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Algebraic Topology (Master)
Summer Semester 2023

Exercise sheet 13

Question 13.1

a) What are the cup pairings on $\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}P^2$?

b) Let M be a manifold of dimension $4n$ or $4n+2$. What can you say about the symmetry of the cup pairing in the middle dimension of $H^*(M)$?

Question 13.2

Let $M = U \cup V$ be a union of two open subspaces We consider cohomology with coefficients in some ring R we omit from our notation.

a) Show that any inclusion $U \to M$ of topological spaces induces a homomorphism on compactly supported cohomology $H^i_c(U \to H^i_c(M)$

b) For fixed compact subset $K \subset U$, $L \subset V$ use the Mayer-Vietoris long exact sequence for relative cohomology together with excision to find a long exact sequence

$$\cdots \to H^{i-1}(M, M \backslash K \cup L) \to H^i(U \cap V, U \cap V \backslash (K \cap L)) \to H^i(U, U \backslash K) \oplus H^i(V, V \backslash L) \to H^i(M, M \backslash K \cup L) \to \ldots$$

c) Deduce the Mayer-Vietoris long exact sequence for compactly supported cohomology:

$$\cdots \to H^i_c(U \cap V; R) \to H^i_c(U; R) \oplus H^i_c(V; R) \to H^i_c(M; R) \to H^{i+1}_c(U \cap V; R) \to \ldots$$

Question 13.3

Show that for n even there is no fixed-point free map $f : \mathbb{C}P^n \to \mathbb{C}P^n$. What can you say for odd n?

Question 13.4

Have another look at questions from previous sheets that you did not quite understand at the time. Try again to solve them.

These questions will be discussed in the class on 12 July 23. You may hand in your solutions the day before.

Questions with an asterisk are more challenging.