Incompressible Navier-Stokes Calculations with Nonreflecting Boundary Conditions

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Many flow problems, such as those concerning channel flows, are considered in unbounded domains. However, in order to bound the complexity of the numerical simulation of such flows artificial boundaries have to be introduced. For numerical purposes boundary conditions at these artificial boundaries should be imposed in such a way that they affect the solutions in such a way that they closely approximate the free-space situations. In the past several attempts on posing nonreflecting outlet boundary conditions for incompressible Navier-Stokes calculation had been made (see for example [6], [3] and the references cited therein). However, in many DNS-codes Neumann-type boundary conditions are implemented at the outlet boundary. This often results in non-physical reflection phenomena near the outlet. In this note we present an approach that exploits the concept of natural boundary conditions for the NSE (see [5], [1]).

For motivation consider a pipe-flow around an obstacle with a prescribed inlet profile $u_{|\Gamma_{in}} = g$ and usual no-slip conditions $u|_\Gamma = 0$ along the rigid part of the boundary (compare Fig. 1). Prescribing *nothing* at the outlet Γ_{out} is then expressed by the variational formulation of the problem:

Find $u = g + v$ and p with $v(t) \in V := \{\phi \in H^1(\Omega)^2; \phi_{|\Gamma \cup \Gamma_{in}} = 0\}$ and $p(t) \in L^2(\Omega)$ for all t satisfying $v(0) = v_0$ and

$$\begin{aligned} (v_t, \phi) + \nu(\nabla v, \nabla \phi) + ((v \cdot \nabla)v, \phi) - (p, div(\phi)) &= 0 \quad \forall \phi \in V & (1)\\ (q, div(v)) &= 0 \quad \forall q \in L^2(\Omega). & (2) \end{aligned}$$

Now suppose that (1),(2) posesses some smooth solution (v, p). Testing in (1) with functions ϕ vanishing on $\partial\Omega$ yields the classical momentum equation

$$v_t - \nu \Delta v + (v \cdot \nabla)v + \nabla p = 0 \quad in\ \Omega \times (0, T],$$

so that testing with functions ϕ having values on Γ_{out} gives the natural boundary condition

$$\nu \partial_\eta v - p\eta = 0 \quad on\ \Gamma_{out} \times (0, T],$$

where η denotes the outward unit-normal to $\partial\Omega$. As a consequence, any smooth solution of (1), (2) satisfies

$$\begin{aligned} v_t - \nu \Delta v + (v \cdot \nabla)v + \nabla p &= 0 && in\ \Omega \times (0, T]\\ div(v) &= 0 && in\ \Omega \times (0, T]\\ v &= 0 && on\ (\Gamma \cup \Gamma_{in}) \times (0, T]\\ \partial_\eta v &= p\eta && on\ \Gamma_{out} \times (0, T]\\ v(0) &= v_0 \end{aligned} \tag{3}$$

System (3) is the starting point for our numerical investigations. For its numerical integration we use the following semi-implicit scheme developed by Chorin in [2]:
Given velocity values v_{m-1} and pressure values p_{m-1} at time step $m - 1$:

1. **Prediction**. Solve for $\tilde{v}$

$$\begin{aligned} \frac{\tilde{v} - v_{m-1}}{\Delta t} - \nu \Delta v_{m-1} + (v_{m-1} \cdot \nabla)v_{m-1} + \nabla p_{m-1} &= 0 && in\ \Omega\\ \tilde{v} &= 0 && on\ \Gamma \cup \Gamma_{in}\\ \nu \partial_\eta \tilde{v} &= p_{m-1}\eta && on\ \Gamma_{out} \end{aligned} \tag{4}$$

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2. **Projection**. Solve for v_m by iteration

$$
\begin{array}{rcll}
\frac{v_m - \tilde{v}}{\Delta t} & = & \nabla(p_m - p_{m-1} & in\ \Omega \\
div(v_m) & = & 0 & in\ \Omega \\
v_m & = & 0 & on\ \Gamma \cup \Gamma_{in} \\
\nu \partial_\eta v_m & = & p_m \eta & on\ \Gamma_{out}.
\end{array} \tag{5}
$$

These equations are integrated over finite volumes, where the staggered grids of Harlow and Welch [4] are used for the discretization of the velocity and pressure fields in space and time. As Fig. 1 shows the numerical method, at least for moderate Reynolds numbers, works very well as the results are reasonably independent on the position of the artificial boundary Γ_{out} even for "short" computational domains.

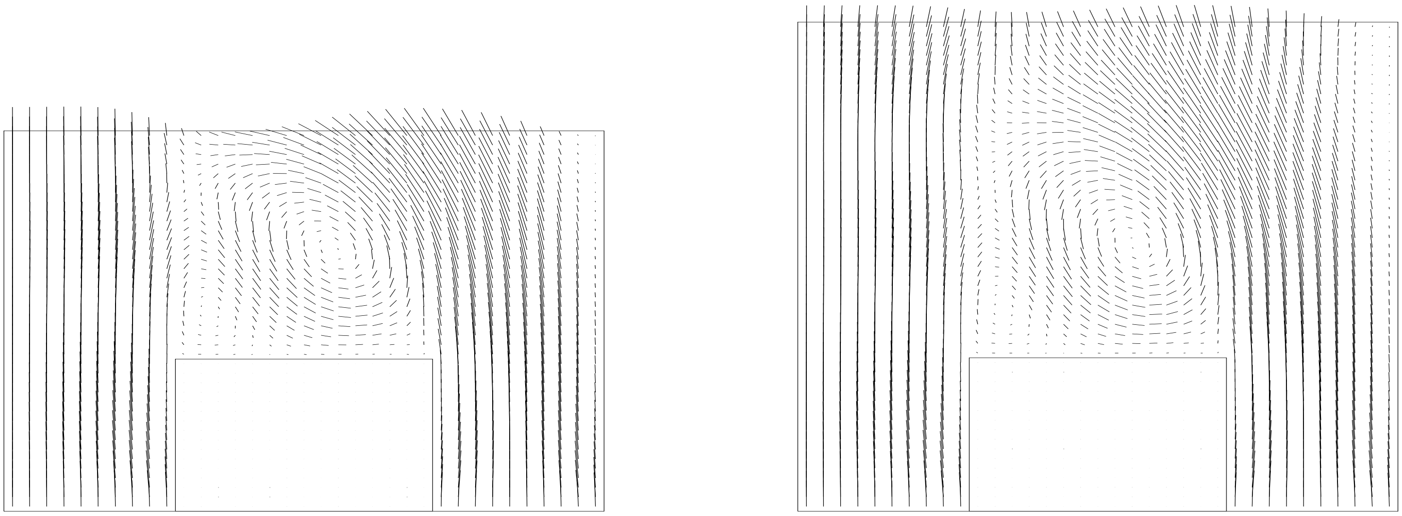

Figure 1: Numerical results after 12000 time-steps, $\nu = \frac{1}{Re} = \frac{1}{500}$

The objective of our numerical investigations is to develop suitable outlet boundary conditions for LES and DNS in boundary-control problems for channel- and pipe-flows. This work is supported by the **Universitärer Forschungsschwerpunkt 8 - Kontrolle turbulenter Scherströmungen** at the Technical University of Berlin, Germany.

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