happens to morphisms. We'd have to replace by arbitrarily long zig-zags $* \to * \leftarrow * \to * \leftarrow \cdots \to *$ where all right-to-left maps are quasi-isomorphisms. But if we do not have a set of objects then we do not have sets of morphisms and things get messy.

Definition. Given an abelian category $\mathscr{A}$ we define the *homotopy category* $K(\mathscr{A})$ to be the category with the same objects as $\mathbf{Ch}(\mathscr{A})$ but with morphisms equal to the homotopy classes of chain maps.

There are different boundedness conditions we can put on chain complexes, and hence on $\mathbf{Ch}(\mathscr{A})$ and $K(\mathscr{A})$. Let $\mathbf{Ch}_b(\mathscr{A})$ to be the category of *bounded chain complexes*, i.e. those A_* such that $A_n = 0$ for all but finitely many n. $\mathbf{Ch}^b(\mathscr{A})$ is the category of bounded cochain complexes. We also define $\mathbf{Ch}_+(\mathscr{A})$, resp. $\mathbf{Ch}_-(\mathscr{A})$, to be the categories of chain complexes that are bounded below, resp. above. $K_+(\mathscr{A})$, $K^-(\mathscr{A})$ etc. are defined similarly.

Definition. Given a category $\mathscr{A}$ and a class of morphisms S we define the *localization* of $\mathscr{A}$ at S to be a category $\mathscr{B}$ with a functor $Q : \mathscr{A} \longrightarrow \mathscr{B}$ such that $Q(s)$ is an isomorphism for any $s \in S$ and which is universal with this property: Any $\mathscr{A} \longrightarrow \mathscr{C}$ that sends all $s \in S$ to isomorphisms factors through Q.

Definition. We define the *derived category* $D(\mathscr{A})$ as the localization of $K(\mathscr{A})$ at the class of quasi-isomorphisms. Write $Q_{\mathscr{A}} : K(\mathscr{A}) \longrightarrow D(\mathscr{A})$ for the natural functor. $D^b(\mathscr{A})$ is defined similarly from $K^b(\mathscr{A})$, etc.

Theorem 12. *$D(\mathscr{A})$ exists.*

Not a proof. See [GM] III.2 or [W] 10.3 for details of the proof.

I'll just give some comments about the shape of the proof: As above, localization means that we throw in an inverse f^{-1} for every quasi-isomorphism f. This is a lot like Ore localization for (noncommutative) rings, if you've met that. The content is in working out suitable conditions for the existence of a localization, and checking that they are satisfied in our case. A class of morphisms is called *localising* if not lectured

- S contains the identities and is closed under composition.
- Given morphisms $A' \xrightarrow{s} A \leftarrow B$ with $s \in S$ there are morphisms $A' \leftarrow B \xrightarrow{t} B$ with $t \in S$ making the obvious diagram commute. And dually given the second pair of morphisms there exists the first one.
- Given any morphism f, g the existence of $s \in S$ with $sf = sg$ is equivalent to the existence of $t \in S$ with $ft = gt$.

The second condition allows us to write any morphism in the derived category as a 2-term zig-zag $* \leftarrow * \rightarrow *$.

This is sometimes called a *calculus of fractions*.

These conditions hold for quasi-isomorphisms in the homotopy category, but not in the category of chain complexes. □

Remark. $D^b(\mathscr{A})$ is equivalent to the subcategory of $D(A)$ given by complexes whose cohomology is concentrated in bounded degrees. This does not hold on the level of homotopy categories.

Remark. We will deal with the existence of localisations later when we work with model categories, and hopefully prove a generalisation of this result.

Example 13. Let X be a scheme. Let $Coh(X)$ be the abelian category of coherent sheaves on X. We define the *derived category* of X, $D^b(X)$, as the derived category of $\mathbf{Ch}^b(Coh(X))$.

As $Coh(X)$ has neither injectives nor projectives we typically enlarge the category to do computations. Let $QCoh(X)$ is the abelian category of quasi-coherent sheaves on X. If X is Noetherian then $D^b(X)$ is equivalent to the subcategory of $D^b(QCoh(X))$ consisting of complexes whose cohomology sheaves are coherent.

Because of its definition $D(\mathscr{A})$ is a bit hard to work with. For example, it's an additive category, but that is not obvious from the definition! It is also not abelian.

However we have the following very useful result:

Theorem 13. *Given a complex of injectives $I \in K^+(\mathscr{A})$ and any cochain complex A we have $Hom_{K(A)}(A, I) \cong Hom_{D(\mathscr{A})}(A, I)$.*

Sketch of proof. The crucial ingredient is the fact that $\mathrm{Hom}_{K(\mathscr{A})}(-, I)$ sends quasi-isomorphism to isomorphisms in **Ab**. This allows us to uniquely replace $A \xleftarrow{\sim} B \to I$ by $A \longrightarrow I$. Now consider a quasi-isomorphism $f : A \longrightarrow B$. We consider the short exact sequence $B \longrightarrow \mathrm{cone}(f) \longrightarrow A[-1]$. Now we show that $\mathrm{Hom}(-, I)$ sends acyclic complexes to acyclic complexes. In fact, let $g : C \longrightarrow I[i]$ be any chain map from an acyclic C to shifted I. Then we need to construct a homotopy to the zero map, which we can do term by term using injectivity of the I^i. It follows that $\underline{\mathrm{Hom}}(A, I)$ and $\underline{\mathrm{Hom}}(B, I)$ are quasi-isomorphic, hence for their zeroeth homology groups we have $\mathrm{Hom}(A, I) \cong \mathrm{Hom}(B, I)$. □

The following corollary allows us to compute hom-sets in the derived category.

Corollary 14. *Assume $\mathscr{A}$ has enough injectives. Then for objects $A, B \in \mathscr{A}$ considered as complexes concentrated in degree 0 we have $Hom_{D(\mathscr{A})}(A, B[i]) = Ext^i(A, B)$.*

Proof. Both sides equal $\mathrm{Hom}_{D(\mathscr{A})}(A, I[i]) = \mathrm{Hom}_{K(\mathscr{A})}(A, I[i]))$ where I is an injective resolution of B. □

Remark. This result remains true with the same proof for A, B any bounded below complex as long as we define $\mathrm{Ext}^i(A, B)$ suitably, i.e. via level-wise injective resolution of B.

Corollary 15. *Assume $\mathscr{A}$ has enough injectives. Consider the subcategory $K^+(Inj(\mathscr{A}))$ of $K^+(\mathscr{A})$ that consists of levelwise injective complexes. Then $K^+(Inj(\mathscr{A}))$ is equivalent to $D^+(\mathscr{A})$.*

Proof. Full faithfulness follows from Theorem 13. To show inclusion is essentially surjective we have to injectively resolve complexes, which we'll do in Lemma 35. (Or you can do it now as an exercise.) □

4.2. Total derived functors

We can now interpret derived functors differently: They lift functors to the derived category. But we need to change our definition a little:

Definition. Let $F : \mathscr{A} \longrightarrow \mathscr{B}$ is a left exact functor. By Corollary 15 we can choose an equivalence of categories $\Phi : D^+(\mathscr{A}) \longrightarrow K^+(\mathrm{Inj}(\mathscr{A}))$.

Then the right derived functor $RF : D^+(\mathscr{A}) \longrightarrow D^+(\mathscr{B})$ is defined as $Q_{\mathscr{B}} \circ F \circ \Phi$.

If we need to disambiguate we will call the R^iF the *classical derived functors* and RF the *total derived functor*.

By Theorem 6 we can use any injective resolution of A to compute $RF(A)$.

Proposition 16. *There is a natural transformation $\epsilon : Q_{\mathscr{B}} \circ F \longrightarrow RF \circ Q_{\mathscr{A}}$ between functors $K^+(\mathscr{A}) \longrightarrow D^+(\mathscr{B})$. Moreover, the pair (RF, ϵ) is initial, i.e. given $G : D^+(\mathscr{A}) \longrightarrow D^+(\mathscr{B})$ and $\phi : Q_{\mathscr{B}} \circ F \longrightarrow G \circ Q_{\mathscr{A}}$ there is a unique natural transformation $\rho : RF \longrightarrow G$ such that $\rho \circ \epsilon = \phi$.*

Not a proof. [GM] III.6.8 or [W] 10.5.6. □

Remark. In fact, this characterisation of the derived functor via the universal property is arguably the correct definition of the derived functor. Not only is it nice and conceptual, but it is also available if $\mathscr{A}$ does not have enough injectives or projectives! Note that [GM] develop the theory of derived functors without reference to injective objects, using *adapted objects*, see Section 3.3.

4.3. Triangles

not lectured

Definition. Given a chain map $f : A \longrightarrow B$ we define the *cylinder*, $\mathrm{cyl}(f)$ to be the complex which is $A_n \oplus A_{n-1} \oplus B_n$ in degree n and has differential

$$\begin{pmatrix} d_A & \mathbf{1}_A & \\ & -d_A & \\ & -f & d_B \end{pmatrix} : A_n \oplus A_{n-1} \oplus B_{n-1} \longrightarrow A_{n-1} \oplus A_{n-2} \oplus B_{n-1}$$

It can be checked ([W] 1.5.6) that the natural inclusion $B \longrightarrow \mathrm{cyl}(f)$ is a chain homotopy equivalence with inverse $(a, a', b) \mapsto f(a') + b$.

Note that we have a short exact sequence of complexes $A \longrightarrow \mathrm{cyl}(f) \longrightarrow \mathrm{cone}(f)$. If $A \xrightarrow{f} B \longrightarrow C$ is a short exact sequence we have compatible quasi-isomorphisms.

The sequences $A \longrightarrow \mathrm{cyl}(f) \longrightarrow \mathrm{cone}(f) \longrightarrow A[-1]$ are called *strict exact triangles* and the sequences isomorphic to them in the derived category are called *exact triangles*. These generalise short exact sequences. They satisfy certain axioms, making the derived category into a *triangulated category*.

Remark. Derived categories and triangulated categories are very fruitful objects of study. However, inverting all quasi-isomorphisms throws out a lot of information and sometimes one wants to keep it and work with an *enhanced derived category*. That could mean working with categories enriched in chain complexes, a.k.a. *differential graded categories*, or with some other notion of *stable infinity categories*.

Proposition. *Derived functors preserve exact triangles.*

Not a proof. E.g. [GM] III.6. □

II. Applications and Examples

5. Ext and extensions

Ext-groups and extensions.

5.1. Ext^1 classifies extensions

The right derived functors of $\mathrm{Hom}(A, -)$ are called $\mathrm{Ext}^i(A, -)$. In fact they agree with the derived functors of $\mathrm{Hom}(-, B)$ and we will use that. There is an elementary proof, but we will prove it using spectral sequences in Proposition 32

Let us consider objects A, B and C in $\mathscr{A}$. Then composition induces a product $\mathrm{Ext}^i(A, B) \otimes \mathrm{Ext}^j(B, C) \longrightarrow \mathrm{Ext}^{i+j}(A, C)$. We choose injective resolutions $B \longrightarrow I$ and $C \longrightarrow J$ and consider the composition:

$$\mathrm{Hom}_{K(\mathscr{A}))}(A, I[i]) \otimes \mathrm{Hom}_{X(\mathscr{A})}(I[i], J[i+j]) \longrightarrow \mathrm{Hom}_{K(\mathscr{A})}(A, J)$$

In particular $\mathrm{Ext}^*(A, A)$ is an algebra.

We will now develop a very nice interpretation of Ext^1, relating to extensions. Recall that a short exact sequence $A \longrightarrow B \xrightarrow{g} C$ in $\mathscr{A}$ can be viewed as an extension of C by A. We call it split if $C \cong A \oplus B$.

To relate this to Ext-groups us apply $R\mathrm{Hom}(C, -)$ and see what happens. We get a boundary map $\partial : \mathrm{Hom}(C, C) \longrightarrow \mathrm{Ext}^1(C, A)$. So what is $\partial(\mathbf{1})$?

Lemma 17. *The class $\partial(\mathbf{1}_C)$ is* 0 *if and only if the s.e.s. defining ∂ is split.*

Proof. If the sequence is split the map induced by g is surjective. For the converse note that exactness means $\mathbf{1}$ lifts to a map $C \longrightarrow B$ if $\partial(\mathbf{1}) = 0$. □

We call $\partial(\mathbf{1})$ the *obstruction* to splitting.

We say two extensions $A \longrightarrow B \longrightarrow C$ and $A \longrightarrow B' \longrightarrow C$ are *equivalent* if there is an isomorphism $B \longrightarrow B'$ that makes the obvious diagram commute.

Theorem 18. *There is a bijection between $Ext^1(C,A)$ and extensions of C by A up to isomorphism.*

Example 14. Consider short exact sequences of the form

$$\mathbb{Z}/p \xrightarrow{j} A \xrightarrow{k} \mathbb{Z}/p$$

Now if the sequence is split $A \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$. Otherwise $A \cong \mathbb{Z}/p^2$ and then $j : 1 \mapsto ap$, with $a \in \{1, \ldots, p-1\}$. This gives p extensions. On the other hand we can easily see that $\mathrm{Ext}^1(\mathbb{Z}/p, \mathbb{Z}, p) \cong \mathbb{Z}/p$.

Proof of Theorem 18. The above construction gives us a map $\Psi : \{\text{extensions}\} \longrightarrow \mathrm{Ext}^1$. Let us consider the beginnings of a projective resolution, $0 \longrightarrow M \xrightarrow{i} P \longrightarrow C \longrightarrow 0$ where P is projective. Then a class in $\mathrm{Ext}^1(C,A)$ is represented by a map $\alpha : M \longrightarrow A$. We define B to be the pushout of i and α, i.e. the cokernel of $(\alpha, -i) : M \longrightarrow A \oplus P$. Write j_A and j_P for the obvious maps from A and P to B.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow^{\alpha} & & \downarrow^{j_P} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{j_A} & B = (A \oplus P)/M & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Then as i is an injection, A is a subobject of B: consider $j_A(a) = 0$, i.e. $(a, 0) = (\alpha(m), -i(m))$, which forces m and hence a to be 0. Next

$$B/A \cong \mathrm{coker}((j_A + \alpha, i) : A \oplus M \longrightarrow A \oplus P) \cong P/M \cong C$$

so we get an extension.

Now assume we are given α' homologous to α. That means $\alpha' = \alpha + fi$ for some $f : P \longrightarrow A$. We need an isomorphism between $B = (P \oplus A)/(i \oplus \alpha)M$ and $B' = (P \oplus A)/(i \oplus \alpha')M$. Clearly $\begin{pmatrix} \mathbf{1} & f \\ 0 & \mathbf{1} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{1} & -f \\ 0 & \mathbf{1} \end{pmatrix}$ are inverse isomorphisms.

So there is a map $\Phi : \mathrm{Ext}^1 \longrightarrow \{\text{extensions}\}$.

We can show Ψ is surjective by checking $\Psi \circ \Phi = \mathbf{1}$. Starting with α, we construct $A \longrightarrow B \longrightarrow C$ and want to show that $\partial(\mathbf{1}_C) = [\alpha]$. We can do that by chasing $\mathbf{1}_C$ through the following diagram as is done in the proof of the snake lemma:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(M,A) & \longrightarrow & \mathrm{Hom}(M,B) & \longrightarrow & \mathrm{Hom}(M,C) & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(P,A) & \longrightarrow & \mathrm{Hom}(P,B) & \longrightarrow & \mathrm{Hom}(P,C) & \longrightarrow & 0 \end{array}$$

Let us actually go the other way. We know $\alpha : M \longrightarrow A$ is sent to $j_A \circ \alpha : M \longrightarrow B$ which equals $j_P \circ i : M \longrightarrow B$, which is the differential of $j_P : P \longrightarrow B$ which maps to the projection map $P \longrightarrow C$ which does indeed represent $\mathbf{1}_C$. Retracing the steps backwards shows that $\alpha = \partial(\mathbf{1}_C)$.

Next we consider the converse. Assume we are given an extension $0 \longrightarrow A \xrightarrow{j} B \xrightarrow{k} C \longrightarrow 0$. Then we get a map $\beta : P \longrightarrow B$ by projectivity of P. This leads to $M \longrightarrow B$ which factors through $\alpha : M \longrightarrow A$ by universality of the kernel. By construction this is $\partial(\mathbf{1}_C)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & P & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow^{\alpha} & & \downarrow^{\beta} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{k} & C & \longrightarrow & 0 \end{array}$$

We now show that B is the pushout $(P \oplus A)/M$. That proves $\Phi \circ \Psi = \mathbf{1}$. Clearly $B/(P \oplus A) = 0$, showing surjectivity of the natural map $j \oplus \beta$. To show injectivity consider the kernel of $j \oplus \beta$. If $j(a) + \beta(p) = 0$ we must have $k\beta(p) = 0$, hence $p = im$ for some $m \in M$. Note that $a = \alpha(m)$ by construction of α. □

5.2. More on extensions

Remark. Note that B we would like to think of B as the cone over "$C[-1] \longrightarrow A$". In the derived category there is indeed a map from $C[-1]$ to A and B can indeed be defined as a cone. Explicitly, on the level of cochain complexes, we consider the map $(\alpha, 0) : (M \xrightarrow{-i} P) \longrightarrow A]$ (here M and A are in degree 0). Then the cone is the complex $(M \xrightarrow{(i,-\alpha)} P \oplus A)$ which is quasi-isomorphic to its homology, which is precisely how we defined B. (Note here that the cohomological cone has underlying complex $C[-1][+1] \oplus A$ and the differential i changes sign twice.)

There is also an addition defined directly on extensions, called the Baer sum. It is the only thing you can define: Take an extensions $A \xrightarrow{i} B \xrightarrow{p} C$ and $A \xrightarrow{i'} B' \xrightarrow{p'} C$. Then $B \oplus B'$ is an extension of $C^{\oplus 2}$ by $A^{\oplus 2}$, so we need to kill a copy each of A and C. We define $Q = \ker(B \oplus B' \xrightarrow{p-p'} C)$ and note that there are two natural inclusions $A \longrightarrow Q$ induced by i and i'. To identify them we quotient by the skew diagonal and define $B'' = \operatorname{coker}(A \xrightarrow{(i,-i')} Q)$ which fits into $A \longrightarrow B'' \longrightarrow C$.

We can write this as a pullback along the diagonal $\Delta : C \longrightarrow C \oplus C$ followed by a pushout along the addition map $+ : A \oplus A \longrightarrow A$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \oplus A & \xrightarrow{i} & B \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0 \\
 & & \| & & \uparrow & \llcorner & \uparrow \Delta & & \\
0 & \longrightarrow & A \oplus A & \longrightarrow & Q & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow + & \lrcorner & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & B'' & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Moreover, this addition agrees with addition in Ext^1. The proof is not hard, we begin with the diagonal map (α, α') representing the extension $B \oplus B'$ and then we compose with diagonal and addition map to get $\alpha + \alpha'$ representing B''. See also [W] 3.4.5.

Remark. Now we can define Ext groups via extensions, even if we do not have enough projectives. (For example in the category of finite abelian groups.)

Note that $\mathrm{Ext}^n(C, A) = \mathrm{Ext}^1(C, A[n-1])$ classifies extensions of C by $A[n-1]$. These are given by complexes now. To prove this we can go through the proof, replacing the partial resolution by a longer one. To be precise we replace P_* by a complex of projectives of length n such that $M \longrightarrow P_* \longrightarrow C$ is exact. Then a map in $\mathrm{Ext}^n(C, A)$ is again represented by a map $M \longrightarrow A$. We can go through the proof and interpret everything in terms of complexes, for example f is understood to be a chain map, and the chain map β is constructed degree by degree. The proof goes through! Moreover, we can replace A by a complex. Now we need to resolve C further to be able to write down a chain map $P_* \longrightarrow A$, where $P_* \simeq C$.

Now an extension of $C[1-n]$ by A can also be viewed as complexes of length n with first and last homology group A respectively C, i.e. these give rise to exact sequences with $n+2$ terms (up to isomorphism):

$$0 \to A \to B_1 \to B_2 \to \ldots \to B_n \to C \to 0$$

Two such sequences are equivalent if there is a family of maps $f_i : B_i \longrightarrow B_i'$ compatible with each other and the identities on A and C. Together with a suitable definition of Baer sums this allows us to define higher Ext-groups without projectives.

Remark. We can go even further and consider $\mathrm{Ext}^1(C, A)$ for two complexes C and A. Let us say C is bounded below. We then define $\mathrm{Ext}^1(C, A) = \mathrm{Hom}_{\mathbf{Ch}(\mathscr{A})}(P, A[1])$, where $P \longrightarrow C$ is a quasi-isomorphism and P is level-wise projective. We will see in Lemma 35 that such a resolution exists. Now short exact sequences $A \longrightarrow B \longrightarrow C$ in $\mathbf{Ch}(\mathscr{A})$ are classified by $\mathrm{Ext}^1(C, A)$.

not lectured

Remark. This is just one example of the rich story of cohomology classes with interpretations as moduli. Indeed, many natural mathematical questions about families of objects have precise homological answers:

- Extensions of a group G by an abelian group A are given by $H^2(G,A) := \mathrm{Ext}^2_{\mathbb{Z}G}(\mathbb{Z},A)$, cf. [W] 6.6.
- Line bundles on a variety are given by $H^1(X,\mathscr{O}^*)$.
- Deformations of a complex manifold are described by $H^1(X,\mathscr{T}_X)$ where $\mathscr{T}_X$ is the tangent sheaf on X.
- The space of deformations of an algebra A over $k[t]/(t^2)$ is given by the second Hochschild cohomology group $HH^2(A,A)$.
- In fact, one can define $HH^2(X)$ for an algebraic variety which describes both geometric and non-commutative deformations of X and encompasses the two previous examples.
- The Brauer group $Br(K)$ of a field is defined as follows: Its elements are central simple K-algebras, i.e. K-algebras which are simple rings with centre K, up to the equivalence relation that $M_n(D) \cong M_m(D)$ for any division algebra D. It turns out that tensor product provides a group operation. Then $Br(X) = H^2(G;K_s^*)$, the (profinite) group cohomology of the Galois group with coefficients in the separable closure K_s of K, cf. [W] 6.11.17. not lectured

6. Group cohomology

Group homology and cohomology. The bar complex. Eilenberg-MacLane spaces.

6.1. Group cohomology

Let G be a (discrete) group and write G-Mod for the category of k-representations of G, equivalent to the category of kG-modules. Unless stated otherwise we will be concerned with the case $k = \mathbb{Z}$.

Definition. There is a natural functor $G\text{-Mod}_k \longrightarrow k\text{-Mod}$, defined by taking invariants $A \mapsto A^G := \{a \in A \mid ga = a \; \forall g \in G\}$. This functor is left exact and hence has right derived functors.

Dually there is the functor of taking coinvariants, $A \mapsto A_G := A/\{ga - a \mid g \in G, a \in A\}$.

One notes that $A^G = \mathrm{Hom}_{kG}(k, A)$ and $A_G = k \otimes_{kG} A$ and has the following easy proposition.

Proposition 19. *We have adjunctions $-_G \dashv T \dashv -^G$ where $T : k\text{-Mod} \longrightarrow kG\text{-Mod}$ is the trivial module functor $M \mapsto M$ considered as a trivial kG-module.*

Proof. These are just the usual adjunctions between Hom and $\otimes$ once one notes that $TM = \mathrm{Hom}_k(k, M) = M \otimes_k k$ where k is the trivial kG-module. □

Definition. The *group cohomology $H^*(G, M)$* of a group G with coefficients in a G-modules M is given by the right derived functors of taking invariants, $R^i(-^G)(M)$. Dually *group homology $H_*(G, M)$* is given by the left derived functors of taking coinvariants.

By the above group cohomology and homology are just Ext and Tor in kG-Mod.

Example 15. Here is an example that uses Theorem 18: Let G be a finite group. Then if M, N are any representations over a field with characteristic not dividing $|G|$ then $\mathrm{Ext}^1_G(M, N) = 0$ by Maschke's theorem.

Let us compute one concrete example.

Example 16. Let C_n have generator g and let A be any C_n-module. We compute $H(C_n, A)$. To do this we need to resolve the C_n-module $\mathbb{Z}$. Consider the augmentation map $\mathbb{Z}C_n \longrightarrow \mathbb{Z}$. Write $N = 1 + g + \cdots + g^{n-1}$. Then we claim that the complex

$$\ldots \longrightarrow \mathbb{Z}C_n \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{(1-g)} \mathbb{Z}C_n \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{(1-g)} \mathbb{Z}C_n \xrightarrow{\epsilon} \mathbb{Z}$$

is exact, where tho maps are given by multiplication with the elements N and $1-g$ respectively. We obtain this resolution by splicing together short exact sequences which we get by examining the kernels and images of our differentials.

- First note that $\mathrm{Im}(N) = \ker(1-g)$. It is clear that $\mathbb{Z}N \subset \ker(1-g)$. On the other hand suppose $(1-g)\sum a_i g^i = 0$. Comparing coefficients all a_{g^i} must be the same, hence $a = a_0 N$.
- Next consider $\ker(\epsilon) = \mathrm{Im}(1-g)$. The second assertion is clear as $g^i - 1$ span $\ker(\epsilon)$, and each $g^i - 1$ is a multiple of $g-1$.
- Finally, $\ker(1-\epsilon) = \ker(N)$. Certainly $\ker(\epsilon) \subset \ker N$. On the other hand suppose $Na = 0$. Then $0 = \epsilon(Na) = \epsilon(N)\epsilon(a) = |G|\epsilon(a)$ which forces $a \in \ker(\epsilon)$.

Then we can easily compute that

$$H^*(C_n, A) = \begin{cases} A^{C_n} & \text{if } * = 0 \\ \{a \mid Na = 0\}/(1-g)A & \text{if } * = 1, 3, 5, \ldots \\ A^{C_n}/N.A & \text{if } * = 2, 4, 6 \ldots \end{cases}$$

Dually $H_*(C_n, A)$ is A_{C_n} in degree 0 and even and odd degree terms are swapped. ($H_2 = H^1$ and $H_1 = H^2$.)

For a concrete example, let $A = \mathbb{Z}$ and we have

$$H^*(C_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{if } * = 1, 3, 5, \ldots \\ \mathbb{Z}/n & \text{if } * = 2, 4, 6 \ldots \end{cases}$$

Remark. If the group G is equipped with a topology we should work in the category of continuous representations. Two important examples are Lie groups and pro-finite groups like Galois groups (considering discrete modules).

Remark. Entirely analogously we can consider Lie algebra cohomology $H^*(\mathfrak{g}, -)$ as $\mathrm{Ext}_{\mathscr{U}(\mathfrak{g})}(k, -)$, where $\mathscr{U}(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra. We also define Hochschild cohomology for bimodules of an algebra A as $HH^*(A, -) =$

$\mathrm{Ext}_{A\otimes A^{\mathrm{op}}}(A,-)$. Note that an associative algebra does not have a unique trivial representation. Instead we study the endomorphism algebra $A\otimes A^{\mathrm{op}}$ with its canonical representation A. (This is not quite the right definition if A is an algebra over a ring k that is not a field.)

6.2. The bar complex

There is an important canonical resolution for the trivial $\mathbb{Z}G$-module $\mathbb{Z}$.

Definition. Define the *unnormalized bar complex* $B^u = B^u(G)$ as follows: B^u_n is the free $\mathbb{Z}G$-module generated by symbols $[g_1 \otimes \cdots \otimes g_n]$ where $g_i \in G$. B^u_0 is just $\mathbb{Z}G$, and we write 1 as $[]$ for convenience. The differentials are given as $d = \sum_{i=0}^n (-1)^i d^i$, where

$$\begin{aligned} d^0([g_1 \otimes \cdots \otimes g_n]) &= g_1[g_2 \otimes \cdots \otimes g_n] \\ d^i([g_1 \otimes \cdots \otimes g_n] &= [g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n]) \\ d^n([g_1 \otimes \cdots \otimes g_n]) &= [g_1 \otimes \cdots g_{n-1}] \end{aligned}$$

It is straightforward to check that this is indeed a complex, i.e. $d^2 = 0$.

Definition. We then define the *(normalised) bar complex* $B_*(G)$ as the quotient complex of $B^u_*(G)$ by the subcomplex generated by elements $[g_1 \otimes \cdots \otimes g_n]$ with at least one g_i equal to the identity.

One can check that B_* is equivalent to the complex with $B_0(G) = \mathbb{Z}G$, $B_n(G)$ the free $\mathbb{Z}G$-module generated by symbols $[g_1|\cdots|g_n]$ where $g_i \in G \setminus \{1\}$. The differential is given as for B^u_* except that $d^i = 0$ if $g_i g_{i+1} = 1$.

Example 17. As the definition of the bar complex is a bit daunting, here are some concrete computations:

- $d([g]) = (g-1)[]$.
- $d([f|g]) = f[g] - [fg] + [f]$
- $B_n(\mathbb{Z}/2) = \mathbb{Z}/2[g|\cdots|g]$, where g is the generator of $\mathbb{Z}/2$, and the bar complex is isomorphic to the resolution we used to compute homology of $\mathbb{Z}/2$. (Check the differentials!)

We define the *augmented bar complex* as $\cdots \longrightarrow B_1 \xrightarrow{d} B_0 \xrightarrow{\epsilon} \mathbb{Z}$ where ϵ is just the augmentation map $\mathbb{Z}G \longrightarrow \mathbb{Z}$ given by $g \mapsto 1$.

Theorem 20. *The normalised and unnormalized bar complexes give resolutions of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$.*

Proof. The proofs are very similar, we do the unnormalized case.

We will show that the normalised bar complex is contractible, i.e. we will produce a chain homotopy between identity and zero map on the complex $B_*(G) \longrightarrow \mathbb{Z}$. This is equivalent to showing that the augmentation map $B_* \longrightarrow \mathbb{Z}$ is a chain homotopy equivalence. (Note that this is an equivalence, as chain complexes, not as $\mathbb{Z}G$-modules.)

We define $s_{-1} : \mathbb{Z} \longrightarrow B_0$ by $s_{-1}(1) = []$ and $s_n : B_n \longrightarrow B_{n+1}$ by $s_n(g_0[g_1|\cdots|g_n]) = [g_0|g_1|\cdots|g_n]$.

Now it is easy to see that $\epsilon \circ s_{-1} = \mathbf{1}_{\mathbb{Z}}$ and $s_{-1} \circ \epsilon = ds_0$. Moreover unravelling definitions we find $ds_n = \mathbf{1}_{B_{n-1}} + s_{n-1}d$. □

Let now A be a right $\mathbb{Z}G$-module. (We could set up the mirror of the bar complex to deal with left $\mathbb{Z}G$-modules.) Then we have $H_*(G; A) = H_*(A \otimes B_*(G))$.

Corollary 21. *For any group G we have $H_1(G; \mathbb{Z}) = G/[G, G]$.*

Proof. By the above $H_1(G; \mathbb{Z})$ is the free abelian group generated by symbols $[g]$ modulo the ideal $\langle [1], [f] - [fg] + [g] \rangle$. Now we can write down a map ϕ from G to H_1 given by $g \mapsto [g]$. The map sends fg to $[fg] = [f] + [g]$, hence it is a homomorphism and must factor through $[G, G]$. On the other hand there is a map $\theta : H_1 \longrightarrow G$ given on the generators by $[g] \mapsto G/[G, G]$ and extended linearly. This is well defined, as $[f] - [fg] + [g] \mapsto f(fg)^{-1}g = [f, g^{-1}] \in [G, G]$. The two maps are clearly inverses. □

We finish this section with a view to algebraic topology.

Definition. There are connected spaces $K(G, 1)$, unique up to homotopy, whose homotopy groups are $\pi_1 = G$ and $\pi_{>1} = 0$. These are called *Eilenberg-MacLane spaces*.

Theorem 22. $H^*_{sing}(K(G, 1), k) = H^*(G, k)$.

This is very reassuring.

Idea of proof. Here are two ways one can prove this, we might learn a little bit more about one of them later on.

We can construct $K(G, 1)$ as the quotient of some contractible EG by a properly discontinuous G-action. (This is an Eilenberg-MacLane space thanks to the long exact sequence of homotopy groups.) But then $\mathbb{Z}G$ acts freely on $\mathrm{Sing}_*(EG)$ and the invariants are singular chains on the quotient.

We will later meet simplicial sets, which give an explicitly combinatorial model for topological spaces. There is a classifying space construction in simplicial sets and one can compute homotopy groups of the classifying space to be the homotopy groups of Eilenberg-MacLane spaces. On the other hand the classifying space looks a lot like tho bar complex and indeed the (co)homology groups are given by the bar complex above. □

7. Homological dimension

Global/homological dimension. Global dimension theorem. Koszul resolutions and Hilbert's theorem on syzygies.

7.1. Homological dimension

Definition. The *projective dimension* $pd_R(M)$ of a a module M over a ring R is the smallest n such that there is a resolution $0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow M$ with P_i projective. The injective dimension $id_R(M)$ dimension is defined dually.

Definition. The *(left) global dimension* $gd(R)$ of a ring R, also called its *homological dimension* is defined as the $\sup_{M \in R\text{-Mod}}(pd_R(M))$.

Right and left global dimension need not agree. We'll restrict attention to the right global dimension as most of our examples are commutative anyways. (The two notions agree for left and right Noetherian rings, see [W] 4.1)

Theorem 23 (Global dimension theorem)**.** *The following numbers (possibly ∞) agree for any ring:*

1. $gd_1 = \sup_{M \in R\text{-}Mod}(pd_R(M)) = gd(R)$
2. $gd_2 = \sup_{M \in R\text{-}Mod}(id_R(M))$
3. $gd_3 = \sup\{pd(R/J) \mid J \subset R \text{ is an ideal}\}$
4. $gd_4 = \sup\{d \mid Ext^d_R(A, B) \neq 0 \text{ for some modules } A, B\}$

Example 18. It follows that our computation in the last lecture show $gd(\mathbb{Z}C_n) = \infty$. Also note that if k is a field then $gd(k) = 0$. A question on the first example sheet shows that $gd(\mathbb{Z}) = 1$.

Lemma 24. *For any R-module M the following are equivalent:*

1. $id(M) \leq d$
2. $Ext^n_R(N, M) = 0$ *for any* $n > d$ *and* $N \in R\text{-}Mod$
3. $Ext^{d+1}(N, M) = 0$ *for any* $N \in R\text{-}Mod$
4. *If* $0 \longrightarrow M \longrightarrow I^0 \longrightarrow \ldots \longrightarrow I^{d-1} \longrightarrow Q \longrightarrow 0$ *is a resolution with* I^i *injective than* Q *is also injective.*

Proof. Exercise. (You just need four implications.) □

Lemma 25 (Baer's criterion)**.** *A module I is injective if and only if every map $J \longrightarrow I$ from an ideal of R can be extended to a map from R.*

Proof. E.g. [W] 2.3.1. The proof uses Zorn's lemma. □

Proof of Theorem 23. The lemma and its dual show that the quantities gd_1, gd_2 and gd_4 agree. Also gd_3 is clearly smaller or equal than the others. So let us assume $d = gd_3$ is finite and take an arbitrary module M. Take a partial injective resolution of M, $M \longrightarrow I^0 \longrightarrow \ldots \longrightarrow I^d \longrightarrow Q \longrightarrow 0$, where the I^i are injective. It suffices to show M is also injective. Note that for any ideal $J \subset R$ we know $\mathrm{Ext}^1(R/J, Q) = \mathrm{Ext}^{d+1}(R/J, M) = 0$ by assumption.

But the short exact sequence $Q \longrightarrow R \longrightarrow R/J$ shows that

$$\mathrm{Hom}(R, Q) \longrightarrow \mathrm{Hom}(J, Q) \longrightarrow \mathrm{Ext}^1(R/J, Q)$$

is exact in the middle. So the map on the left is surjective for all ideals. By Baer's criterion this shows Q is injective. □

Example 19. Recall that a ring R is *semi-simple* if every ideal is a direct summand. It's easily seen this is equivalent to R having global dimension zero. (Consider the s.e.s. $I \longrightarrow R \longrightarrow R/I$.) This is also equivalent, by Theorems 23 and 18, to saying that all extensions of R-modules are split. We say the category of R-modules is semi-simple.

You may recall that semi-simple rings are precisely direct sums of matrix algebras over division rings, by Wedderburn's theorem.

Remark. Regular local rings have finite global dimension and local rings with finite global dimension are regular. One important application of this result is that localisations of regular local rings are regular. We don't have time to do it, but if you like commutative algebra have a look at the proof in [W] 4.4, for example. It's a serious proof, but it seems to be by far the easiest way (but not the only way) to show this pure commutative algebra result.

7.2. Hilbert's theorem

One of the crucial examples (predating most of homological algebra!) is the following:

Theorem 26 (Hilbert's syzygy theorem)**.** *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field. Then $gld(R) = n$.*

Note that we (like Hilbert) will prove that the "free dimension" of R is n, which is a priori stronger than saying the global (projective) dimension is n. It turns out that the notions are almost equivalent (see the second example sheet).

Remark. In fact, projective modules over R are necessarily free, but that is a hard theorem. (Conjectured by Serre, proven by Suslin and Quillen independently.)

Remark. Modules over R are also known as quasi-coherent sheaves on $\mathbb{A}^n_k$ and thus very interesting objects for algebraic geometers.

We will now prove the theorem, which needs some warming up. To compute with modules over $R = k[x_1, \dots, x_n]$ the following class of resolutions is very useful. (Actually, it is useful in many other settings as well.)

Definition. Let $\mathbf{t} = (t_1, \dots, t_m)$ a sequence of central elements of a ring S. Then let $K(t_i)$ be the complex $S.e_i \xrightarrow{t_i} S$ where the differential is $d : e_i \mapsto t_i$. Here e_i has degree 1.

We define the *Koszul complex* $K(\mathbf{t})$ as follows: $K_p = \bigwedge^p(\oplus_{i=1}^m S.e_i)$ and the differential is

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \mapsto \sum_{j=0}^{p} (-1)^j t_j e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_p}$$

Example 20. Here is $K(x, y)$ for $S = \mathbb{C}[x, y]$:

$$0 \longrightarrow S.e_x \wedge e_y \xrightarrow{(y,-x)} S.e_x \oplus S.e_y \xrightarrow{\binom{x}{y}} S.1 \longrightarrow 0$$

In fact we would like to say $K(\mathbf{t}) := K(t_1) \otimes_S \cdots \otimes_S K(t_m)$. To make this precise we have to define the tensor product of complexes.

Definition. Let A, B be complexes. We let $(A \otimes B)_n = \oplus_i (A_i \otimes B_{n-i})$ and $d_n^{A\otimes B} = d^A \otimes \mathbf{1} + (-1)^i \otimes d^B$. This is the *(total) tensor product complex.*

Let's go back to the Koszul complex. We say $\mathbf{t}$ is a *regular sequence* on an S-module M if $t_i \in S$ is never a zerodivisor in $M/(t_1, \ldots, t_{i-1}) := M \otimes_S S/(t_1, \ldots, t_{i-1})$.

Lemma 27. *Let* $\mathbf{t}$ *be a regular sequence on* M*. Then* $K(\mathbf{t}) \otimes_S M$ *is a resolution of* $M/(t_1, \ldots, t_m)$*.*

Proof. The result holds for $m = 1$ as t_1 is not a zerodivisor. Now we proceed by induction: Let $\mathbf{t}' = (t_1, \ldots, t_{m-1})$. Then let $C = K(\mathbf{t}') \otimes_S M$, which resolves $M/\mathbf{t}' := M/(t_1, \ldots, t_{m-1})$ by assumption. We are now looking for the homology groups of $K(t_m) \otimes_S C$, which we can arrange as a *double complex* with two columns and m rows, and with vertical and horizontal differentials. One can prove directly (or use the spectral sequence for a double complex, to be introduced shortly) that its homology is $H_0(C)/t_m H_0(C) = M/\mathbf{t}$, concentrated in degree 0. □

Let's go back to $R = k[x_1, \ldots, x_n]$ as in the theorem.

Example 21. We can compute $\mathrm{Ext}_R(k, k)$ or $\mathrm{Tor}^R(k, k)$ using a Koszul resolution. In fact $\mathrm{Ext}^*_R(k, k) \cong \wedge^* k^{\oplus n}$.

Proof of Theorem 26. The trick is to consider a resolution of R itself over the bigger ring $S = R[y_1, \ldots, y_n]$. There is $R' = k[y_1, \ldots, y_n] \subset S$ and in fact $S = R \otimes_k R'$.

Let $\mathbf{t} = (t_1, \ldots t_n)$ with $t_i = x_i - y_i$. We can rewrite S as $R[t_1, \ldots, t_n]$ and this shows $K(\mathbf{t})$ is a resolution of $R'' := S/(t_1, \ldots, t_m)$. Of course $R'' \cong R$. Now $K(\mathbf{t}) \longrightarrow R''$ is actually a split exact sequence of R-modules (and R'-modules). As the sequence is split exact, it stays exact after applying any additive functor, so given any R-module M we can apply $- \otimes_R M$ to obtain an exact sequence, which is an S-module resolution of $M \cong R'' \otimes_R M$. We want to show this is a resolution by free R-modules. Note that all t_i are 0 on $S/\mathbf{t}$ and hence on M. So the R-module structure and R'-module structure on the module M are the same. Hence we can compute $S \otimes_R M = R' \otimes_k M$, which is a free $R' \cong R$-module. So $K(\mathbf{t}) \otimes_R M$ is a natural resolution of M with $n + 1$ terms, showing $pd(M) \leq n$. But M was arbitrary.

Finally the computation of $\mathrm{Ext}_R(k, k)$ shows that the global dimension is at least n. □

III. Spectral Sequences

8. Introducing spectral sequences

Intuition for spectral sequences. Example. Filtrations.

8.1. Motivation

We'll be working with cochain complexes. Before giving the definition, we try to motivate it by considering a desired application.

Definition. A *double complex* C^{**} is a collection of objects C^{ij} together with differentials $d_h^{ij} : C^{ij} \longrightarrow C^{i+1,j}$ and $d_v^{ij} : C^{ij} \longrightarrow C^{i,j+1}$ which satisfy $d_h^2 = d_v^2 = 0$ and $d_h d_v = -d_v d_h$.

There are two total complexes associated to a double complex: The direct sum total complex $\mathrm{Tot}^{\oplus} C$ is defined as the complex (D^*, d) where $D^n = \oplus_{i+j=n} C^{ij}$ and $d = d_h + d_v$. The direct product total complex $\mathrm{Tot}^{\prod} C$ has $(\mathrm{Tot}^{\prod} C)^n = \prod_{i+j=n} C^{ij}$ and the same differential.

Note that the total complexes will only exist if $\mathscr{A}$ has all direct sums, resp. all direct products. This is certainly the case in R-Mod.

Convention. Throughout this chapter we will assume we work in the abelian category R-Mod.

Example 22. This looks familiar from our definition of the total tensor product. We can first define a double complex $(A \otimes B)_{pq} = A_p \otimes B_q$ with horizontal differential $d^h = d^A \otimes \mathbf{1}$ and vertical differential $d^v = (-1)^p \mathbf{1} \otimes d^B$. Then $(A \otimes B)_* = \mathrm{Tot}^{\oplus}((A \otimes B)_{**})$.

How can we compute cohomology of the total complex? Let's assume our complex is concentrated in the first quadrant.

Consider an element x in $H^n(\mathrm{Tot}\, C)$. We can represent it as $x = \sum_{i=0}^{m} x_i$ where $x_i \in C^{p+i,n-p-i}$. (Here p is minimal such that $x_0 \neq 0$. Let $q = n - p$.) Now if x

is to be exact, we need $d_v x_0 = 0$, and $d_h x_i = d_v x_{i+1}$, and $d_h x_m = 0$. So x_0 is an element in H_v^* and we have some extra condition about its horizontal differential.

It is useful to introduce a *filtration* on C. We are going to define filtrations properly soon, here we just define $F^n C = \bigoplus_{j \leq n} C^{j*}$. So $x \in F^p C$.

We would like to approximate x by only considering the first few x_i. If we quotient $F^p C$ by $F^{p+r} C$ we have $[d_h x_{r-1}] = 0$. Note the quotient by F^{p+1} precisely kills off the horizontal differential.

If we want to reconstruct $x \in F^p C / F^{p+r+1} C$ we need to consider $d_h x_{p+r-1} \in F_{p+r}$, and show it is a suitable boundary. This means that if we start by analysing an object in C^{pq} we are led to consider if an object in $C^{p+r,q-r+1}$ vanishes. And the implicit map from C^{pq} to $C^{p+r,q-r+1}$ is just induced by d.

So there is a differential associated to considering $F^p C / F^{p+r} C$ and taking homology repeatedly for larger and larger r should approximate the total complex.

In our example looking at a double complex, this means we first compute homology with respect to d^v and then with respect to d^h, and then with respect to some new differential of degree $(2, -1)$.

Hopefully this helps explain where the following definition comes from:

Definition. A *(cohomology) spectral sequence* starting at E_a in an abelian category $\mathscr{A}$ is the following data:

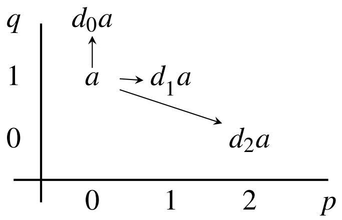
- A collection E_r^{pq} of objects of $\mathscr{A}$, where $r \geq a$ and $p, q \in \mathbb{Z}$,
- a collection of morphisms $d_r = d_r^{pq} : E_r^{pq} \longrightarrow E_r^{p+r,q-r+1}$ satisfying $(d_r)^2 = 0$,
- isomorphisms $E_{r+1}^{pq} \cong \ker d_r^{pq} / \operatorname{Im} d_r^{p-r,q+r-1}$.

The spectral sequence is usually denoted E_r^{**}.

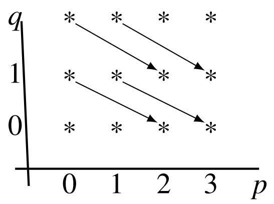
Here is the first page of a spectral sequence:

$$
\begin{array}{c|cccc}
q & E_0^{02} & E_0^{12} & E_0^{22} & E_0^{23} \\
 & \uparrow & \uparrow & \uparrow & \uparrow \\
1 & E_0^{01} & E_0^{11} & E_0^{12} & E_0^{13} \\
 & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & E_0^{00} & E_0^{10} & E_0^{20} & E_0^{30} \\
\hline
 & 0 & 1 & 2 & p
\end{array}
$$

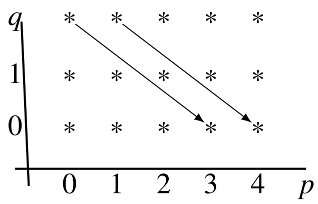
Here is a picture of the first few differentials, slightly abusing notation:



This is a schematic of E_2 with differentials:



And here is a schematic of E_3 with differentials:



Dually, there are homology spectral sequences.

We typically want to investigate what happens as r becomes large. Let us collect some terminology.

Definition. A spectral sequence is *regular* if for every (p, q) there exists an r such that $d_s^{pq} = 0$ for all $s \geq r$. We say a spectral sequences *degenerates* at E_r if all $d_{s \geq r}$ are 0.

Definition. If E_r^{pq} becomes eventually stable, i.e. there exists r_0 such that all E_r^{pq} agree for $r \geq r_0$ then we define $E_\infty^{pq} := E_{r_0}^{pq}$.

If the spectral sequence does not become stable we can still consider the kernels and images of the differentials d_r and use them to construct $B_r \subset B_{r+1} \subset Z_{r+1} \subset Z_r \subset E_a$. So let $B_\infty = \cup B_r$ and $Z_\infty = \cap Z_r$ and define $E_\infty^{pq} = Z_\infty^{pq}/B_\infty^{pq}$.

A spectral sequence is a tool for computing something complicated, e.g. the cohomology of a total complex, in terms of simpler objects, e.g. the vertical and horizontal cohomology.

8.2. First examples

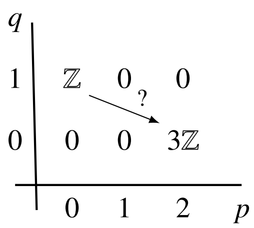
Example 23. Consider the following double complex.

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{3} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/3 \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

Here all unnamed maps are the identity respectively the natural projection.

I claim this gives rise to a spectral sequence. The underlying doubly graded object is E_0 of a spectral sequence. The vertical differential is d^0. Then we can define d^1 to be induced by the horizontal differential, and it must be trivial.

Here is the E_2 term of the sequence.

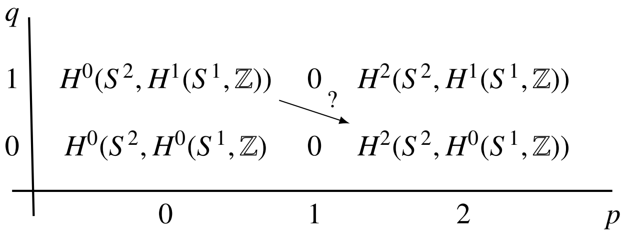


What should the differential do? Let x denote a generator of E_0^{01}. We would like to approximate $dx = 0 + d_h x$. But $d_h x = d_v 3y$, where y generates E_0^{10}. And $d_h y$ is the generator z of E_0^{20}. Hence d_2 should send $[x] \in E_2^{01}$ to $[3z] \in E_2^{20}$. Hence the spectral sequence degenerates to 0. It is easy to see that the double complex is exact.

The next examples are from topology. (Don't worry about the details unless you want to.)

Example 24. Assume given a (Serre) fibration $F \longrightarrow E \longrightarrow B$ of topological spaces with fiber F and with base simply connected base B. It is a theorem that there is a cohomological spectral sequence with $E_2 = H^p(B, H^q(F))$, which degenerates and "converges" to $H^n(E)$. We'll define convergence shortly, in practice we often have $\oplus_p E_\infty^{p,n-p} = H^n(E)$. This is called the *Leray-Serre spectral sequence*. Note that over a field $H^p(B, H^q(F))$ is just $H^p(B) \otimes H^q(F)$.

The projection $S^2 \times S^1 \longrightarrow S^2$ is a fibration with fiber S^1, as is the Hopf fibration $S^1 \longrightarrow S^3 \longrightarrow S^2$. The two spectral sequences have the some objects in E_2:



But the differential $d^2 : E_2^{01} \cong E_2^{20}$ is 0 for the product and is an isomorphism for the Hopf fibration.

8.3. Filtrations and convergence

Most spectral sequences arise from filtrations, and we restrict attention to those.

Definition. A *(decreasing) filtration* on an object C of $\mathscr{A}$ is a collection of subobjects F^nC satisfying $F^nC \subset F^{n-1}C$. The filtration is *exhaustive* if $C = \cup F^n$ and *Hausdorff* if $\cap F^n = 0$.

Remark. If we are not working in R-Mod we can still define union and intersection as direct limit and inverse limit, respectively, of the diagram F^iC. All we need is that $\mathscr{A}$ is complete and cocomplete.

There are also increasing filtrations, typically denoted F_nC, which can be defined completely analogously (and are typically encountered in homological examples).

For a filtration of complexes we define the following:

Example 25. Let C be a complex. There are two important filtrations we can always define. The *canonical filtration* on C is defined by

$$(\tau^{\geq n}C)^m = \begin{cases} C^m & \text{if } m > n \\ Z^m & \text{if } m = n \\ 0 & \text{if } m < n \end{cases}$$

The *stupid filtration* is defined by.

$$(F^nC)^m = \begin{cases} C^m & \text{if } m \geq n \\ 0 & \text{if } m < n \end{cases}$$

Definition. The *associated graded object* or just the *associated* of (C, F) is defined as the graded object which is $Gr_F^p C = F^pC/F^{p+1}C$ in degree p.

The associated graded of the stupid filtration is $Gr_F^p(C) = C^p$. The associated graded of the canonical filtration is $Gr_\tau^p(C) \simeq H^p(C)$.

We still need to define what it means for a spectral sequence to converge, which turns out to be a little subtle. We begin by thinking about the case of a filtered complex C. We want to compute cohomology of the filtered complex starting with the associated graded complex.

Example. A double complex is filtered by columns, say. The cohomology of the associated graded is vertical cohomology.

Getting the full homology seems too much to ask, but the filtration on C associates a filtration on cohomology, and we can often find the associated graded of homology of C.

Definition. A spectral sequence E_r *weakly converges* to H^* if there is a filtration on every H^n such that $E_\infty^{pq} \cong \mathrm{Gr}^p H^{p+q}$. We often write this as $E_a^{pq} \Rightarrow H^*$

This allows for lots of really silly examples unless the filtration on the H_n is exhaustive and Hausdorff. Even then we cannot tell the different between $H^0 = \mathbb{C}[t]$ and $H^0 = \mathbb{C}[[t]]$, and that is not good enough. So we sharpen the definition one more time:

Definition. The *completion* of a filtered complex is $\hat{C} = \varprojlim C/F^nC$ and a filtration is complete if $C \cong \hat{C}$.

Recall that $\varprojlim C/F^nC$ is the limit of the diagram

$$\cdots \longrightarrow C/F^{n+1}C \longrightarrow C/F^nC \longrightarrow C/F^{n-1}C \longrightarrow \cdots$$

For example $\mathbb{C}[[t]] = \varprojlim \mathbb{C}[t]/(t^n)$.

Definition. E_r *converges* to H^* if the spectral sequence is regular and weakly converges to H^*, and if the filtration on every H^n is exhaustive and complete (and thus Hausdorff).

Example 26. If a spectral sequence degenerates and is bounded then it converges to $H^* = \oplus_p E_\infty^{p,*-p}$. Just use the filtration coming from the first grading, $F^r = \oplus_{p\geq r} E_\infty^{p,*-p}$.

Remark. If the case the filtration on H^* is exhaustive and Hausdorff, but not necessarily complete one sometimes says E_r *abuts* to H^* or *approaches* H^*.

To make the subject even more confusing the nomenclature is not quite standardised. We follow [W]. [GM] say "converges" where we say approaches, and McCleary: A User's Guide to Spectral Sequences uses "convergent" for our weakly convergent.

9. The spectral sequence of a filtration

Convergence theorem for the spectral sequence of a filtration. Double complexes. Balancing.

9.1. The convergence theorem

Let us now define boundedness for filtrations and for spectral sequences, the two are closely related.

Definition. A filtration on C^* is *bounded* if for every n there are integers $a \geq b$ such that $F^a C^n = 0$ and $F^b C^n = C^n$. F is *bounded below* if for every n there is an a such that $F^a C^n = 0$. F is *bounded above* if for every n there is a b such that $F^b C^n = C^n$.

Definition. A spectral sequence is *bounded* if, for some r, for each n there are only finitely many terms of total degree n in E_r. It is *bounded below* if for each n the terms of total degree n vanish for large p. A spectral sequence that vanishes in the second quadrant is bounded below, but not bounded. It is *bounded above* if terms of total degree n vanish for small p.

Note a bounded below spectral sequence is regular. A bounded above spectral sequence need not be.

We are now ready to state the main theorem of this chapter.

Theorem 28. *For every filtered cochain complex (C, F) there is a spectral sequence with $E_1^{pq} = H^{p+q}(Gr_F^p(C^{p+q}))$.*

1. *If F is bounded then the spectral sequence is bounded and converges to H^*C.*
2. *If F is bounded below and exhaustive than the spectral sequence is bounded below and converges to H^*C.*
3. *If F is complete and exhaustive and the spectral sequence is regular then the spectral sequence converges weakly to H^*C. If E_r is moreover bounded above then it converges to H^*C*

Not a proof. I'm not going to prove this, but I will define the spectral sequence. First we define the first page: $E_0^{pq} = F^p C^{p+q} / F^{p+1} C^{p+q}$. We see that this shows the spectral sequence is bounded/bounded above/bounded below if the filtration is bounded/bounded above/bounded below.

The crucial definition is the next one:

$$A_r^{pq} := \{x \in F^p C^{p+q} \mid dx \in F^{p+r} C^{p+q+1}\}$$

These are approximate cycles, and as r becomes larger and larger they approximate the actual cycles. Now Z_r^{pq} is the image of A_r^{pq} in $E_0^{pq} = F^p C^{p+q}/F^{p+1}$, and $B_{r+1}^{p+r,q-r+1}$ is the image of the differential dA_r^{pq} in $F^{p+r}C^{p+q+1}/F^{p+r-1}$. Now we can define $E_r^{pq} = Z_r/B_r$ and there is a differential induced by the differential d of C. To be precise, pick c representing an element in E_r^{pq}. So $dc \in F^{p+r}C^{p+q+1}$ and we need to check it represents an element of $Z^{p+r,q+r-1}$ and that d factors through B^{pq}.

Similarly we define E_∞.

The next step is to show that E_{r+1} is the homology of E_r. Then we need to worry about convergence, i.e. we must identify E_∞ with $\operatorname{Gr} H^*$. For the bounded below case this is arguably just book-keeping (a lot of it, though). The complete convergence case needs some ideas, and the derived functor of $\varprojlim$ makes an appearance, details can be found in Section 5.5 of [W]. □

Remark. It can be seen from the definitions that given a filtered complex (C, F) the spectral sequences associated to C, to $\hat{C}$ and to $C/\cap_i F_i C$ and to $\cup F_i C$ all agree.

We can also define maps of spectral sequences. But since we care about the behaviour of spectral sequences only as r becomes large, we can be a bit lax:

Definition. A map between spectral sequences E and E' is a collection of maps $f_r^{pq} : E_r^{pq} \longrightarrow E_r'^{pq}$, for $r \geq b$ for some fixed b, that commute with the d_r.

Theorem 29. *A map $f : C \longrightarrow D$ of filtered complexes induces a map f_r of spectral sequences compatible with the induced map on homology. If the filtrations are complete and exhaustive and there is some r such that f_r^{pq} is an isomorphism for all p and q then $H^*(f) : H^*(C) \longrightarrow H^*(D)$ is an isomorphism.*

Idea of proof. Note that the conditions don't force convergence of the spectral sequences, but we still get the quasi-isomorphism. The trick is to consider a filtration on $\operatorname{cone}(f)$, the associated spectral sequence degenerates to 0, so is certainly bounded above. See [W] 5.5.11. □

Corollary 30. *Consider a map of filtered complexes inducing quasi-isomorphisms on the associated graded complexes. If the filtrations are complete and exhaustive then the map is a quasi-isomorphism.*

9.2. Double complexes and balancing

Let us now go back to double complexes. There are two obvious filtrations and often it is useful to play them against each other.

Let C^{**} be a double complex. We define two filtrations, ${}^{I}F$ by columns and ${}^{II}F$ by rows. That is, $({}^{I}F^{n}C)^{pq} = C^{pq}$ if $p \geq n$ and 0 otherwise. We let ${}^{I}E_r$ be the spectral sequence associated to the filtration ${}^{I}F$ and ${}^{II}E_r$ the spectral sequence associated to ${}^{II}F$.

Proposition 31. *Let C^{**} be a first quadrant double complex. Then ${}^{I}E_r$ and ${}^{II}E_r$ are bounded and convergent and we have:*

$${}^{I}E_2^{pq} = H_h^p H_v^q(C) \Rightarrow H^{p+q}(\mathrm{Tot}^{\oplus}(C))$$

and

$${}^{II}E_2^{pq} = H_v^p H_h^q(C) \Rightarrow H^{p+q}(\mathrm{Tot}^{\oplus}(C))$$

Here $H_v(C)$ denotes cohomology of C^{**} with respect to d_v. Then $H_v^q(C)$ is indeed a complex with a differential induced by d_h, and the p-th cohomology group of this complex is ${}^{I}E_2$.

Proof. This follows from Theorem 28. We just have to check that d_2 is indeed induced by d_h resp. d_v, which is straightforward. Note the change of index. □

The spectral sequence ${}^{I}E_r$ finishes the proof of Lemma 27, by arranging $K(\mathbf{t}') \otimes K(t_m)$ as a double complex with m rows and two columns. (Because the grading was homological you don't quite get a first quadrant spectral sequence, but that is easily overcome.) ⚭

If the complex is larger we have to worry about the difference between $\mathrm{Tot}^{\oplus} C$ and $\mathrm{Tot}^{\prod} C$.

I still owe you the balancing results for Ext and Tor. This is easy now:

Proposition 32. *Let B and A be R-modules. Then $L(-\otimes_R B)(A) \cong L(A \otimes_R -)(B)$ and $RHom_R(A,-)(B) \cong RHom_R(-,B)(A)$.*

Proof. We do the proof for Tor, the proof for Ext is similar. Take resolutions $P_* \longrightarrow A$ and $Q_* \longrightarrow B$ and consider the double complex $P \otimes Q$. The total tensor product chain complex is $\mathrm{Tot}^{\oplus}(P \otimes Q)$. The two (homological) spectral sequences of the double complex converge and the E^2 terms are $H_*(P \otimes_R B)$ and $H_*(A \otimes_R Q)$ respectively. Hence the spectral sequences actually *collapse*, they are concentrated in a single row, resp. column! This means that there are no extension problems and we have ${}^{I}E^2_{p0} \cong H_p(\mathrm{Tot}) \cong {}^{II}E^2_{0p}$, proving the theorem. □

Example. Next consider an application to topology, ignore this example unless you know some differential geometry. Consider a smooth manifold M and recall that the *de Rham cohomology* of M is the cohomology of the de Rham complex, the complex of smooth differential forms $\mathscr{A}^*(M)$, defined in any course on differential geometry. not lectured

Now assume M has a cover $\mathfrak{U}$ by contractible open sets with contractible intersections. To M is associated a category $Op(M)$ whose objects are open subsets of M and whose morphisms are inclusions. Then the *constant preseheaf* $\underline{\mathbb{R}}$ is defined as the constant functor $Op(M)^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ with value $\mathbb{R}$.

The *Čech cohomology* is the cohomology of the complex $C^*(\mathfrak{U}, \mathbb{R})$ which is defined via

$$C^q(\mathfrak{U}, \mathbb{R}) = \bigoplus_{i_1 < \cdots < i_q} \underline{\mathbb{R}}(U_{i_1} \cap \cdots \cap U_{i_q})$$

The differential is induced by the alternating sum of inclusions.

Now note that we can define the de Rham complex for every open subset $U \subset M$. Thus we can define double complex with $\oplus \mathscr{A}^p(U_{i_1} \cap \cdots \cap U_{i_q})$ in the (p,q) position and with vertical differential given by the de Rham differential and horizontal differential given by the Čech differential.

By the Poincaré lemma ${}^{I}E_r$ degenerates on the second page, E_1 is the Čech complex, and E_2 is Čech cohomology.

Next, since $\mathscr{A}^p$ is a fine sheaf on M, we have that ${}^{II}E_r$ also collapses on the second page, E_1 is the de Rham complex of M and E_2 is de Rham cohomology.

As the spectral sequences converge to the same complex the two cohomologies agree. This example is nicely explained in Bott and Tu: Differential Forms in Algebraic Topology.

10. The Grothendieck spectral sequence

Grothendieck spectral sequence. Cartan-Eilenberg resolutions. Hochschild-Serre spectral sequence.

10.1. The theorem

Next we prove a result that gives us several important spectral sequences. It answers that natural question how to compute the composition of derived functors.

Theorem 33 (Grothendieck spectral sequence)**.** *Let $\mathscr{A} \xrightarrow{G} \mathscr{B} \xrightarrow{F} \mathscr{C}$ be left exact functors between abelian categories. Suppose that $\mathscr{A}$ and $\mathscr{B}$ have enough injectives and that G sends injective objects to injective objects. Then for every object $A \in \mathscr{A}$ there is a convergent first quadrant spectral sequence $E_2^{pq} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A)$.*

Remark. We will see in the proof that it is enough to assume G sends injectives to F-acyclic objects, as defined in Section 3.3.

To simplify life we are still just thinking about spectral sequences in R-Mod, so we may assume that $\mathscr{C} = R$-Mod for simplicity.

While the statement of the theorem does not involve them, it is clear that the right context for this theorem are derived functors of complexes. Recall that in the Section 4.2 we extended our definitions of derived functors to complexes. The idea is that if $\mathscr{A}$ has enough injectives then every complex A^* in $\mathbf{Ch}^{\geq 0}$ is quasi-isomorphic to a level-wise injective complex , say $A^* \simeq I^*$. We'll finally prove that now. Then one can consider $RG(A) := G(I)$. The cohomology groups are sometimes called *hyperderived functors* and written $\mathbb{R}^qG(A) = H^qGI$.

To deduce the Grothendieck spectral sequence we will construct a resolution that is pieced together from resolutions of the A^p. Here goes:

Definition. A *(right) Cartan-Eilenberg resolution* for a cochain complex A^* in $\mathscr{A}$ is a double complex I^{**} of injective objects in $\mathscr{A}$ together with a map $e^* : A^* \longrightarrow I^{*0}$ such that the induced maps $B^p(e)$ and $H^p(e)$ on boundary and cohomology are injective resolutions.

We will think of A as aligned vertically in what follows, as in the following diagram.

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ 0 & \to & I^{10} & \to & I^{11} & \to & I^{12} & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & I^{00} & \to & I^{01} & \to & I^{02} & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & A^0 & \to & A^1 & \to & A^2 & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Lemma 34. *In the above setting e^p and $Z^p(e^*)$ are also injective resolutions. If A is bounded below then there is a quasi-isomorphism $A^* \longrightarrow \mathrm{Tot}^{\prod}(I^{**})$.*

Proof. The snake lemma applied to $B^p(I) \longrightarrow Z^p(I) \longrightarrow H^p(I)$ augmented by $B^p(A) \longrightarrow Z^p(A) \longrightarrow H^p(A)$ shows that $Z^p(I)$ is a resolution of $Z^p(A)$. Similarly considering the obvious augmentation of $Z^p(I) \longrightarrow I \longrightarrow B^{p+1}(I)$ shows that I^{p*} resolves A^p. Next we will show that $e : A \longrightarrow \mathrm{Tot}^{\prod}(I^{**})$ is a quasi-isomorphism. We consider the double complex C given as the augmented double complex with -1st column equal to A. Filtering the double complex by columns gives a bounded filtration. Hence by Theorem 28 the spectral sequence converges. Moreover $E_\infty = 0$ as the rows of E_1 are exact. So $H(\mathrm{Tot}^{\prod C}) = 0$, but on the other hand cone(e) is isomorphic to C, by unwrapping definitions. So e is a quasi-isomorphism by Lemma 3. □

Lemma 35. *If $\mathscr{A}$ has enough injectives then every bounded below complex A^* has a Cartan-Eilenberg resolution.*

Proof. Let B^{p*} and H^{p*} denote injective resolutions of $B^p(A)$ and $H^p(A)$. Consider the short exact sequence $B^p(A) \longrightarrow Z^p(A) \longrightarrow H^p(A)$. By Lemma 9 (horseshoe) there is a compatible injective resolution Z^{p*} of $Z^p(A)$. Similarly by the short exact sequence $Z^p(A) \longrightarrow A^p \longrightarrow B^{p+1}(A)$ there is a compatible injective resolution I^{p*}. The I^{pq} will give us the desired resolution once we define the correct differential.

To define the horizontal differential we multiply the differential of I^{p*} by $(-1)^p$. It remains to define the vertical differential. From Lemma 9 we get natural maps lifting inclusion and projection and we piece them together to give $I^{p-1,*} \longrightarrow B^{p,*} \longrightarrow Z^{p,*} \longrightarrow I^{p*}$ and this composition is d_v. As we can factor $d : A^{p-1} \longrightarrow A^p$ similarly we see that $e : A^* \longrightarrow I^{*0}$ is a chain map. Since the second and third map are injections we also have that $B^p(I, d_v) = B^p$. Then the short exact sequences $Z \longrightarrow I \longrightarrow B$ and $B \longrightarrow Z \longrightarrow H$ imply that $Z^p(I, d_v) = Z^p$ and that $H^p(I, d_v) = H^p$. It follows that $B^p(e)$ and $H^p(e)$ are injective resolutions. □

Remark. By some twist of fate the boundedness assumption is not needed for left, i.e. projective, Cartan Eilenberg resolutions. The associated spectral sequence is bounded above and converges.

Given a bounded below complex A^* we can now explicitly define its hyperderived functors $\mathbb{R}^p GA$ as $H^p \operatorname{Tot}^{\prod} GI$ for any Cartan-Eilenberg resolution $A \longrightarrow I$.

It follows from Theorem 13 that this is well-defined.

Proof of Theorem 33. Choose an injective resolution $A \longrightarrow I$ and Cartan-Eilenberg resolution $G(I) \longrightarrow J$. The first double quadrant double complex $F(J)$ has two associated (convergent) spectral sequences.

The filtration by rows yields ${}^{II}E_2^{pq} = H_v^p H_h^q FJ = H^p R^q F(GI)$. As the GI are injective (or just F-acyclic) we get collapse of the spectral sequence in the row $q = 0$, to $H^p R^0 FG(I) = H^p(FG)(I)$. As the spectral sequence converges to $H^{p+q} \operatorname{Tot}^{\prod} FJ = (\mathbb{R}^{p+q}F)(GI)$, we have $\mathbb{R}^p F(GI) \cong H^p(FG)(I) = R^p(FG)(A)$.

The filtration by columns yields ${}^{I}E_2^{pq} = H_h^p H_v^q(FJ) = H_h^p F H_v^q(J)$. Since all the vertical cycles, boundaries and cohomology groups of J are injective, J is vertically split and the exact functor F commutes with vertical cohomology. Now $H_h^p F H_v^q J = R^p F(H^q GI)$ as $H_v^* J$ is a resolution for $H^* GI$. Finally $R^p F(H^q GI) = R^p F R^q G(A)$ and since ${}^{I}E_\infty$ must agree with ${}^{II}E_\infty = R^{p+q}(FG)(A)$ we have proved Grothendieck's spectral sequence. □

10.2. Applications

Corollary 36 (Hochschild-Serre spectral sequence)**.** *Let G be a group with normal subgroup N and let A be a G-module. Then there is a convergent first quadrant spectral sequence $E_2^{pq} = H^p(G/N, H^q(N, A)) \Rightarrow H^{p+q}(G, A)$.*

Note that $-^N$ sends G-modules to G/N-modules, hence so do the derived functors: Just consider a resolution by G-modules. Even if A is trivial as G-module, its N-cohomology need not be trivial over G/H. In practise the precise action can be tricky to compute. (In particular since the whole point of the spectral sequence was to not resolve things as G-modules.)

Proof. We apply the Grothendieck spectral sequence to the composition

$$G\text{-Mod} \xrightarrow{-^N} G/N\text{-Mod} \xrightarrow{-^{G/N}} \mathbf{Ab}$$

Unwrapping the definitions the composition is $-^G$. It remains to show that $-^N$ sends injectives to injectives. We note that it is right adjoint to the forgetful functor, see Proposition 19, which is exact. (As it has both adjoints, or simply since exactness in R-modules is determined on the underlying sets.) Hence we are done by the next lemma. □

Lemma 37. *Left adjoints of exact functors preserve projectives and right adjoints of exact functors preserve injectives.*

Proof. Let's do injectives. Consider an adjunction $F \dashv U$ with F left exact (right exactness is automatic and irrelevant) and let I be an injective object and $A \longrightarrow B$ any injection. Then $\mathrm{Hom}(FB, I) \longrightarrow \mathrm{Hom}(FA, I)$ is onto, and by naturality of the adjunction isomorphisms $\mathrm{Hom}(B, UI) \longrightarrow \mathrm{Hom}(A, UI)$ is surjective, showing UI is injective. □

The next result is about sheaves on topological spaces, but note that the spectral sequence we write down is a spectral sequence in **Ab**. not lectured

Corollary (Leray spectral sequence)**.** *Let $f : X \longrightarrow Y$ be a continuous map of topological spaces and let $\mathscr{F}$ be a sheaf on X. There is a spectral sequence $E_2^{pq} = H^p(Y, R^q f_* \mathscr{F}) \Rightarrow H^{p+q}(X, \mathscr{F})$.*

Proof. The pushforward f_* has a left adjoint f^{-1}, which is exact, hence it is left-exact and preserves injectives. Note that $\Gamma_X = \Gamma_Y \circ f_*$. □

IV. Homotopical Algebra

11. Interlude on simplicial sets

Definitions. Examples. Dold-Kan. Realisation.

11.1. Definition and examples

Simplicial sets are a bit technical, but they are extremely useful tools providing explicit combinatorial models, both in classical algebraic topology and for higher or derived algebra and geometry.

Definition. Let us first define the *simplex category* $\mathbf{\Delta}$ which has objects $[n] = (0 < 1 < \cdots < n)$, and which has morphisms given by nondecreasing functions. Then we define a *simplicial object* in a category $\mathscr{C}$ as a functor $\mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathscr{C}$ from the opposite category of $\mathbf{\Delta}$ to $\mathscr{C}$. There is a natural category $\mathbf{s}\mathscr{C}$ of simplicial objects in $\mathscr{C}$, with morphisms given by natural transformations.

We can dually define a *cosimplicial object* in $\mathscr{C}$ as a functor from $\mathbf{\Delta}$ to $\mathscr{C}$.

Example 27. The most important case is when $\mathscr{C}$ is just the category of sets. We denote the category of simplicial sets by **sSet**. Also important is **sAb** the category of simplicial abelian groups.

Example 28. Given a topological space X you may have met the singular simplicial set $\mathrm{Sing}(X)$.

We write a simplicial set as A_* where A_n is $A([n])$. Next we need to understand the data coming from morphisms. We notice two families of morphisms in $\mathbf{\Delta}$, corresponding to leaving out respectively repeating the i-ith term.

Definition. Define the i-th *face map* $\epsilon_i : [n-1] \longrightarrow [n]$ to be the injection only leaving out $i \in [n]$, and define the i-th *degeneracy map* $\eta_i : [n+1] \longrightarrow [n]$ to be the surjection mapping two elements to i.

A straightforward, if tedious, check shows the following identities:

$$\begin{aligned}
\epsilon_j\epsilon_i &= \epsilon_i\epsilon_{j-1} \quad \text{if } i < j \\
\eta_j\eta_i &= \eta_i\eta_{j+1} \quad \text{if } i \leq j \\
\eta_j\epsilon_i &= \begin{cases} \epsilon_i\eta_{j-1} & \text{if } i < j \\ \mathbf{1} & \text{if } i = j, j+1 \\ \epsilon_{i-1}\eta_j & \text{if } i > j+1 \end{cases}
\end{aligned}$$

Proposition 38. *Any map $\alpha : [n] \longrightarrow [m]$ can be factored uniquely as a composition of degeneracy maps followed by a composition of face maps, i.e. $\alpha = \epsilon_{i_1} \ldots \epsilon_{i_s}\eta_{j_1} \ldots \eta_{j_t}$.*

Not a proof. E.g. [W] 8.1.2. □

Hence it suffices to understand $\partial_i := A(\epsilon_i) : A_n \longrightarrow A_{n-1}$ and $\sigma_i := A(\eta_i) : A_n \longrightarrow A_{n+1}$ to complete our understanding of A. These maps satisfy the following equations.

$$\begin{aligned}
\partial_i\partial_j &= \partial_{j-1}\partial_i \quad \text{if } i < j \\
\sigma_i\sigma_j &= \sigma_{j+1}\sigma_i \quad \text{if } i \leq j \\
\partial_i\sigma_j &= \begin{cases} \sigma_{j-1}\partial_i & \text{if } i < j \\ \mathbf{1} & \text{if } i = j, j+1 \\ \sigma_j\partial_{i-1} & \text{if } i > j+1 \end{cases}
\end{aligned}$$

Example 29. We define the *standard n-simplex* $\Delta[n]$ as the image in **sSet** of $[n]$ under the contravariant Yoneda embedding: $\Delta[n] = \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$. In particular $\Delta[n]_i = \mathrm{Hom}_{\mathbf{\Delta}}([i], [n])$ and $\partial_i = \mathrm{Hom}_{\mathbf{\Delta}}(\epsilon_i, [n])$. This simplicial set is universal in the sense that $A_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n], A)$, by the Yoneda lemma. We call A_n the *n-simplices* of A.

Example 30. Let us consider $\Delta[1]$ in a little more detail. Any map $[n] \longrightarrow [1]$ in $\mathbf{\Delta}$ sends the first k terms to 0 and the remaining $n - k$ terms to 1 and we write it as a sequence of k zeros and $n - k$ ones. So

$$\begin{aligned}
\Delta[1]_0 &= \{0, 1\} \\
\Delta[1]_1 &= \{00, 01, 11\} \\
\Delta[1]_2 &= \{000, 001, 011, 111\} \\
\Delta[1]_3 &= \{0000, 0001, 0011, 0111, 1111\}
\end{aligned}$$

All elements except 0, 1 and 01 have repeats and hence occur in the image of some σ_j. These are called *degenerate simplices*. We think of $\Delta[1]$ as consisting of 2 vertices, 1 edge, and a lot of degenerate simplices.

Similarly we find 7 nondegenerate simplices in $\Delta[2]$: 3 vertices, 3 edges and 1 face.

11.2. Chain complexes

Many useful chain complexes can be seen as arising from simplicial abelian groups. Let us make that precise.

Definition. Let A_* be a simplicial object in $\mathscr{A}$. We define the *associated chain complex* CA_* as follows: $CA_n := A_n$ and $d_n = \sum(-1)^i\partial_i$. The simplicial identities imply that $d^2 = 0$.

Example 31. Given a topological space X we can view singular chains as associated to the simplicial abelian group $\mathbb{Z}\mathrm{Sing}_*(X)$, which is just the free group on $\mathrm{Sing}_n(X)$ in every degree, with face and degeneracy maps extended in the obvious way.

Remark. The Čech complex and the Koszul complex come from *semi-simplicial sets*, which are just simplicial sets without the degeneracy maps. They are sometimes easier to understand.

Definition. The *normalised chain complex* NA of A_n is defined as follows: $NA_n = \cap_{i=0}^{n-1}\ker(\partial_i) \subset A_n$ and the differential is $d_n = (-1)^n\partial_n$.

Theorem 39 (Dold-Kan)**.** *If $\mathscr{A}$ is an abelian category then $N : \mathbf{s}\mathscr{A} \longrightarrow \mathbf{Ch}_{\geq 0}(\mathscr{A})$ is part of an equivalence of categories.*

About the proof. The idea of the proof is to explicitly write down a functor $\Gamma : \mathbf{Ch}_{\geq 0}(\mathscr{A}) \longrightarrow \mathbf{s}\mathscr{A}$ and check that $\Gamma \circ N$ and $N \circ \Gamma$ are naturally equivalent to the identity functors. To get started, let $\Gamma(A)_n = \oplus_{n\to k}A_k$ where the direct sum is over all surjections. □

11.3. Topological spaces and more examples

Next we consider a cosimplicial topological space, i.e. a functor $\mathbf{\Delta} \longrightarrow \mathbf{Top}$.

Example 32. Consider the *geometric n-simplices* $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum x_i = 1)$. Picking standard bases for all $\mathbb{R}^n$ we can number the elements of all Δ^n_0. Then any map $\alpha : [m] \longrightarrow [n] \in \mathbf{\Delta}$ induces a map $\alpha_* : \Delta^m \longrightarrow \Delta^n$ by using α on the set of vertices and extending linearly. Hence the Δ^n form a cosimplicial topological space in a natural way.

We can now view $\mathrm{Sing}(X)$ as the simplicial set we obtain by applying the functor $\mathrm{Hom}(\Delta^*, -)$ to X. (Hom-sets out of a cosimplicial set naturally form a simplicial set!)

Definition. There is a functor from simplicial sets to topological spaces called *realisation* defined as follows. We will again use Δ^*, the cosimplicial topological space of geometric n-simplices. As A_n is just a set we can write $A_n \times \Delta^m$ for the topological space obtained by taking a disjoint union of Δ^m's, indexed by A_n.

Then define $|A| = \amalg_n A_n \times \Delta^n / \sim$ where the equivalence relation identifies $(\alpha^*(x), y) \in A_m \times \Delta^m$ and $(x, \alpha_*(y)) \in A_k \times \Delta^k$ for any $\alpha : [m] \longrightarrow [k]$. Here α_* is the map defined in the example above, and α_* is the map $A(\alpha) : A_k \longrightarrow A_m$ that is part of the structure of A as a simplicial set. We can concisely write this definition as follows:

$$|A| = coeq\left(\amalg_{\alpha: m \to k} \Delta^m \times A_k \rightrightarrows \amalg_n \Delta^n \times A_n\right)$$

Example 33. The realisation of $\Delta[n]$ is Δ^n.

Remark. Simplicial sets are a combinatorial model for topological spaces. In fact, realisation and the singular simplicial set functor give an adjunction $|\cdot| \dashv \mathrm{Sing}$ and induce a kind of equivalence between topological spaces and simplicial sets. We'll define the precise kind of equivalence later.

Here is one of the most important classes of simplicial sets.

Example 34. Let G be a group. We define a simplicial set BG by $BG_n = G^{\times n}$ with faces and degeneracies given as follows:

$$\sigma_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n)$$

and

$$\partial_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } i = 1, \ldots, n-1 \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n \end{cases}$$

The realisation of this simplicial set is called the *classifying space* of G and is a $K(G, 1)$. Also note that the chain complex associated to the free $\mathbb{Z}$-module on BG_* is the complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B^u_*(G)$ used to compute homology of G in Section 6.2.

Example 35. One can generalise this construction. A group is nothing but a category with one object all of whose morphisms are isomorphisms. So let $\mathscr{C}$ be an arbitrary (small) category. We define $B\mathscr{C}_0$ to be the set of objects of $\mathscr{C}$ and $B\mathscr{C}_n$ to be the set of all composable n-tuples of morphisms in $\mathscr{C}$. Then there are face and degeneracy maps as above! This construction is called the *nerve* of a category, and after applying the realisation functor we obtain the *classifying space* of a category.

This is a rather crude functor from categories to simplicial sets. There are many more sophisticated connections and the use of simplicial sets to study categories has been extremely fruitful.

12. Introducing Model Cats

Definition. Chain complexes as a model category. Small object argument.

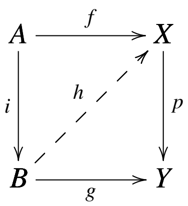
12.1. Definition

In this course we have been doing homotopy theory, just with chain complexes rather than spaces. We have been working with abelian and derived categories. Many of the key ideas work in more general, less abelian setting: Model categories.

The definitions are involved, so it is useful to keep in mind the basic example, which is our basic example throughout the course: Bounded below chain complexes over a ring R. If you know some homotopy theory, you can think of model categories as formalising the common homotopy theory of chain complexes and topological spaces.

We will need the following definitions:

Definition. A map i has the *left lifting property* (LLP) with respect to a map p if given any f, g with $pf = ig$ there is a lift h with $hi = f$ and $ph = g$. In the same situation we say p has the *right lifting property* (RLP) with respect to i



(IV.1)

For example, in an abelian category, the map from 0 to a projective module has the LLP with respect to all surjections.

Definition. A map $f : A \longrightarrow B$ is a *retract* of a map $g : A' \longrightarrow B'$ if there exist factorisations of the identity $A \longrightarrow A' \longrightarrow A$ and $B \longrightarrow B' \longrightarrow B$ making the obvious diagram commute.

Now we can define model categories:

Definition. A *model category* is a category $\mathscr{M}$ with special classes $\mathscr{W}$ (*weak equivalences*), $\mathscr{F}$ (*fibrations*) and $\mathscr{C}$ (*cofibrations*) of morphisms such that the axioms MC1 to MC5 hold. We call $\mathscr{F} \cap W$ the *acyclic fibrations* and $\mathscr{C} \cap W$ the *acyclic cofibrations*.

MC 1 Small limits and colimits exist in $\mathscr{M}$.

MC 2 If f and g are maps such that gf is defined and if two out of f, g, gf are in $\mathscr{W}$ then so is the third. (This is called the "two-out-of-three" property.)

MC 3 If f is a retract of g and g is in $\mathscr{F}$, $\mathscr{C}$ or $\mathscr{W}$ then so is f.

MC 4 (i) Any cofibration has the LLP with respect to all acyclic fibrations and (ii) any acyclic cofibration has the LLP with respect to all fibrations.

MC 5 Any map f can be functorially factored in two ways: (i) $f = pi$, where p is a acyclic fibration and i is a cofibration. (ii) $f = qj$ where q is a fibration and j is a acyclic cofibration.

Note that MC4(i) is equivalent to saying any acyclic fibration has RLP with respect to all cofibrations, similarly for MC4(ii).

Functorial factorisation means that there is a functor from the category $\mathscr{M}^I$of morphisms in $\mathscr{M}$, with morphisms given by commutative diagrams, to the category $\mathscr{M}^I \times \mathscr{M}^I$ satisfying the above condition.

Here is our main example:

Example 36. The category $\mathbf{Ch}_R$ of nonnegative chain complexes of R-modules is a model category if we define the following:

- $\mathscr{W}$ is the class of all quasi-isomorphisms,
- $\mathscr{F}$ consists of all chain maps f such that f_n is surjective whenever $n > 0$
- $\mathscr{C}$ consists of all chain maps f such that every f_n is injective with projective cokernel.

Note that fibrations need not be surjective in degree 0.

There is some redundancy in our definition:

Lemma 40. *Let $\mathscr{M}$ be a model category. Then the cofibrations are precisely the maps which have the LLP with respect to acyclic fibrations.*

The three analogous versions are also true.

Proof. Let $f : K \longrightarrow L$ have LLP with respect to all acyclic fibrations. By MC5(i) we can write $f = pi : K \longrightarrow L' \longrightarrow L$ where i is a cofibration and p an acyclic fibration. By assumption we can lift $\mathbf{1} : L \longrightarrow L$ to a map $h : L \longrightarrow L'$ with $ph = \mathbf{1}$. But then f is a retract of i and hence a cofibration by MC3. □

Next we will do our main example in some detail.

Remark. All three classes of maps are in fact closed under composition and contain the identity maps. For $\mathscr{F}$ and $\mathscr{C}$ this follows from Lemma 40. It follows from MC2 that $\mathscr{W}$ is closed under compositions. To show $\mathbf{1}_A$ is a weak equivalence use MC5 to write $\mathbf{1}_A = pi$ where $p \in \mathscr{W}$. Then $p = \mathbf{1}_A \cdot p$ and we can apply MC2. not lectured

12.2. Chain complexes as a model category

Theorem 41. *The category* $\mathbf{Ch}_R$ *with the model structure defined in Example 36 is indeed a model category.*

Remark. This is called the projective model structure.

For nonpositive chain complexes (or nonnegative cochain complexes) we use the dual injective model structure: cofibrations are monic in nonzero degrees and fibrations are level-wise epic with injective kernel.

One can also put an injective or projective model structure on unbounded chain complexes, but the definition, due to Spaltenstein, is more subtle. See [Hov].

Partial proof. MC1: Construct limits and colimits levelwise.

MC2: Clear.

MC3: Retracts of isomorphisms, monics and epics are isomorphisms, monics and epics, respectively, by the 5-lemma. Retracts of projectives are projectives as a retract is a direct summand.

MC4: The crucial ingredient is of course the lifting property of projectives, but we need to do some technical work, see [DS].

MC5: This is by far the hardest axiom to check. We will use the small object argument, which generalises far beyond chain complexes.

The idea is that there are sets (not classes!) of *generating cofibrations* $\mathscr{I}$ and *generating acyclic cofibrations* $\mathscr{J}$ and to see if p is a fibration, respectively an acyclic fibration, it is enough to check the RLP with respect to $\mathscr{J}$, respectively $\mathscr{I}$.

More precisely: Define $S(n)$ for $n \geq 0$ to be the chain complex which is R in degree n and 0 elsewhere. Let $S(-1) = 0$. Let $D(n)$ for $n \geq 1$ be the chain complex which is R in degrees $n-1$ and n and 0 elsewhere, with $d_n = \mathbf{1}_R$. Let $D(0) = R$ concentrated in degree 0. Then define $\mathscr{I} = \{i_n : S(n-1) \longrightarrow D(n)\}_{n\geq 0}$ and $\mathscr{J} = \{j_n : 0 \longrightarrow D(n)\}_{n\geq 1}$ to be the families of the obvious maps. The following is not hard to check.

Lemma 42. *A map p is a fibration if and only if it has RLP with respect to $\mathscr{J}$. It is an acyclic fibration if and only if it has RLP with respect to $\mathscr{I}$.*

Proof. 3rd example sheet. □

Now consider a map $a : X \longrightarrow Y$. We want to factor as $X \xrightarrow{\iota^\infty} Y^\infty \xrightarrow{\pi^\infty} Y$ where π^∞ has the RLP with respect to $\mathscr{I}$. We enforce this by attaching extra cells to X for every possible diagram like IV.1 with $i \in \mathscr{I}$!

The attachment will be in the form of a huge colimit of colimits, and it will only have the correct lifting property if the domains of $\mathscr{I}$ and $\mathscr{J}$ are *small objects*. Hence the method of proof is called the *small object argument*.

Let's define what we mean by "small". Consider a diagram $B : N \longrightarrow \mathscr{C}$ from the category of natural numbers with morphisms given by the relation "$\leq$". Then $\operatorname{colim}_N = \varinjlim$. Then by the universal property of colim there is a natural morphism $\operatorname{colim}_n \operatorname{Hom}_{\mathscr{C}}(A, B(n)) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(A, \operatorname{colim}_n B(n))$ which is not in general an isomorphism.

Definition. An object $A \in \mathscr{C}$ is called *sequentially small* if

$$\operatorname*{colim}_n \operatorname{Hom}_{\mathscr{C}}(A, B(n)) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(A, \operatorname*{colim}_n B(n))$$

is an isomorphism for all diagrams B.

Lemma 43. *The $S(n)$ are sequentially small.*

Proof. 3rd example sheet. Think of colim_n as a union to get some intuition. □

Now we fix $a : X \longrightarrow Y$ and consider, for every $i_n : S(n-1) \longrightarrow D(n) \in \mathscr{I}$ all of the following diagrams:

$$\begin{array}{ccc} S(n-1) & \xrightarrow{f} & X \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle a} \\ D(n) & \xrightarrow[g]{} & Y \end{array}$$

We write $M(n)$ for the set of all f, g making this diagram commute. We now take a very large coproduct. For every $\mathscr{I}$ we take a copy of $S(n-1)$ and a copy of $D(n)$ for every $(f, g) \in M(n)$. We take the coproduct over all $\mathscr{I}$ over all $M(n)$ and there is a natural map from $\amalg S(n-1)$ to $\amalg D(n)$. Moreover, the f induce a natural map, which we write as $\sum f$, from the domain to X. Then we form the pushout.

$$\begin{array}{ccc} \amalg_{i_n \in \mathscr{I}} \amalg_{M(n)} S(n-1) & \xrightarrow{\sum f} & X \\ {\scriptstyle \amalg_{\mathscr{I}} \amalg_{M(n)} i_n}\downarrow & & \downarrow{\scriptstyle \iota^1} \\ \amalg_{i_n \in \mathscr{I}} \amalg_{M(n)} D(n) & \xrightarrow[G]{} & Y^1 \end{array}$$

Note that Y^1 comes with a natural map π^1 to Y, and for every pair (f, g) in $M(n)$, the map $g = \sum g \circ inc_g : B(n) \longrightarrow Y$ factors as $\pi^1 \circ G \circ inc_g$, where inc_g is the inclusion of $D(n)$ in the (f, g)-th place of $\amalg_{\mathscr{I}} \amalg_{M(n)}$, and $\sum g$ is the canonical map $\amalg D(n) \longrightarrow Y$. So any map $g : D(n) \longrightarrow Y$ lifts through π^1.

$$\begin{array}{ccccc} S(n-1) & \xrightarrow{inc_f} & \amalg S(n-1) & \xrightarrow{\sum f} & X \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle \amalg i_n} & & \downarrow{\scriptstyle \iota^1} \\ D(n) & \xrightarrow[inc_g]{} & \amalg D(n) & \xrightarrow[G]{} & Y^1 \\ & & & \searrow^{\sum g} & \downarrow{\scriptstyle \pi^1} \\ & & & & Y \end{array}$$

Then we repeat the procedure with Y^1 in place of X, to obtain another pushout $\iota^2 : Y^1 \longrightarrow Y^2$. We repeat to obtain a family Y^n and take the direct limit over all of them, $Y^\infty := \mathrm{colim}_n\, Y^n$. By the universal property this comes with maps $\iota^\infty : X \longrightarrow Y^\infty$ and $\pi^\infty : Y^\infty \longrightarrow Y$ such that $a = \pi^\infty \circ \iota^\infty$

New we show that π^∞ has the RLP with respect to any $i_n : S(n-1) \longrightarrow D(n)$. Indeed, any map $f : S(n-1) \longrightarrow Y^\infty$ must factor through some $\mathscr{Y}^k$, as $S(n-1)$ is small. Then we have the following diagram:

$$\begin{array}{ccccc} S(n-1) & \xrightarrow{f'} & Y^k & \xrightarrow{\iota^{k+1}} & Y^{k+1} \\ \big\downarrow{\scriptstyle i_n} & & \big\downarrow{\scriptstyle \pi^k} & & \big\downarrow{\scriptstyle \pi^{k+1}} \\ D(n) & \xrightarrow[g]{} & Y & \xrightarrow[=]{} & Y \end{array}$$

As described above, by construction $g : D(n) \longrightarrow Y$ lifts to a map $h' = \sum g \circ inc_g : D(n) \longrightarrow Y^{n+1}$, which gives $h : D(n) \longrightarrow Y^\infty$ with $\pi^\infty \circ h = g$

It remains to show that $X \longrightarrow Y^\infty$ is a cofibration. But each Y^{k+1} is in every degree the direct sum of Y^k with copies of R. Similarly for the colimit Y^∞. Alternatively we can observe that cofibrations are closed under pushouts and direct limits, see the 3rd example sheet.

Note that our construction in terms of colimits was entirely functorial as colim is a functor.

Replacing $\mathscr{I}$ by $\mathscr{J}$ we can prove MC5(ii) in exactly the same manner. □

13. Homotopies and the homotopy category

Homotopy via cylinder objects. Sketch of the homotopy category.

Definition. We say an object is *fibrant* if the canonical map to the terminal object is a fibration, and *cofibrant* if the canonical map from the initial object is a cofibration.

By MC5 we can functorially replace any object by a fibrant object, call this functor R. Similarly we call cofibrant replacement Q. We have natural transformations $p_X : QX \longrightarrow X$ and $i_X : X \longrightarrow RX$. Also notice that RQX and QRX are both fibrant and cofibrant, e.g. $0 \longrightarrow QX \longrightarrow RQX$ is a composition of cofibrations.

Now we will address the question how to do homotopy theory in a model category. Unfortunately I only have time to sketch the theory and will not prove the results. See [DS] for all the details. (Note that [DS] do not use functorial factorisation, which makes some of their proofs a bit more involved than they need to be.)

We need a notion for two maps to be homotopic. Recall that $f, g : A \longrightarrow B$ between topological spaces are homotopic if there is a certain map $H : A \times [0, 1] \longrightarrow B$. To generalise this we need a generalisation of the construction of $A \times [0, 1]$ from A.

Definition. A *cylinder object* for an object A in a model category $\mathscr{M}$ is an object $A \wedge I$ such that the natural map $A \amalg A \longrightarrow A$ factors as $A \amalg A \xrightarrow{i} A \wedge I \xrightarrow{q} A$ such that q is a weak equivalence. We say $A \wedge I$ is a *very good cylinder object* if i is a cofibration and q is an acyclic fibration.

Convention. Given maps $f, g : A \longrightarrow B$ we write $f + g$ for the canonical map $A \amalg A \longrightarrow B$.

Example 37. A cylinder object for M in $\mathbf{Ch}_R$ is given by the *mapping cylinder* $\mathrm{cyl}(M)$, defined as follows. We define $\mathrm{cyl}(M)_n$ to be $M_n \oplus M_{n-1} \oplus M_n$ and let the differential be

$$\begin{pmatrix} d & \mathbf{1} & \\ & -d & \\ & -\mathbf{1} & d \end{pmatrix} : M_n \oplus M_{n-1} \oplus M_{n-1} \longrightarrow M_{n-1} \oplus M_{n-2} \oplus M_{n-1}$$

This is the special case of a construction in Section 4.3.

Definition. A *left homotopy* between two maps $f, g : A \longrightarrow B$ in $\mathscr{M}$ via a cylinder object $A \wedge I$ is map $H : A \wedge I \longrightarrow B$ such that $H \circ i = f + g$. Two maps are *left homotopic*, written $f \overset{l}{\sim} g$ if there is a left homotopy between them.

We can define $\pi^l(A, X)$, the equivalence classes of maps from A to B under the equivalence relation generated by left homotopy.

Lemma 44. *1. Left homotopy is an equivalence relation on $Hom(A, B)$ if A is cofibrant.*

2. *If A is cofibrant and $f : X \longrightarrow Y$ is an acyclic fibration then $\phi_* : f \mapsto \phi \circ f$ indices a bijection $\pi^l(A, X) \longrightarrow \pi^l(A, Y)$.*
3. *If X is fibrant and f, g are left homotopic maps $A \longrightarrow X$ then fh and gh are left homotopic for any $h : B \longrightarrow A$.*
4. *If X is fibrant then there is a composition map $\pi^l(B, A) \times \pi^l(A, X) \longrightarrow \pi^l(B, X)$.*
5. *If X is fibrant and $f \overset{l}{\sim} g$ we may assume the homotopy can be realised via a very good cylinder object.*

Dually we can define path objects, typically written $X \longrightarrow X^I \longrightarrow X \times X$, and right homotopies.The dual results hold for right homotopies. We say f and g are homotopic and write $f \sim g$ if they are both left and right homotopic.

The next lemma relates the two notions:

Lemma 45. *: Let $f, g : A \longrightarrow X$ be maps. Then if A is cofibrant $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Dually, if X is fibrant $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$.*

Hence if A is cofibrant and X is fibrant, f and g are homotopic if they are left or right homotopic. We write $\pi(A, X)$ for maps up to homotopy. (If A is not cofibrant or X is not fibrant we let this be the set of maps up to the equivalence relation generated by homotopy.)

Theorem 46. *A map $f : A \longrightarrow X$ between fibrant and cofibrant objects is a weak equivalence if and only if it has a homotopy inverse*

Definition. The *homotopy category $Ho(\mathscr{M})$* of a model category $\mathscr{M}$ is defined to have the same objects as $\mathscr{M}$, and with $\mathrm{Hom}_{Ho(\mathscr{M})}(X, Y) = \pi(QRX, QRY)$.

Recall that given a category $\mathscr{M}$ and a class of morphisms $\mathscr{W}$ we define the *localization* of $\mathscr{M}$ at $\mathscr{W}$ to be a category $\mathscr{B}$ with a functor $Q : \mathscr{M} \longrightarrow \mathscr{B}$ such that $Q(w)$ is an isomorphism for any $w \in \mathscr{W}$ and which is universal with this property: Any $\mathscr{M} \longrightarrow \mathscr{C}$ that sends all $w \in \mathscr{W}$ to isomorphisms factors through Q.

Theorem 47. *The homotopy category $Ho(\mathscr{M})$ is a localization of $\mathscr{M}$ at to the class of weak equivalences. We write $\gamma : \mathscr{M} \longrightarrow Ho(\mathscr{M})$ for the natural functor.*

Sketch of proof. First we have to show Ho takes weak equivalences to isomorphisms, the main ingredient is Theorem 46. Then we need to establish the universal property. Given some $G : \mathscr{M} \longrightarrow \mathscr{C}$ sending $\mathscr{W}$ to isomorphisms we need to construct $\tilde{G} : Ho(\mathscr{M}) \longrightarrow \mathscr{C}$. On objects we may just use G, on morphisms we use the fact that G sends the cofibrant and fibrant replacement functors to isomorphisms and given $f : G \longrightarrow A$ represented by $f' : RQ(A) \longrightarrow RQ(B)$ we define $\tilde{G}(f) = G(p_B)G(i_{QB})^{-1}G(f')G(i_{QA})G(p_A)^{-1}$. We have to work to make sure this is well-defined. For details see [DS] 6.2. □

Example 38. So what are maps in $Ho(\mathbf{Ch}_{\geq 0}(R))$? Consider objects A, B concentrated in degree 0 and n, say. So A and $B[-n]$ are R-modules. Let A, B have cofibrant replacements QA, QB. These are in particular projective resolutions. (Every object is fibrant, so we need not worry about that.) Since left homotopies in $\mathscr{M}$ are just chain homotopies, as you show on the example sheet, we have the following:

$$\mathrm{Hom}_{Ho(\mathbf{Ch})}(A, B) = \pi(QA, QB) = \mathrm{Hom}_{K(R)}(QA, QB) = \mathrm{Ext}^n_R(A, B[-n])$$

Similarly we can see that $Ho(\mathbf{Ch}_{\geq}(R)) \cong D_{\geq 0}(R)$.

In the next section we will lift functors to the homotopy category.

14. Derived functors

Total derived functors and universal property. Existence. Quillen functors.

Definition. Let $F : \mathscr{M} \longrightarrow \mathscr{N}$ be functor where $\mathscr{M}$ is a model category. A *left derived functor* is a pair (LF, t) where $LF : Ho(\mathscr{M}) \longrightarrow \mathscr{N}$ and $t : LF\gamma \longrightarrow F$ is a natural transformation, which is terminal among such pairs, also called "universal from the left". Explicitly, whenever $G : Ho(\mathscr{M}) \longrightarrow \mathscr{N}$ is a functor with a natural transformation $s : G\gamma \longrightarrow F$ then there is a natural transformation $\theta : G \longrightarrow LF$ such that $s = t \circ \theta_\gamma$.

Theorem 48. *Let F be a functor from a model category that sends weak equivalences between cofibrant objects to isomorphisms. Then the left derived functor of F exists and $t_X : LFX \longrightarrow FX$ is an isomorphism for every cofibrant object X.*

Proof. Recall that we have a cofibrant replacement functor $Q : \mathscr{M} \longrightarrow \mathscr{M}$. So let us consider the composition FQ. We have to show this descends to the homotopy category. But by MC2 we know that Q sends weak equivalences to weak equivalences, so FQ sends weak equivalences to isomorphisms and by Theorem 47, i.e. the universal property of $Ho(\mathscr{M})$, we have a factorization $FQ = LF \circ \gamma$.

Next we define a natural transformation $t : LF \circ \gamma \longrightarrow F$ by assigning to $X \in \mathscr{M}$ the map $F(p_x) : LF\gamma(X) = FQ(X) \longrightarrow F(X)$. Recall $p_X : QX \longrightarrow X$ is the natural transformation from cofibrant replacement to the identity. In particular if X is cofibrant $F(p_X)$ is an isomorphism by assumption.

Now we need to show universality. Let $G : Ho(\mathscr{M}) \longrightarrow \mathscr{N}$ be any functor and $s : G\gamma \longrightarrow F$ a natural transformation. We are looking for a unique natural transformation $\theta : G \longrightarrow LF$ such that $s = t \circ \theta_\gamma$. Consider the following diagram:

$$\begin{array}{ccc} G\gamma Q(X) & \xrightarrow{s_{QX}} & LF\gamma(X) = FQ(X) \\ {\scriptstyle G\gamma(p_X)}\downarrow & \overset{\theta_{\gamma X}}{\nearrow} & \downarrow{\scriptstyle t_X = F(p_X)} \\ G\gamma(X) & \xrightarrow[s_X]{} & FX \end{array}$$

We are looking for a lift $\theta_{\gamma X} : G\gamma X \longrightarrow LF\gamma X$ and since γ sends weak equivalences to isomorphisms we can just define $\theta_X = s_{QX} \circ (G\gamma(p_X))^{-1}$. This is a natural transformation $G \longrightarrow LF$ since s and $\gamma(p)$ are natural transformations of functors on the homotopy category. If X is cofibrant then p_X is an isomorphism and the lift is unique. But in the homotopy category $\gamma(X) \cong \gamma(QX)$, so the lift is unique. □

Example 39. Let us apply this to one of our favourite functors, $T = \gamma \circ (N \otimes -) : \mathbf{Ch}_{\geq 0}(R) \longrightarrow Ho(\mathbf{Ch}_{\geq 0}(S)) \cong D_{\geq 0}(S)$ for an $S \otimes R^{\text{op}}$-module N. We have to check that this preserves weak equivalences between cofibrant objects, and actually it suffices to check T preserves acyclic cofibrations between cofibrants, see Lemma 49 below.

Consider $f : A \longrightarrow B$ an acyclic cofibration. Then $C = B/A$ is levelwise projective and acyclic. Hence d_1 is surjective and $C_0 = Z_0(C)$ is projective, giving a splitting and an isomorphism $C \cong \tau_{\geq 1}(C) \oplus D_1(Z_0(C))$ and inductively $C \cong \oplus D_n(Z_{n-1}(C))$. Here we write $D_n(M)$ for $D(n) \otimes_R M$. But all $D_n(M)$ are projective in $\mathbf{Ch}(R)$ if M is projective. Hence $B \cong A \oplus C$. Moreover $T(D_n(M))$ is clearly acyclic. Hence $T(B) \cong T(A) \oplus T(C) \cong T(A)$.

Remark. Note that we cannot compute right derived functors in this way, which makes the injective model structure relevant.

Definition. The *total left derived functor* of a functor F is the left derived functor of $\gamma \circ F$.

Remark. We have been overloading the term "derived functor". The similarity to the total derived functor between derived categories is quite clear. But note that not every model category comes from chain complexes on an abelian category, and not every derived category comes from a model category. (Not all abelian categories are complete and cocomplete!)

We have just used the following, and we are about to use it again.

Lemma 49 (Ken Brown's Lemma.)**.** *Suppose a functor $F : \mathscr{M} \longrightarrow \mathscr{C}$ sends acyclic cofibrations between cofibrant objects to isomorphisms. Then it sends all weak equivalences between cofibrant objects to isomorphisms.*

Proof. This is just playing around with factorisations. Consider $f : A \longrightarrow B$ a weak equivalence between cofibrant objects. Then consider the map $f + \mathbf{1}_B : A \amalg B \longrightarrow B$ and factor it as a cofibration j followed by an acyclic fibration p. As A and B are cofibrant the inclusions i_A, i_B of A and B into $A \amalg B$ are cofibrations.

Now observe $f = p \circ ji_A$, hence ji_A is a weak equivalence by MC2, and it's a cofibration. The same holds for ji_B. Hence $F(ji_B)$ is invertible and then $F(p) = F(\mathbf{1}_B)F(ji_B)^{-1}$ is invertible. Finally $F(f) = F(p)F(ji_A)$ is invertible, too. □

Theorem 50. *Assume given a pair of adjoint functors $F : \mathscr{M} \rightleftarrows \mathscr{N}$ between model categories such that F preserves cofibrations and G preserves fibrations. Then the total derived functors LF and RG exist and form an adjunction $LF : Ho(\mathscr{M}) \rightleftarrows Ho(\mathscr{N}) : RG$.*

Idea of proof. For the existence of the derived functors we need to know that our condition imply the conditions of Theorem 48. This is exactly the content of Lemma 49.

To show there is an adjunction on the level of homotopy categories it is crucial to observe that $F(A \wedge I)$ is a cylinder object for FA in $\mathscr{N}$ if $A \wedge I$ is a very good cylinder object for A. For details see [DS] 9.7. □

We can now define the most interesting notion of functor between model categories.

Definition. An adjunction as in the theorem is called a *Quillen adjunction*, and F a *left Quillen functor*. If the adjunction induces an equivalence $Ho(\mathscr{M}) \cong Ho(\mathscr{N})$ we call it a *Quillen equivalence*.

Note that tensoring with an $S \otimes R^{\mathrm{op}}$-module M is not in general a left Quillen functor unless M is cofibrant as an S-module.

15. More model categories

Topological spaces. Simplicial sets. Quasi-categories.

We now collect some more examples. Sadly, we will not have time to prove anything.

We will mention generating (acyclic) cofibrations in a few cases. The small object argument is used extensively throughout the subject.

Example 40. The category of topological spaces has a model structure defined as follows:

- Weak equivalences are given by weak homotopy equivalences, i.e. maps inducing isomorphisms on all homotopy groups.
- Fibrations are given by Serre fibrations, i.e. maps which have the RLP with respect to all inclusions $A \longrightarrow A \times [0,1]$, where A is a CW-complex.
- Cofibrations are maps with the LLP with respect to all Serre fibrations which are also weak equivalences.

To see if f is a fibration it suffices to check the right lifting property with respect to $\mathscr{J} = \{D^n \longrightarrow D^n \times I\}$, and to check f is an acyclic fibration we check RLP with respect to $\mathscr{I} = \{S^{n-1} \longrightarrow D^n\}$.

Example 41. There is a model structure on simplicial sets closely related to the one on topological spaces. Define the following classes of maps:

- Weak equivalences are those maps whose realisations are weak homotopy equivalences.
- Cofibrations are inclusions.
- Fibrations are defined via the lifting property.

Again we can make this more concrete by exhibiting sets of generating (acyclic) cofibrations. We need to define the following special simplicial sets:

Definition. We define the boundary $\partial\Delta[n]$ of the standard n-simplex by leaving out the non-degenerate n-simplex s corresponding to $\mathbf{1}_{[n]}$ (and its degeneracies). Concretely $\partial\Delta[n]_k$ consists of non-surjective maps $[k] \longrightarrow [n]$.

We define the *k-th horn* $\Lambda_k[n]$ by leaving out the nondegenerate n-simplex $s \in \Delta[n]_n$ and the k-th nondegenerate $n-1$-simplex, $\partial_k(s)$ from $\Delta[n]$. (Of course we also leave out all their degeneracies.)

A simplicial set K is called a *Kan complex* if any map $\Lambda_k[n] \longrightarrow K$ extends to a map $\Delta[n] \longrightarrow K$. The slogan is: "Every horn can be filled."

Then we take generating acyclic cofibrations to be the set of all $\Lambda_k[n] \longrightarrow \Delta[n]$. In particular, Kan complexes are fibrant objects. We take generating cofibrations to be $\partial\Delta[n] \longrightarrow \Delta[n]$.

It is unsurprising that $|\partial\Delta[n]|$ is the boundary of Δ^n and $|\Lambda_k[n]|$ is the boundary of Δ^n with the interior of one face removed. So they are homeomorphic to S^{n-1} and D^{n-1} respectively.

Then $| - | : \mathbf{sSet} \rightleftarrows \mathbf{CGHauss} : \mathrm{Sing}$ is a Quillen adjunction and in fact a Quillen equivalence. Here **CGHauss** is the subcategory of topological spaces consisting of compactly generated Hausdorff spaces. It's a large sub-model category containing almost all the spaces homotopy theorists care about.

Remark. Our definition of weak equivalences might seem a bit disingenuous, but there are several entirely simplicial characterisations of weak equivalences that are equivalent to the one we just gave. For example we can define homotopies between maps of simplicial sets and a weak equivalence is precisely a map $A \longrightarrow B$ inducing an isomorphism between homotopy classes of maps $B \longrightarrow K$ and $A \longrightarrow K$ for any Kan complex K, see e.g. [GM].

Example 42. Interestingly, there are other model category structures on simplicial sets. For example there is Joyal's model structure which has fewer weak equivalences. Fibrant objects are now *weak Kan complexes*, where we only demand that inner horns can be filled (all but $\Lambda_0[n]$ and $\Lambda_n[n]$). Weak Kan-complexes are also called *quasi-categories* or just ∞-*categories* and they form the building blocks of Jacob Lurie's work on higher toposses and higher algebra.

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Remark. Quasi-categories are one of several sensible model for $(\infty, 1)$-categories, i.e. categories with (invertible) morphisms between morphisms and (invertible) morphisms between morphisms between morphisms, etc.

We need a language to compare different categories of $(\infty, 1)$-categories, and Quillen equivalence of model categories is exactly the right language to use.

Simplicial sets with the Joyal model structure are Quillen equivalent to a suitable model structure on categories enriched in simplicial sets or topological spaces. For a category enriched in topological spaces we can think of morphisms between morphisms as paths in the space of morphisms, and of morphisms between morphisms between morphisms as homotopies.

16. Rational homotopy theory

Commutative differential graded algebras. Rational homotopy theory.

This is a slightly extended version of the last lecture of the course, none of the material is examinable.

In this lecture will see a rather surprising equivalence of homotopy categories, that makes parts of algebraic topology completely algebraic. This is Quillen's and Sullivan's rational homotopy theory, cleanly exposited in Bousfield and Gugenheim's AMS Memoir.

Definition. A *commutative differential graded algebra* or *cdga* over a field k, also called a *cdga*, is a chain complex with a graded commutative algebra structure compatible with the differential via a Leibniz-rule: $d(ab) = da.b + (-1)^{|a|}a.db$.

Remark. This is a badly behaved notion if k has positive characteristic.

Example 43. Let M be a differentiable manifold, then the de Rham complex $\Omega^* M$ of differential forms on M with the wedge product is a cdga.

Example 44. $\Omega^*(S^2) \simeq H^*(S^2)$ and $H^*(S^2)$ is a cdga with generators the constant function in degree 0 and the volume form in degree 2.

Consider the category **cdga** of commutative differential graded algebras over $\mathbb{Q}$. With the following classes of morphisms it becomes a model category.

- Weak equivalences are given by quasi-isomorphisms.
- Fibrations are given by surjections.
- Cofibrations are given by the lifting property.

Remark. In fact, many other algebraic structures on chain complexes obtain a model category structure by *transfer of model structure* from chain complexes.

Example 45. Examples of cofibration of cdga's are as follows:

- We write $\Lambda(x_{n_1}, x_{n_2}, \ldots, x_{n_k})$ for the free cdga on k generators in degrees n_i. For example $\Lambda(x_2)$ is a polynomial algebra on a generator in degree 2 and $\Lambda(x_5)$ is an exterior algebra on a generator in degree 5. This is a cofibrant cdga. For example on the second example sheet you have computed cohomology of the Eilenberg-MacLane space as $\Lambda(x_n)$, a free cdga on a single generator in degree n.

- The maps $\Lambda(x) \longrightarrow \Lambda(y, dy)$ given by $x \mapsto dy$ are cofibrations, where $\Lambda(y, dy)$ has two generators and differential $d : y \mapsto dy$.

We can make the homotopy theory of cdgas explicit with the following observation:

Path objects in **cdga** are given by $B \otimes \Lambda(t, dt)$ where $|t| = 0$. I.e. a homotopy between maps f and g from A to B are maps $H : A \longrightarrow B \otimes \Lambda(t, dt)$ such that $\partial_0 H = f$ and $\partial_1 H = g$ where $\partial_0 t = 0, \partial_1 t = 1$.

Now we recall the model category of simplicial sets or equivalently (compactly generated Hausdorff) topological spaces.

We want to define a functor between cdga's and **sSet**. The categories are obviously not Quillen equivalent, but miraculously they are *up to torsion*.

The idea is to generalise the de Rham functor that sends a manifold to the algebra of differential forms. To extend this to all topological spaces we will define polynomial differential forms an simplicial sets. These can naturally be defined using the simplicial dg-algebras $\nabla(n, *)$.

Definition. For every n define $\nabla(n, *)$ to be the dg-algebra over $\mathbb{Q}$ generated by $t_0, \ldots, t_n, dt_0, \ldots, dt_n$ subject to the relations $\sum t_i = 1$ and $\sum dt_i = 0$, and with differential $t_i \mapsto dt_i$. We can think of these as polynomial differential forms on the n-simplex in $\mathbb{R}^{n+1}$.

Now the face and degeneracy maps between n-simplices induce face and degeneracy maps on $\nabla(*, *)$ and make it into a simplicial dg-algebra.

Remark. The precise face and degeneracy maps are:

$$\partial_i t_j = \begin{cases} t_{j-1} & \text{if } i < j \\ 0 & \text{if } i = j \\ t_j & \text{if } i > j \end{cases}$$

$$s_i t_j = \begin{cases} t_{j+1} & \text{if } i < j \\ t_j + t_{j+1} & \text{if } i = j \\ t_j & \text{if } i > j \end{cases}$$

Definition. For a simplicial set K define $A(K)$ to be $\mathrm{Hom}_{\mathbf{sSet}}(K, \nabla)$. To be precise, $A^p(K) = \mathrm{Hom}_{\mathbf{sSet}}(K, \nabla(*, p))$ and the differential and multiplication on ∇ induce the structure of a cdga on $A(K)$. We call this the dg-algebra of *polynomial differential forms* on K. Conversely associate to any dg-algebra D the simplicial set $S(D) := \mathrm{Hom}_{\mathbf{cdga}}(D, \nabla)$.

Note these are contravariant functors.

First it is reassuring that $H^*(AK) \cong H^*(|K|)$ as rings and if M is a manifold $AM \simeq \Omega^* M$.

Theorem 51. *The functors A and S form a Quillen adjunction $A \dashv S$ between* $\mathbf{cdga}^{op}$ *and* $\mathbf{sSet}$.

About the proof. To check we have an adjunction write down the natural map on hom-spaces and it is a bijection.

Then we need to check that A sends (acyclic) cofibrations to (acyclic) fibrations. This is equivalent to checking S sends (acyclic) generating cofibrations to (acyclic) fibrations. But that is not hard to check. □

The proof extends to an adjunction between pointed simplicial sets and *augmented* cdgas, i.e. cdgas A equipped with a map $\epsilon : A \longrightarrow \mathbb{Q}$. We will talk about the pointed/augmented case from now on.

Cofibrant replacement in the category of cdgas are very useful for computations, so let us look at some examples.

Example 46. First we'll describe a cofibrant cdga quasi-isomorphic to AS^n or equivalently to the de Rham complex $\Omega^*(S^n)$ of S^n. (As S^n is a manifold we may as well use the usual de Rham complex.) If n is odd we note the map $\Lambda(x_n) \longrightarrow \Omega^*(S^n)$ that sends x_n to the volume form on S^n is a quasi-isomorphism.

If n is even $\Lambda(x_n)$ is a polynomial algebra, so there is no quasi-isomorphism. If we demand $x_n^2 = 0$ the algebra would no longer be cofibrant. What we can do instead is consider $\Lambda(x_n, y_{2n-1} \overset{d}{\mapsto} x_n^2)$. The map sending x_n to the volume form on y_{2n-1} to 0 is a quasi-isomorphism.

It follows from our computation on the second example sheet that $AK(\mathbb{Q}, n) \simeq \Lambda(x_n)$, and this is already cofibrant.

Remark. In fact any nice cdga has a canonical cofibrant replacement, unique up to isomorphism, called its *minimal model*.

After this detour we will state the main theorem. The aim is to show that we can restrict to nice subcategories $\mathscr{N}$ and $\mathscr{D}$ where A induces an equivalence. The correct categories are as follows:

Definition. Let $\mathscr{D}$ be the category of cofibrant, homologically connected algebras of finite $\mathbb{Q}$-type. A is homologically connected if $H^0(A) = \mathbb{Q}$ and $H^{<0}(A) = 0$. It is of finite $\mathbb{Q}$-type if it satisfies some finiteness condition. (In language to be defined later: if $\pi^k(A)$ is finite-dimensional over $\mathbb{Q}$ for all k.)

Let $\mathscr{N}$ be the category of connected spaces which are nilpotent, rational and of finite $\mathbb{Q}$-type.

We'll define the conditions on $\mathscr{N}$ as we talk about the proof. Roughly speaking: Nilpotent says the fundamental group is nilpotent and acts nilpotently on other homotopy groups. Finite $\mathbb{Q}$-type is again some finiteness-condition. The most interesting condition is rationality, which roughly means that homotopy and integral cohomology groups are actually $\mathbb{Q}$-vector spaces. This seems an odd and strange requirement, an example would be a $K(\mathbb{Q}, 1)$, a space with $\pi_1 \cong \mathbb{Q}$ and no higher homotopy. We can glue such a space out of infinitely many S^1, but it does not seem natural. However, we can think of any space X as being equivalent "up to torsion" to some rational space $X_{\mathbb{Q}}$, its rationalisation. In fact it follows from Quillen's theory that the unit of our adjunction $X \longrightarrow SA(X)$ is such a *rationalisation* of X if X is connected nilpotent. That means it is a map from X to a rational space which induces isomorphisms on rational cohomology and on the nilpotent completions of the fundamental group. We can think of it as replacing X by a rational space with the same rational invariants.

Theorem 52. *The adjunction $A \dashv S$ induces an equivalence of the homotopy categories $Ho(\mathscr{N})$ and $Ho(\mathscr{D}^{op})$.*

The proof comes down to checking $K \simeq SA(K)$ and $D \simeq AS(D)$ for $K \in \mathscr{N}$ and $D \in \mathscr{D}$. Let us get a feel for the proof that $K \simeq SA(K)$ for $K \in \mathscr{S}$. The most important topological input is the Postnikov tower for K. Writing a space as a cell complex is building it up out of spheres, building the homology groups of X step by step. The Postnikov tower builds K out of Eilenberg-MacLane spaces, synthesising the homotopy groups.

To be precise we write $X = \varprojlim X_n$ where X_0 is a point and every $X_n \longrightarrow X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$.

Definition. A space is *nilpotent* if the fundamental group is nilpotent and acts nilpotently on higher homotopy groups.

This is equivalent to the condition that in the Postnikov tower of X we can factor each $X_n \longrightarrow X_{n-1}$ as a finite sequence of principal fibrations $X_{n,i} \longrightarrow X_{n,i-1}$, i.e. pull-backs of the canonical path fibration $\mathscr{P}K(B,n) \longrightarrow K(B,n)$ for an abelian group B. Here $X_{n,0} = \varprojlim_i X_{n-1,i}$.

Definition. Now we can define a space to be *rational* if $X \cong \varprojlim_{n,i} X_{n,i}$ where each $X_{n,i} \longrightarrow X_{n,i-1}$ is a principal fibration with fiber $K(\mathbb{Q},n)$.

Let us now also note that a nilpotent connected space is of finite $\mathbb{Q}$-type if all $H_n(X,\mathbb{Q})$ are finite-dimensional $\mathbb{Q}$-vector spaces.

The theorem is proven by taking the tower of fibrations that is the Postnikov tower and turning it into a tower of cofibrations whose limit is a cofibrant model for AX Concretely, we have to consider the image of a principal fibration under A.

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{P}K(\mathbb{Q},n) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & K(\mathbb{Q},n) \end{array}$$

corresponds to

$$\begin{array}{ccccc} AX & \longleftarrow & A\mathscr{P} & \overset{\sim}{\longleftarrow} & \Lambda(y,dy) \\ \uparrow & & \uparrow & & \uparrow \\ Y & \longleftarrow & AK(\mathbb{Q},n) & \underset{\sim}{\longleftarrow} & \Lambda(x) \end{array}$$

Where we attach a cofibrant replacement of both objects and the map between them on the right. (A does not send fibrations to cofibrations!)

To finish the proof one can inductively prove that our adjunction maps induce weak equivalences at each stage of the process. The so-called Eilenberg-Moore spectral sequence allows us to put these together, but there is some subtlety involved, as the natural map $AK \longrightarrow C^*K$ is not an algebra map. (One side is commutative, one isn't!) But there is an algebra structure that is commutative up to homotopy, which suffices to prove some equality of Tors.

Proposition 53. *For a nilpotent connected CW-complex X with minimal model M we can compute $\pi_n(X) \otimes \mathbb{Q}$ in terms of M.*

About the proof. This comes down to the fact that $X \longrightarrow SA(X)$ is a rationalisation, so $\pi_n(X) \otimes \mathbb{Q} \cong \pi_n(SA(X))$. But the latter is just the space of homotopy classes of maps $S^n \longrightarrow SAX$, equivalent to homotopy classes of maps $AX \longrightarrow AS^n$, which can be computed in terms of a cofibrant replacement, see below. □

In fact the $\pi_n(X) \otimes \mathbb{Q}$ are dual to $\pi^n M$, if $n \geq 2$ which are defined as follows. Together with the augmentation $\epsilon : A \longrightarrow \mathbb{Q}$ comes the augmentation ideal $IA = \ker(\epsilon) \subset A$. We define the *indecomposables* of A to be $QA = IA/IA.IA$.

Using the indecomposables we define *homotopy groups* $\pi^k(B) = H^k(QB)$ of B. And indeed it follows from the proof of the proposition that these are dual to the (rationalised) homotopy groups of X!

By inspecting our example computation we find the following:

Corollary 54. *Let $k \geq 1$. The homotopy group $\pi_k(S^n)$ has rank 0 unless $k = n$ or n is even and $k = 2n - 1$, in which case it has rank 1.*