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Abstract

We consider an optimal control problem of tracking type for Stokes flows in two and three space dimensions. Based on the first order optimality conditions, the solution is approximated by finite elements in space and implicitly in time. We prove optimal error estimates for the resulting fully–discrete scheme and present numerical examples that confirm our results.

Key words. Optimal control, instationary Stokes equations, time-space discretization, error estimates, Newton's method.
AMS(MOS) subject classification. 49M15,49M29,65K10,76D55.

1 Introduction

Optimal control problems in fluid dynamics have received considerable attention over the past years. In general, solutions to such problems can only be obtained numerically, so that the design and analysis of efficient numerical methods is of great interest.
In this paper we extend the concept of semi-discrete solutions introduced in [2] to the fully discrete case. The concept is based on the first order optimality conditions. We shall carry out a rigorous error analysis for this method and present numerical calculations which confirm our results. In order to keep the presentation simple and focus on the main ideas we shall assume that the underlying flow is governed by the Stokes equations and restrict ourselves to tracking type functionals of the form

$$J(y,u) = \frac{1}{2}\int_0^T \int_\Omega |y-z|^2 dxdt + \frac{\alpha}{2}\int_0^T \int_\Omega |u|^2 dxdt \tag{1.1}$$

with a given target flow z and a constant $\alpha > 0$. However, let us mention at this point that our approach can be generalized to the full Navier–Stokes equations (in 2D) as well as to more general functionals of the form

$$J(y,u) = \phi_0(y(T)) + \int_0^T \Theta(y(t),t)dt + \frac{\alpha}{2}\int_0^T \|u(t)\|_U^2 dt \tag{1.2}$$

which include for example functionals regulating the vorticity of the flow. The precise formulation of results together with detailed proofs can be found in [3].
As already mentioned we consider the optimal control problem

$$\min \ J(y,u) \text{ over } (y,u) \tag{1.3}$$

subject to

$$\left\{\begin{array}{rcll} y_t - \nu\Delta y + \nabla p &=& u(t) & \text{in } \Omega\times(0,T) \\ -\operatorname{div} y &=& 0 & \text{in } \Omega\times(0,T) \\ y(.,t) &=& 0 & \text{on } \partial\Omega\times(0,T) \\ y(.,0) &=& y_0 & \text{in } \Omega. \end{array}\right. \tag{1.4}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3) and $T > 0$, y denotes the velocity field and p the pressure of the fluid. Furthermore, u is the control variable. Since (1.4) has a unique solution $y = y(u)$, we may consider the reduced problem

$$\min \ \hat{J}(u) = \min \ J(y(u),u) \text{ over } u. \tag{1.5}$$

In general, two approaches can be distinguished to compute discrete solutions of (1.5); the first approach is called *first discretize then optimize* and works as follows: The terms appearing in the functional $\hat{J}$ and the state equations together with the controls are discretized. The solutions of the resulting finite dimensional optimization problem then are considered as discrete approximations to the continuous optimization problem.
An alternative approach may be called *first optimize then discretize*. In the situation which we consider, (1.5) has a unique minimum u^*, which is characterized by the first order optimality condition

$$\hat{J}'(u^*) = 0 \tag{1.6}$$

and one can try to approximate the solution of this equation. To do so, the state and adjoint variables which appear in equation (1.6) are replaced by their discrete counterparts. This procedure gives rise to an approximation F_h of the gradient $\hat{J}'$ and we are left with the equation

$$F_h(u_h^*) = 0. \tag{1.7}$$

The corresponding solution u_h^* is considered as an approximation of the minimum of the functional $\hat{J}$ in (1.5). We emphasize that in this approach the controls u are not discretized explicitly but that the discrete solution u_h^* happens to be discretized implicitly in terms of the finite element approximation λ_h of the adjoint variable λ. For the Taylor-Hood element as spatial and the implicit Euler scheme as temporal discretization we will prove the error estimate

$$\int_0^T \int_\Omega |u^* - u_h^*|^2 dxdt \ \leq C\big((\Delta t)^2 + h^4\big), \tag{1.8}$$

where h is the space mesh parameter and Δt the time step.
Let us briefly comment on some related contributions. In [5] Gunzburger and Manservisi investigate the *first discretize then optimize* approach for tracking type control of the instationary Navier-Stokes equations. Among other things they prove that discrete solutions converge to solutions of the continuous problem. Similar investigations for the boundary control case are provided by the same authors in [6]. Contributions for the stationary Navier-Stokes equations are given in [4, 10, 9], and semidiscrete Ritz-Galerkin approximations of nonlinear parabolic control problems are investigated in [11, 12].

The paper is organized as follows. Section 2 readdresses the optimal control problem and introduces the concept of discrete solution. Section 3 contains the convergence proof and in Section 4 we present the results of numerical computations.

2 The optimal control problem and its discretization

Let us start by introducing some notation. Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) be a convex, polygonal domain and H and V the closure of $\{v \in C_0^\infty(\Omega) \,|\, \operatorname{div} v = 0\}$ in $L^2(\Omega)^n$ and $H^1(\Omega)^n$ respectively. Furthermore, we denote by $P : L^2(\Omega)^n \to H$ the Leray–operator. We also set $L_0^2(\Omega) := \{q \in L^2(\Omega) \mid \int_\Omega q dx = 0\}$.
For $T > 0$ we define the spaces $Y = L^2(0, T; L^2(\Omega)^n)$ and $Z := H^1(0, T; L^2(\Omega)^n)$ endowed with the norms

$$\|v\|_Y := \Big(\int_0^T \|v\|^2 dt\Big)^{\frac{1}{2}}, \quad \|v\|_Z := \Big(\int_0^T \big(\|v\|^2 + \|v_t\|^2\big) dt\Big)^{\frac{1}{2}}.$$

Here and in what follows we shall use $(\cdot, \cdot)$ for the $L^2(\Omega)$–dot product and $\|\cdot\|$ to denote the corresponding norm. Next, let us fix the initial velocity $y_0 \in V \cap H^2(\Omega)^n$ and $z \in Z$. Utilizing the regularity results of [1] one can prove that for $u \in Y$ the system (1.4) has a unique solution (y, p) satisfying $y \in L^2(0, T; H^2(\Omega)^n) \cap C^0([0, T]; V)$ with $y_t, \nabla p \in L^2(0, T; L^2(\Omega)^n)$. Furthermore

$$\sup_{0 \le t \le T} \|y(t)\|_{H^1}^2 + \int_0^T \big(\|y\|_{H^2}^2 + \|y_t\|^2 + \|\nabla p\|^2\big) dt \le C\big(\|y_0\|_{H^1}^2 + \|u\|_Y^2\big). \tag{2.1}$$

In particular, we can consider $u \mapsto y = y(u)$ as a mapping from Y to Y. This mapping is affine and it is convenient to write it in the form $y(u) = Su + \bar{y}$, where the linear, continuous operator $S : Y \to Y$ assigns to each u the solution of (1.4) corresponding to $y_0 = 0$ and $\bar{y}$ solves (1.4) for $u = 0$. Note, that if $u \in Z$, then the regularity properties of the solution improve to $y \in C^0([0, T]; V \cap H^2(\Omega)^n)$, $p \in C^0([0, T]; L_0^2(\Omega) \cap H^1(\Omega))$ with $y_t \in C^0([0, T]; L^2(\Omega)^n) \cap L^2(0, T; H_0^1(\Omega)^n)$ and

$$\sup_{0 \le t \le T} \big(\|y(t)\|_{H^2}^2 + \|y_t(t)\|^2 + \|\nabla p(t)\|^2\big) + \int_0^T \|\nabla y_t\|^2 dt \le C\big(\|y_0\|_{H^2}^2 + \|u\|_Z^2\big). \tag{2.2}$$

Let us now return to our optimal control problem (1.5):

Theorem 2.1. The problem

$$\min \hat{J}(u) \quad \text{s.t.} \quad u \in Y \text{ and } y = y(u) \text{ satisfies (1.4)} \tag{2.3}$$

has a unique solution u^*.

Proof. Existence of a minimum u^* follows from the form of $\hat{J}$ together with (2.1). Furthermore, it is not difficult to verify the following identity:

$$\langle \hat{J}'(u), \phi \rangle = \int_0^T \int_\Omega (y(u) - z) \cdot S\phi \, dx dt + \alpha \int_0^T \int_\Omega u \cdot \phi \, dx dt, \qquad u, \phi \in Y. \tag{2.4}$$

A straightforward calculation shows that

$$\begin{aligned} \hat{J}(u) - \hat{J}(u^*) &= \frac{1}{2} \int_0^T \int_\Omega |y(u) - y(u^*)|^2 dx dt + \frac{\alpha}{2} \int_0^T \int_\Omega |u - u^*|^2 dx dt \\ &\quad + \int_0^T \int_\Omega (y(u^*) - z) \cdot (y(u) - y(u^*)) dx dt + \alpha \int_0^T \int_\Omega u^* \cdot (u - u^*) \, dx dt \\ &= \frac{1}{2} \int_0^T \int_\Omega |y(u) - y(u^*)|^2 dx dt + \frac{\alpha}{2} \int_0^T \int_\Omega |u - u^*|^2 dx dt \end{aligned}$$

since $y(u) - y(u^*) = S(u - u^*)$ and $\hat{J}'(u^*) = 0$. This implies uniqueness of the minimum u^*. $\square$

Our aim is to use the condition $\hat{J}'(u^*) = 0$ in order to find the solution of (2.3). Starting from (2.4) we shall rewrite this expression in a well–known way: for a given $f \in Y$ we denote by w the solution of the backward Stokes problem

$$\left\{ \begin{array}{rcll} -w_t - \nu \Delta w + \nabla \xi &=& f & \text{in } \Omega \times (0, T) \\ -\operatorname{div} w &=& 0 & \text{in } \Omega \times (0, T) \\ w(., t) &=& 0 & \text{on } \partial\Omega \times (0, T) \\ w(., T) &=& 0 & \text{in } \Omega. \end{array} \right. \tag{2.5}$$

Similar to (2.1) we have

$$\sup_{0 \le t \le T} \|w(t)\|_{H^1}^2 + \int_0^T \big(\|w\|_{H^2}^2 + \|w_t\|^2 + \|\nabla \xi\|^2 \big) dt \le C \|f\|_Y^2. \tag{2.6}$$

Let us introduce the operator $R : Y \to Y$ by $w = Rf$. It is not difficult to verify using integration by parts that

$$\int_0^T \int_\Omega Rf \cdot u \, dx dt = \int_0^T \int_\Omega f \cdot Su \, dx dt \qquad \forall u, f \in Y. \tag{2.7}$$

Applying this identity to (2.4) we obtain the representation

$$\langle \hat{J}'(u), \phi \rangle = \int_0^T \int_\Omega \big(R(y(u) - z) + \alpha u \big) \cdot \phi \, dx dt$$

for all $\phi \in Y$, so that it is natural to look for a zero of the mapping $F : Y \to Y$ defined by

$$F(u) := R\big(y(u) - z\big) + \alpha u. \tag{2.8}$$

This mapping again is affine with constant derivative

$$\langle F', \phi \rangle = RS\phi + \alpha\phi, \quad \phi \in Y. \tag{2.9}$$

Note further that in view of (2.7) F' is selfadjoint and that

$$\int_0^T \int_\Omega \langle F', \phi \rangle \cdot \phi dx dt = \int_0^T \int_\Omega \big(|S\phi|^2 + \alpha|\phi|^2\big) dx dt, \tag{2.10}$$

which shows that F' is injective. Since $\phi \mapsto RS\phi$ is a compact operator on Y, the Fredholm alternative implies that F' is invertible. Also observe that $F(u) = \langle F', u \rangle + R(\bar{y} - z)$ so that we have the following formula for the zero u^* of F:

$$u^* = (F')^{-1}\big(R(z - \bar{y})\big). \tag{2.11}$$

Let us now turn to our numerical method, which will follow the above ideas at the discrete level. We begin by explaining our notation. Let $\mathcal{T}_h$ be a triangulation of Ω with maximum mesh size $h := \max_{T \in \mathcal{T}_h} \text{diam}(T)$. We assume that $\mathcal{T}_h$ is uniformly regular, i.e. that each polyeder $T \in \mathcal{T}_h$ contains a ball of radius $\kappa_1 h$ and is contained in a ball of radius $\kappa_2 h$, where the constants κ_1 and κ_2 are independent of h. Our discretization in space uses the Taylor–Hood element, so that the velocity space $H_0^1(\Omega)^n$ is approximated by

$$X_h = \{v_h \in C^0(\bar{\Omega})^n \,|\, v_{h|T} \in P^2(T)^n \text{ for all } T \in \mathcal{T}_h, v_{h|\partial\Omega} = 0\},$$

while the pressure space $L_0^2(\Omega)$ is approximated by

$$M_h = \{q_h \in C^0(\bar{\Omega}) \,|\, q_{h|T} \in P^1(T) \text{ for all } T \in \mathcal{T}_h, \int_\Omega q_h dx = 0\}.$$

It is well–known that the spaces X_h and M_h satisfy the Babuška–Brezzi condition

$$\sup_{v_h \in X_h} \frac{(\text{div} v_h, p_h)}{\|\nabla v_h\|} \geq c_0 \|p_h\| \quad \forall p_h \in M_h.$$

The discrete analogue of V is given by

$$V_h := \{v_h \in X_h \,|\, (\text{div}\, v_h, q_h) = 0 \text{ for all } q_h \in M_h\}.$$

The L^2–projection $P_h : L^2(\Omega)^n \to V_h$ is defined by

$$(v, \phi_h) = (P_h v, \phi_h), \qquad \forall \phi_h \in V_h$$

and satisfies (cf. [7, I,Lemma 4.3])

$$\|v - P_h v\| + h\|\nabla(v - P_h v)\| \le ch^2 \|D^2 v\| \qquad \forall v \in V \cap H^2(\Omega)^n. \tag{2.12}$$

Furthermore, there exists an interpolation operator $i_h : L^2_0(\Omega) \to M_h$ with the property that

$$\|p - i_h p\| \le ch\|\nabla p\| \qquad \forall p \in L^2_0(\Omega) \cap H^1(\Omega). \tag{2.13}$$

Finally, let $t_n = n\Delta t, n = 0, ..., N$, where $\Delta t > 0$ is the time step and $N\Delta t = T$.

The discretization of (1.4) then takes the form: given $u \in Y$ and $y_h^0 = P_h y_0$, find $(y_h^n, p_h^n) \in X_h \times M_h$, $1 \le n \le N$ such that

$$\begin{aligned}
&(2.14)\quad (\frac{y_h^{n+1} - y_h^n}{\Delta t}, \phi_h) + \nu(\nabla y_h^{n+1}, \nabla \phi_h) - (\operatorname{div} \phi_h, p_h^{n+1}) &=& \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (u(t), \phi_h) dt \\
&(2.15)\quad -(\operatorname{div} y_h^{n+1}, q_h) &=& 0
\end{aligned}$$

for all $\phi_h \in X_h$, $q_h \in M_h$ and $0 \le n \le N-1$. This problem has a unique solution, which we extend as a function $y_h(u) : [0,T] \to X_h$ as follows: $y_h(u)(0) = y_h^0$ and

$$y_h(u)(t) := y_h^n, \qquad t \in (t_{n-1}, t_n],\ n = 1, ..., N.$$

Just as in the continuous case we write $y_h(u) = S_h u + \bar{y}_h$, where $S_h : Y \to Y$ assigns to each u the solution of (2.14), (2.15) corresponding to $y_h^0 = 0$ and $\bar{y}_h$ solves (2.14), (2.15) for $\bar{y}_h^0 = P_h y_0$, $u = 0$. Again we think of $S_h u$ and $\bar{y}_h$ as piecewise constant extensions of the values at t_n.

Next, we introduce a linear operator $R_h : Y \to Y$, which will approximate R. Given $f \in Y$ and $w_h^N = 0$ we denote by $(w_h^n, \xi_h^n) \in X_h \times M_h$, $0 \le n \le N-1$, the unique solution of the discrete backward Stokes problem:

$$\begin{aligned}
&(2.16)\quad -(\frac{w_h^{n+1} - w_h^n}{\Delta t}, \phi_h) + \nu(\nabla w_h^n, \nabla \phi_h) - (\operatorname{div} \phi_h, \xi_h^n) &=& \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (f(t), \phi_h) dt \\
&(2.17)\quad -(\operatorname{div} w_h^n, q_h) &=& 0
\end{aligned}$$

for all $\phi_h \in X_h$, $q_h \in M_h$ and $n = N-1, ..., 0$. We then set

$$(R_h f)(t) := w_h^{n-1}, \qquad t \in [t_{n-1}, t_n),\ n = 1, ..., N$$

and $(R_h f)(T) := w_h^N = 0$. Inserting $\phi_h = w_h^n$ into (2.16) one deduces

$$-\frac{1}{2}\|w_h^{n+1}\|^2 + \frac{1}{2}\|w_h^n\|^2 + \frac{1}{2}\|w_h^{n+1} - w_h^n\|^2 + \nu\Delta t\|\nabla w_h^n\|^2 = \int_{t_n}^{t_{n+1}} (f(t), w_h^n) dt,$$

and a discrete Gronwall argument yields

$$\max_{n=0,...,N} \|w_h^n\|^2 + \sum_{n=0}^{N} \Delta t\, \|\nabla w_h^n\|^2 \le C \int_0^T \|f\|^2 dt,$$

in particular,

$$\|R_h f\|_Y \le C\|f\|_Y \qquad \forall f \in Y \text{ uniformly in } h,\ \Delta t. \tag{2.18}$$

We now define an approximation $F_h : Y \to Y$ to F by

$$F_h(u) := R_h\big(y_h(u) - z\big) + \alpha u. \tag{2.19}$$

Definition 2.1. We call $u_h^* \in Y$ a discrete solution of the optimality system, if $F_h(u_h^*) = 0$.

Even though F_h is defined on Y, the discrete solution satisfies $u_h^* = -\frac{1}{\alpha}\lambda_h^*$, where $\lambda_h^* = R_h(y_h(u_h^*) - z)$, so that u_h^* turns out to be implicitly discretized in terms of the adjoint variable.
For our error analysis and practical computations it is more convenient to rewrite $F_h(u_h^*) = 0$ in the same way as in the continuous case. Clearly, we have

$$\langle F_h', \phi\rangle = R_h S_h \phi + \alpha\phi, \qquad \phi \in Y \tag{2.20}$$

and we can rewrite the equation $F_h(u_h^*) = 0$ as

$$\langle F_h', u_h^*\rangle = R_h(z - \bar{y}_h). \tag{2.21}$$

Furthermore, F_h' is selfadjoint with

$$\int_0^T \int_\Omega \langle F_h', \phi\rangle \cdot \phi dxdt = \int_0^T \int_\Omega \big(|S_h\phi|^2 + \alpha|\phi|^2\big) dxdt \tag{2.22}$$

for all $\phi \in Y$. Similarly as above one then argues that F_h' is invertible and also that

$$\|(F_h')^{-1}\|_{L(Y,Y)} \le \frac{1}{\alpha} \qquad \text{uniformly in } h, \Delta t. \tag{2.23}$$

The discrete solution u_h^* is then simply given by

$$u_h^* = (F_h')^{-1} R_h(z - \bar{y}_h). \tag{2.24}$$

Our main result can now be formulated as follows:

Theorem 2.2. There exists a unique discrete solution u_h^* in the sense of Definition 2.1 and we have the estimate

$$\|u^* - u_h^*\|_Y \le C\big(\Delta t + h^2\big). \tag{2.25}$$

The proof of (2.25) is presented in the next section, while numerical computations in §4 based on (2.21) will confirm our result.

3 Proof of Theorem 2.2

We start by deriving an estimate for $Su - S_h u$.

Lemma 3.1. We have for all $u \in Z$

$$\|Su - S_h u\|_Y \le C\big(\Delta t + h^2\big)\|u\|_Z. \tag{3.1}$$

Proof. The proof uses techniques employed in [7], which we include for the convenience of the reader. Recall that $v = Su$ satisfies $v(0) = 0$ in Ω, $v = 0$ on $\partial\Omega \times (0,T)$ as well as

$$\begin{aligned} v_t - \nu\Delta v + \nabla p &= u(t) \quad \text{in } \Omega \times (0,T) \\ -\operatorname{div} v &= 0 \quad\quad\;\; \text{in } \Omega \times (0,T), \end{aligned}$$

while we have for $v_h = S_h u$: $v_h^0 = 0$ in Ω and

$$\begin{aligned} \Big(\frac{v_h^{n+1} - v_h^n}{\Delta t}, \phi_h\Big) + \nu(\nabla v_h^{n+1}, \nabla\phi_h) - (\operatorname{div}\phi_h, p_h^{n+1}) &= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (u(t), \phi_h)dt \\ -(\operatorname{div} v_h^{n+1}, q_h) &= 0 \end{aligned}$$

for all $\phi_h \in X_h$ and $q_h \in M_h$. Since

$$\int_{t_n}^{t_{n+1}} u(t)dt = \int_{t_n}^{t_{n+1}} \big(v_t - \nu\Delta v + \nabla p\big)dt = v(t_{n+1}) - v(t_n) + \int_{t_n}^{t_{n+1}} \big(-\nu\Delta v + \nabla p\big)dt \quad \text{in } L^2(\Omega)^n,$$

the difference $e := v - v_h$ satisfies $e^0 = v(0) - v_h^0 = 0$ and

$$\Big(\frac{e^{n+1} - e^n}{\Delta t}, \phi_h\Big) + \nu(\nabla e^{n+1}, \nabla\phi_h) = \langle g, \phi_h\rangle \tag{3.2}$$

$$-(\operatorname{div} e^{n+1}, q_h) = 0 \tag{3.3}$$

for all $\phi_h \in X_h$, $q_h \in M_h$, where

$$\langle g, \phi_h\rangle = (\operatorname{div}\phi_h, p_h^{n+1}) + \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (\phi_h, \nabla p(t))dt + \frac{\nu}{\Delta t}\int_{t_n}^{t_{n+1}} (\nabla(v(t_{n+1}) - v(t)), \nabla\phi_h)dt. \tag{3.4}$$

We insert $\phi_h = P_h e^{n+1}$ into (3.2) and multiply by Δt. Observing that $(e^{n+1} - e^n, P_h e^{n+1}) = (P_h e^{n+1} - P_h e^n, P_h e^{n+1})$ and using integration by parts together with (3.3), (2.12) and (2.13) we obtain

$$
\begin{aligned}
&\frac{1}{2}\|P_h e^{n+1}\|^2 - \frac{1}{2}\|P_h e^n\|^2 + \frac{1}{2}\|P_h e^{n+1} - P_h e^n\|^2 + \nu\Delta t\,\|\nabla P_h e^{n+1}\|^2 \\
&= \nu\Delta t\big(\nabla P_h e^{n+1}, \nabla(P_h v^{n+1} - v^{n+1})\big) + \Delta t\langle g, P_h e^{n+1}\rangle \\
&\le \nu\Delta t\,\|\nabla P_h e^{n+1}\|\,\|\nabla(v^{n+1} - P_h v^{n+1})\| + \Big|\int_{t_n}^{t_{n+1}} \big(P_h e^{n+1} - e^{n+1}, \nabla p(t)\big)dt\Big| \\
&\quad + \Big|\int_{t_n}^{t_{n+1}} \big(\operatorname{div} e^{n+1}, p(t) - i_h p(t)\big)dt\Big| + \nu\int_{t_n}^{t_{n+1}} \|\nabla\big(v(t_{n+1}) - v(t)\big)\|\,\|\nabla P_h e^{n+1}\|dt \\
&\le C\Delta t\, h\|v^{n+1}\|_{H^2}\|\nabla P_h e^{n+1}\| + C\Delta t\,\|v^{n+1} - P_h v^{n+1}\| \sup_{0\le t\le T}\|\nabla p(t)\| \\
&\quad + C\Delta t\, h \sup_{0\le t\le T}\|\nabla p(t)\|\,\|\nabla e^{n+1}\| + C\Delta t\,\|\nabla P_h e^{n+1}\|\int_{t_n}^{t_{n+1}}\|\nabla v_t\|dt \\
&\le \delta\,\Delta t\|\nabla P_h e^{n+1}\|^2 + C_\delta h^2\Delta t\big(\sup_{0\le t\le T}\|\nabla p(t)\|^2 + \sup_{0\le t\le T}\|v(t)\|_{H^2}^2\big) \\
&\quad + C_\delta(\Delta t)^2\int_{t_n}^{t_{n+1}}\|\nabla v_t\|^2 dt.
\end{aligned}
$$

If we choose δ small enough and sum from $n = 0, ..., l$ $(l \le N-1)$ we deduce taking into account (2.2) and $e^0 = 0$

$$
\begin{aligned}
&\|P_h e^{l+1}\|^2 + \Delta t\sum_{n=0}^{l+1}\|\nabla P_h e^n\|^2 \\
&\le C\big((\Delta t)^2 + h^2\big)\Big(\sup_{0\le t\le T}\|\nabla p(t)\|^2 + \sup_{0\le t\le T}\|v(t)\|_{H^2}^2 + \int_0^T\|\nabla v_t\|^2 dt\Big) \\
&\le C\big((\Delta t)^2 + h^2\big)\|u\|_Z^2.
\end{aligned}
$$

Combining this estimate with (2.12) and (2.2) we finally deduce

$$
\max_{n=0,...N}\|e^n\|^2 + \Delta t\sum_{n=0}^{N}\|\nabla e^n\|^2 \le C\big((\Delta t)^2 + h^2\big)\|u\|_Z^2. \tag{3.5}
$$

The remainder of the proof is standard and consists in the solution of a dual problem, i.e. find (r^n, ξ^n), $0 \le n \le N-1$ such that

$$
\begin{aligned}
-\frac{r^{n+1} - r^n}{\Delta t} - \nu\Delta r^n + \nabla\xi^n &= e^{n+1} && \text{in } \Omega \\
-\mathrm{div} r^n &= 0 && \text{in } \Omega \\
r^n &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

for $n = N-1, ..., 0$ with $r^N = 0$ in Ω. The solution satisfies $(r^n, \xi^n) \in (V \cap H^2(\Omega)^n) \times (L_0^2(\Omega) \cap H^1(\Omega))$ and it is not difficult to show that

$$
\max_{n=0,...,N}\|r^n\|_{H^1}^2 + \Delta t\sum_{n=0}^{N}\big(\|r^n\|_{H^2}^2 + \|\nabla\xi^n\|^2\big) \le C\Delta t\sum_{n=0}^{N}\|e^n\|^2. \tag{3.6}
$$

Using summation by parts, (3.2) and (3.3) we obtain

$$
\begin{aligned}
&\sum_{n=0}^{N-1} \|e^{n+1}\|^2 \\
&= \sum_{n=0}^{N-1} \big(e^{n+1}, -\frac{r^{n+1}-r^n}{\Delta t} - \nu\Delta r^n + \nabla\xi^n\big) \\
&= \sum_{n=0}^{N-1} \big((\frac{e^{n+1}-e^n}{\Delta t}, r^n) + \nu(\nabla e^{n+1}, \nabla r^n) - (\mathrm{div} e^{n+1}, \xi^n)\big) \\
&= \sum_{n=0}^{N-1} \Big((\frac{e^{n+1}-e^n}{\Delta t}, r^n - P_h r^n) + \nu(\nabla e^{n+1}, \nabla(r^n - P_h r^n)) - (\mathrm{div} e^{n+1}, \xi^n - i_h\xi^n) + \langle g, P_h r^n\rangle\Big) \\
&= \sum_{n=0}^{N-1} \Big((\frac{v^{n+1}-v^n}{\Delta t}, r^n - P_h r^n) + \nu(\nabla e^{n+1}, \nabla(r^n - P_h r^n)) - (\mathrm{div} e^{n+1}, \xi^n - i_h\xi^n) + \langle g, P_h r^n\rangle\Big)
\end{aligned}
$$

where the last identity follows from the fact that $\big(\frac{v_h^{n+1}-v_h^n}{\Delta t}, r^n - P_h r^n\big) = 0$. If we argue in a similar way as above we may estimate

$$
\begin{aligned}
&|\langle g, P_h r^n\rangle| \\
&\quad\le \frac{1}{\Delta t}\Big|\int_{t_n}^{t_{n+1}} \big(r^n - P_h r^n, \nabla p(t)\big)dt\Big| + \frac{\nu}{\Delta t}\Big|\int_{t_n}^{t_{n+1}} \big(\nabla(v(t_{n+1}) - v(t)), \nabla P_h r^n\big)\Big| \\
&\quad\le \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \|r^n - P_h r^n\|\,\|\nabla p(t)\|dt + \nu\|\nabla P_h r^n\|\int_{t_n}^{t_{n+1}} \|\nabla v_t\|dt \qquad (3.7) \\
&\quad\le Ch^2\|r^n\|_{H^2}\sup_{0\le t\le T}\|\nabla p(t)\| + C\sqrt{\Delta t}\Big(\int_{t_n}^{t_{n+1}} \|\nabla v_t\|^2 dt\Big)^{\frac{1}{2}}\big(\|\nabla r^n\| + h\|r^n\|_{H^2}\big).
\end{aligned}
$$

Let us insert (3.7) into the above identity and multiply by Δt. Recalling (3.5), (3.6) and

(2.2) we finally deduce

$$
\begin{aligned}
&\sum_{n=0}^{N-1} \Delta t\, \|e^{n+1}\|^2 \\
\le\ & Ch^2 \sum_{n=0}^{N-1} \sqrt{\Delta t}\, \|r^n\|_{H^2} \Big(\int_{t_n}^{t_{n+1}} \|v_t\|^2 dt \Big)^{\frac{1}{2}} + Ch \sum_{n=0}^{N-1} \Delta t\, \|\nabla e^{n+1}\| \big(\|r^n\|_{H^2} + \|\nabla \xi^n\| \big) \\
+\ & C\Delta t \sum_{n=0}^{N-1} \sqrt{\Delta t} \big(\|\nabla r^n\| + h \|r^n\|_{H^2} \big) \Big(\int_{t_n}^{t_{n+1}} \|\nabla v_t\|^2 dt \Big)^{\frac{1}{2}} \\
+\ & Ch^2 \sup_{0 \le t \le T} \|\nabla p(t)\| \sum_{n=0}^{N-1} \Delta t\, \|r^n\|_{H^2} \\
\le\ & Ch \Big(\sum_{n=0}^{N-1} \Delta t\, \|\nabla e^{n+1}\|^2 \Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{N-1} \Delta t \big(\|r^n\|^2_{H^2} + \|\nabla \xi^n\|^2 \big) \Big)^{\frac{1}{2}} \\
+\ & Ch^2 \Big(\int_0^T \|v_t\|^2 dt \Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{N-1} \Delta t\, \|r^n\|^2_{H^2} \Big)^{\frac{1}{2}} + Ch^2 \sup_{0 \le t \le T} \|\nabla p(t)\| \sqrt{T} \Big(\sum_{n=0}^{N-1} \Delta t\, \|r^n\|^2_{H^2} \Big)^{\frac{1}{2}} \\
+\ & C\Delta t \Big(\int_0^T \|\nabla v_t\|^2 dt \Big)^{\frac{1}{2}} \Big(\sqrt{T} \max_{n=0,\dots,N} \|r^n\|_{H^1} + h \big(\sum_{n=0}^{N-1} \Delta t\, \|r^n\|^2_{H^2} \big)^{\frac{1}{2}} \Big) \\
\le\ & C(h^2 + \Delta t) \|u\|_Z \Big(\sum_{n=0}^{N} \Delta t\, \|e^n\|^2 \Big)^{\frac{1}{2}}.
\end{aligned}
$$

This concludes the proof of the lemma. $\square$

In an analogous way one can prove that for all $f \in Z$

$$\|\bar{y} - \bar{y}_h\|_Y \le C\big(\Delta t + h^2\big) \|y_0\|_{H^2} \tag{3.8}$$

$$\|Rf - R_h f\|_Y \le C\big(\Delta t + h^2\big) \|f\|_Z \tag{3.9}$$

We are now in position to prove Theorem 2.2. Taking the difference of (2.11) and (2.24) gives

$$
\begin{aligned}
u^* - u_h^* &= (F_h')^{-1} \Big(F_h' (F')^{-1} \big(R(z - \bar{y}) \big) - R_h (z - \bar{y}_h) \Big) \\
&= (F_h')^{-1} \Big((F_h' - F')(F')^{-1} \big(R(z - \bar{y}) \big) + (R - R_h)(z - \bar{y}) + R_h (\bar{y}_h - \bar{y}) \Big).
\end{aligned}
$$

Let us abbreviate $\tilde{v} := (F')^{-1} \big(R(z - \bar{y}) \big)$. In view of (2.9) we may write

$$\alpha \tilde{v} + RS\tilde{v} = R(z - \bar{y}),$$

which implies that $\tilde{v} \in Z$. Furthermore, (2.6) shows

$$\|\tilde{v}\|_Z \le \frac{1}{\alpha} \|RS\tilde{v}\|_Z + \frac{1}{\alpha} \|R(z - \bar{y})\|_Z \le C\|\tilde{v}\|_Y + C\|z - \bar{y}\|_Y \le C\|z - \bar{y}\|_Y. \tag{3.10}$$

Finally, (2.23), (2.18), (3.1), (3.8), (3.9) and (3.10) yield

$$\begin{aligned}
\|u^* - u_h^*\|_Y &\leq C\|R_h S_h \tilde{v} - RS\tilde{v}\|_Y + C\|(R - R_h)(z - \bar{y})\|_Y + \|R_h(\bar{y}_h - \bar{y})\|_Y \\
&\leq C\|R_h\|_{L(Y,Y)}\|S\tilde{v} - S_h\tilde{v}\|_Y + C\|(R - R_h)S\tilde{v}\|_Y + C(\Delta t + h^2)\big(\|z - \bar{y}\|_Z + \|y_0\|_{H^2}\big) \\
&\leq C(\Delta t + h^2)\big(\|\tilde{v}\|_Z + \|S\tilde{v}\|_Z + \|z\|_Z + \|y_0\|_{H^2}\big) \\
&\leq C(\Delta t + h^2)(\|z\|_Z + \|y_0\|_{H^2})
\end{aligned}$$

proving the estimate (2.25).

Remark 3.1. As already mentioned in the Introduction, our approach can be used to derive error estimates in a more general setting involving the Navier–Stokes equations (in 2D) as state equations and cost functionals of the form (1.2). It should be noted that in this case the corresponding mapping $F(u)$ is in general not affine and might be quite complicated, so that one cannot expect the equation $F(u) = 0$ to have a unique solution. Thus, it is only possible to obtain local results: a solution u_h^* of the equation $F_h(u) = 0$ exists in a neighborhood of a regular solution u^* of $F(u) = 0$ and the difference $u^* - u_h^*$ can be estimated. This is achieved by applying a quantified version of Newton's theorem, see [3] for details.

4 Numerical results

In practice, (2.21) is used in order to find the discrete solution u_h^*. Let us introduce the finite–dimensional subspace

$$Y_h := \{u : \Omega \times [0,T] \to \mathbb{R} \,|\, u_{|[t_j,t_{j+1})} \in X_h, j = 0, ..., N-1;\ u(T) = 0\}$$

of Y. We may identify Y_h with $\mathbb{R}^{N \cdot M}$ as follows:

$$Y_h \ni u_{h|[t_j,t_{j+1})} = \sum_{i=1}^{M} u_i^{(j)} \psi_i, \quad 0 \leq j \leq N-1 \quad \longleftrightarrow \quad \underline{u} = (u^{(0)}, ..., u^{(N-1)})^t \in \mathbb{R}^{N \cdot M},$$

where $\psi_1, ..., \psi_M$ is a basis of X_h. Note that $u_h^* \in Y_h$ and since $F_h'(Y_h) \subset Y_h$, $R_h(z - \bar{y}_h) \in Y_h$ it is sufficient to consider $F'_{h|Y_h}$. Using the above equivalence this leads to the linear system

$$A\underline{u}^* = \underline{b}, \tag{4.1}$$

where A is symmetric and positive definite and $\underline{b} = (b^{(0)}, ..., b^{(N-1)})^t$ with $R_h(z - \bar{y}_h)_{|[t_j,t_{j+1})} = \sum_{i=1}^M b_i^{(j)} \psi_i$. The system (4.1) is solved with the help of the conjugate gradient method with starting value $\underline{u} = 0$. Each step requires a multiplication by A, which is carried out as follows: given $\underline{u} = (u^{(0)}, ..., u^{(N-1)})^t \in \mathbb{R}^{N \cdot M}$, solve

$$\begin{aligned}
\Big(\frac{v_h^{n+1} - v_h^n}{\Delta t}, \phi_h\Big) + \nu(\nabla v_h^{n+1}, \nabla \phi_h) - (\operatorname{div} \phi_h, p_h^{n+1}) &= \sum_{i=1}^{M} u_i^{(n)} (\psi_i, \phi_h) \\
-(\operatorname{div} v_h^{n+1}, q_h) &= 0
\end{aligned}$$

for all $\phi_h \in X_h, q_h \in M_h$, $0 \le n \le N-1$ with $v_h^0 = 0$. Afterwards solve

$$\begin{aligned} -(\frac{\lambda_h^{n+1}-\lambda_h^n}{\Delta t}, \phi_h) + \nu(\nabla\lambda_h^n, \nabla\phi_h) - (\operatorname{div}\phi_h, \xi_h^n) &= (v_h^{n+1}, \phi_h) \\ -(\operatorname{div}\lambda_h^n, q_h) &= 0 \end{aligned}$$

for all $\phi_h \in X_h$, $q_h \in M_h$, $n = 0, ..., N-1$ with $\lambda_h^N = 0$. If we write $\lambda_h^n = \sum_{i=1}^M \lambda_i^{(n)}\psi_i$ then

$$A\underline{u} = (\lambda^{(0)}, ..., \lambda^{(N-1)})^t + \alpha\underline{u}.$$

For further computational details see [8, Chapter 6].
In our computations we chose $\Omega = (0,1)\times(0,1)$ and $T = 0.1$. The initial value of the uncontrolled flow is given by

$$y_0(x) \;=\; e\begin{bmatrix} (\cos 2\pi x_1 - 1)\sin 2\pi x_2 \\ -(\cos 2\pi x_2 - 1)\sin 2\pi x_1 \end{bmatrix}$$

with e denoting the Euler number, and the desired state z is chosen as $0 \in \mathbb{R}^2$. We present calculations for $\alpha = 1.e0$, $\alpha = 1.e-2$, $k = 1$ and $\nu = 1/\mathrm{Re} = 1/10$. To verify the estimate (2.25) numerically we concentrate on the contribution from spatial discretization. This can be justified by numerical experiments which show that the major part of the total error results from the finite element discretization and not from time stepping. For what follows let h denote the finite element grid size. We introduce the experimental order of convergence for positive error functionals $E(h)$ with $h > 0$ as follows: Given two grid sizes $h_1 \neq h_2$, let

$$\mathrm{EOC} := \frac{\ln E(h_1) - \ln E(h_2)}{\ln h_1 - \ln h_2}.$$

It follows from this definition that if $E(h) = \mathcal{O}(h^\gamma)$ $(h \to 0)$ then EOC $\approx \gamma$.

Example 4.1. In Table 1 the numerical results for $\alpha = 1.e0$ are presented. As a substitute for the exact solution u^* of the control problem we choose the numerical solution computed on a grid containing 65536 triangles with 131585 velocity and 33025 pressure nodes. The number of unknowns in the discretized control problem therefore has the magnitude 8.5×10^6, including the primal, adjoint and control variables. The cost of the uncontrolled flow on this grid is $\hat{J}(0) = 3.29555403e-1$. We set

$$E_h^J := |J(y^*, u^*) - J(y_h^*, u_h^*)|, \quad E_h^u := \left(\int_0^T \|u^* - u_h^*\|^2 dt\right)^{\frac{1}{2}} \text{ and } E_h^y := \sup_{t\in[0,T]} \|y^* - y_h^*\|$$

and define

$$\mathrm{EOC}_h^J := \frac{\ln E_h^J - \ln E_{h/2}^J}{\ln 2},$$

EOC_h^u and EOC_h^y analogously. Table 1 shows the values of the cost functional and the error functionals on different grids. Table 2 contains the results for the experimental

Grid size h	$J(y_h^*, u_h^*)$	E_h^J	E_h^u	E_h^y
1/2	3.21458015e-1	7.366013e-3	5.0060187e-3	3.42701017e-1
1/4	3.26197855e-1	2.626173e-3	2.46836485e-3	2.40351324e-1
1/16	3.28649405e-1	1.74623e-4	6.2461678e-4	6.14284855e-2
1/32	3.28812670e-1	1.1352e-5	1.56829874e-4	1.54192716e-2
1/64	3.28823300e-1	7.2e-07	3.90447133e-5	3.8379829e-3
1/128	3.28823985e-1	4.3e-08	9.21571074e-6	9.03339275e-4
1/256	3.28824028e-1	0.e-0	0.e-0	0e-0

Table 1: Values of cost and error functionals on different levels, $\alpha = 1.e - 0$

Grid size h	EOC_h^J	EOC_h^u	EOC_h^y
1/2	-	-	-
1/4	1.49	1.02	0.51
1/16	3.91	1.98	1.97
1/32	3.94	1.99	1.99
1/64	3.98	2.01	2.01
1/128	4.07	2.08	2.09
1/256	-	-	-

Table 2: Experimental order of convergence, $\alpha = 1.e - 0$

order of convergence. The results confirm the estimate (2.25). A proof that the order of convergence for E_h^y is 2 is sketched in [3]. We observe convergence of order 4 for E_h^J which may be justified by the fact that $\hat{J}'(u^*) = 0$.

Example 4.2. We now investigate the case $\alpha = 1.e - 2$ which results in much stronger control action than in Example 4.1. A comparison of the evolution of the cost for the uncontrolled and the controlled flow on a grid containing 16384 triangles, 33025 velocity and 8321 pressure nodes is shown in Fig. 1. The numerical solution of the optimal control problem on this grid is used here as reference solution. Tables 3 and 4 show the same quantities as Tables 1 and 2, respectively. The results indicate insensitivity of the behaviour of convergence with respect to the parameter α. Again, the estimate (2.25) is confirmed.

Let us close this section with mentioning that the numerical computation of the reference control in Example 4.1 takes several hours on a PC with 1GH Pentium III^{TM} CPU.

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Grid size h	$J(y_h^*, u_h^*)$	E_h^J	E_h^u	E_h^y
1/2	2.66113311e-1	5.8086777e-3	371449068e-1	3.20580596e-1
1/4	2.69856644e-1	2.065344e-3	1.91338069e-1	2.28803060e-1
1/16	2.71785002e-1	1.36986e-4	4.84453105e-2	5.84771616e-2
1/32	2.71913156e-1	8.832e-6	1.21008313e-2	1.46043156e-2
1/64	2.71921460e-1	5.28e-7	2.86114730e-3	3.43990516e-3
1/128	2.71921988e-1	0.e-0	0.e-0	0.e-0

Table 3: Values of cost and error functionals on different levels, $\alpha = 1.e - 2$

Grid size h	EOC_h^J	EOC_h^u	EOC_h^y
1/2	-	-	-
1/4	1.49	0.96	0.48
1/16	3.91	1.98	1.97
1/32	3.96	2.00	2.00
1/64	4.06	2.08	2.09
1/128	-	-	-

Table 4: Experimental order of convergence, $\alpha = 1.e - 2$

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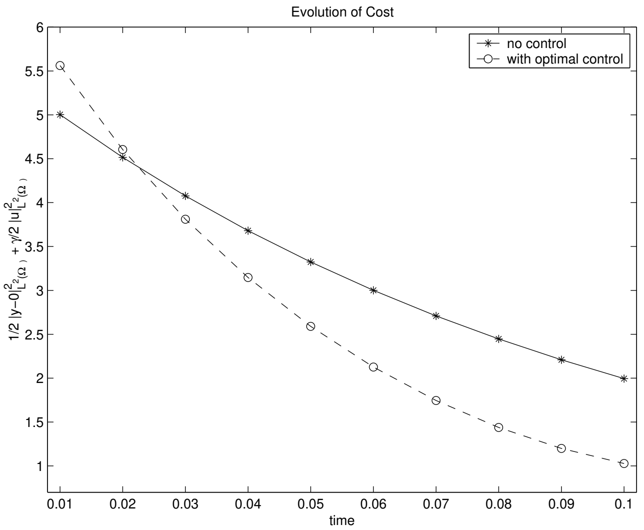

Figure 1: Evolution of the cost functional for $t \in [0, T]$: Comparison of uncontrolled and controlled cost

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