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Abstract

Instantaneous control is applied to the control of the instationary Navier-Stokes system. This control technique is closely related to receding horizon control and allows for an interpretation as suboptimal closed loop controller, whose parameters may be adjusted so as to stabilize the nonlinear equation under consideration. Besides convergence analysis for the distributed control case numerical examples for the continuous and discrete-in-time control laws are presented.

Keywords: Optimal control, Instantaneous control, Closed loop control, Navier-Stokes equations.

1 Introduction

This research is devoted to the study of instantaneous control applied to the instationary Navier-Stokes system. As a model the following distributed optimal control problem is considered;

$$
\textbf{(P)} \qquad \left\{ \begin{array}{rcll}
\min J(y,u) & = & \frac{1}{2}\int\limits_0^T \int\limits_\Omega |y-z|^2\,dxdt \; + \; \frac{\alpha}{2}\int\limits_0^T |u|^2_{\mathcal{U}}\,dt & \\
\text{s.t.} & & & \\
y_t - \nu\Delta y + (y\nabla)y + \nabla p & = & \mathcal{B}u & \text{in } (0,T)\times\Omega, \\
-\text{div } y & = & 0 & \text{in } (0,T)\times\Omega, \\
y & = & 0 & \text{on } (0,T)\times\partial\Omega, \\
y(0) & = & \phi & \text{in } \Omega.
\end{array} \right.
$$

Here, the control target is to match the given desired state z in the L^2-sense by adjusting the body force $\mathcal{B}u$. In this context $\mathcal{B}$ denotes an abstract control extension operator and $\Omega \subset \mathbb{R}^2$ denotes a bounded domain. The first term in the cost functional measures the physical objective, the second one the size of the control, where the parameter $\alpha > 0$ plays the role of a weight. Instantaneous control is a suboptimal control approach which yields controls whose quality compares to that of the optimal ones, but which can be computed at significantly lower numerical costs. Moreover, the method allows an interpretation as closed loop control approach. The main aim of this work is to analyze the stability properties of this suboptimal control approach in terms of the parameters involved in the method.

If the control goal is primarily to match the desired state without considering the minimization of the energy and work linear body force feedback control can be used instead of optimal control [10, 20, 21]. In this context instantaneous control serves a dual purpose – to construct a closed loop feedback control law which steers the state to the desired one, and to compute an open loop control policy to approximate the optimal open loop strategy at considerable control costs. The method works as follows. The uncontrolled Navier-Stokes equations are discretized with respect to time. Then, at selected time slices an instantaneous version of the cost functional is approximately minimized w.r.t. a stationary quasi-Navier-Stokes system, whose structure depends on the chosen discretization method. The control obtained is used to steer the system to the next time slice, where the procedure is repeated.

Instantaneous control is closely related to receding horizon control (rhc) or model predictive control (mpc) with finite time horizon [9, 24, 25]. Its difference to (rhc) and (mpc) can best be explained in terms of a chess analogy proposed by Bewley et al. in [3]: (rhc) and (mpc) compute the next n, say, optimal moves and apply the first one to steer the system to the next time slice. Instantaneous control computes an approximation to the next optimal move by applying exactly one step of the gradient method to the solution of the related instantaneous optimal control problem and applies the approximate control to advance to the next time slice. In terms of the chess analogy applying instantaneous control would correspond to playing Blitz. Clearly, it can not be expected that the instantaneous control technique is a candidate to stabilize arbitrary dynamical systems, compare the citations above. However, as shall be shown for the instationary Navier-Stokes equations, it is capable of enhancing the stability properties of dissipative systems. In its nature instantaneous control is a discrete-in-time feedback control approach and, as shall be shown, can be interpreted as the stable time discretization of a continuous closed-loop control law, compare also [14, 13].

The optimal control problem **(P)**, among other things is investigated by Abergel and Temam in [1], where optimality systems and existence of optimal controls are discussed. As solution algorithm a gradient technique is proposed, see also [2, 11]. In [16] Kunisch and the author investigate second order methods for distributed optimal control of the instationary Navier-Stokes equations, including first order necessary and second order sufficient optimality conditions.

Instantaneous control among other things was applied to control the Burgers equation with stochastic forcing in [6]. Thereafter this control technique was successfully applied to compute suboptimal controls for a great variety of fluid mechanical control problems [4, 5, 12, 15, 22, 23] and in the recent past also as real-time control approach to the cooling of steel [28]. As far as the author knows there are only few stability investigations for its application to the control of infinite-dimensional systems. For the Burgers equation the related analysis can be found in [17]. Stability analysis of the approach for finite-dimensional systems with distributed controls can be found in [14]. The analysis of special case of mpc of the Navier-Stokes equations can be found in [19].

In the present work the results obtained in [17] are extended to the instationary Navier-Stokes system in two space dimensions, compare also [13]. The main result of this works reads as follows.

Main result: Given a sufficiently smooth desired state z, instantaneous control of the instationary Navier-Stokes system can be regarded as time discretization of a closed loop feedback policy K which steers the system exponentially fast to z, i.e. with S denoting the Stokes operator and $b(y)$ the nonlinearity of the Navier-Stokes equations

$$y_t + \nu S y + b(y) = K(y),$$

the solution of this system satisfying $|y(t) - z(t)|_{H^1} \le ce^{-\gamma t}$ with some positive constants c and γ.

The paper is organized as follows. Section 2 is devoted to the study of **(P)**, which serves as model problem for the validation of instantaneous control. We briefly discuss existence and second order sufficiency for solutions of this control problem. Section 3 is devoted to the derivation of the instantaneous control approach and the formulation of the algorithm. In Section 4 we relate the discrete-in-time control approach to a continuous closed-loop control law and prove both stability of the continuous and the discrete controller. Finally, in Section 5 numerical examples are presented, which illustrate the theoretical results.

Throughout this work c and C denote global generic constants whose dependencies are mentioned when necessary.

2 Optimal control

For given $T > 0$ let $Q = (0,T) \times \Omega$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain. We set $V = \{v \in H_0^1(\Omega)^2,\ \text{div}\ v = 0\}$, $H = \text{clos}_{L^2(\Omega)^2}\{v \in C_0^\infty(\Omega)^2,\ \text{div}\ v = 0\}$ and identify the Hilbert space H with its dual H'. On H the common inner product is used, and V is endowed with the inner product

$$(\varphi, \psi)_V = (\varphi', \psi')_H \text{ for } \varphi, \psi \in V.$$

Moreover, with Z denoting a Hilbert space, $L^p(Z)$ $(1 \le p \le \infty)$ denotes the space of measurable abstract functions $\varphi : (0,T) \to Z$, which are p-integrable $(1 \le p < \infty)$, or essentially bounded on $(0,T)$ $(p = \infty)$, respectively.

As control space $L^2(\mathcal{U})$ is taken, where $\mathcal{U}$ denotes the Hilbert space of abstract controls. The space $\mathcal{U}$ also is identified with its dual. Furthermore, $\mathcal{B} : \mathcal{U} \to V'$

$$\mathcal{B} : \mathcal{U} \to V' \tag{1}$$

denotes the control extension operator which is assumed to be bounded. In order to formulate the weak form of the instationary Navier-Stokes equations let

$$W := W(V) = \{\varphi \in L^2(V) : \varphi_t \in L^2(V')\}$$

supplied with the common inner product. Further, introduce

$$b(u,v,w) := \int_\Omega (u\cdot\nabla)v\,w\,dx.$$

Then,

Lemma 2.1.

$$(2) \qquad b(u,v,w) \le C \begin{cases} |u|_H^{\frac12}|u|_V^{\frac12}|v|_H^{\frac12}|v|_V^{\frac12}|w|_V & \forall\, u,v,w\in V \\ |u|_H^{\frac12}|u|_V^{\frac12}|v|_V^{\frac12}|\Delta v|_H^{\frac12}|w|_H & \forall\, u\in V, v\in V\cap H^2(\Omega)^2, w\in H \\ |u|_H|v|_V|w|_H^{\frac12}|\Delta w|_H^{\frac12} & \forall\, u\in H, v\in V, w\in V\cap H^2(\Omega)^2 \\ |u|_H^{\frac12}|\Delta u|_H^{\frac12}|v|_V|w|_H & \forall\, u\in V\cap H^2(\Omega)^2, v\in V, w\in H, \end{cases}$$

with a positive constant C.

Proof. In [27]. □

Now, for $y\in L^2(V)$ the function $b(y)$ defined by

$$(3) \qquad \langle b(y),v\rangle_{V',V} := -b(y,y,v) \quad \text{for all } v\in V$$

is an element of V' for almost all $t\in(0,T)$ and $b(y)\in L^1(V)$ [26, Lemma 3.1].

For controls $u\in L^2(\mathcal{U})$ the solenoidal form of the Navier-Stokes equations reads: Find the state $y\in W$ such that

$$(4a) \qquad \begin{aligned} &\frac{d}{dt}(y(t),\varphi)_H + \nu(y(t),\varphi)_V \\ &\qquad = \langle b(y)+\mathcal{B}u(t),\varphi\rangle_{V',V} \text{ for all } \varphi\in V \text{ and a.e. } t\in[0,T] \end{aligned}$$

and

$$(4b) \qquad (y(0),\chi)_H = (\phi,\chi)_H \text{ for all } \chi\in H.$$

With Re denoting the Reynolds number, 1/Re=: $\nu>0$ is the viscosity parameter. The proof of the following theorem can be found in [26].

Theorem 2.2. For any $\phi\in H$ and for every control $u\in L^2(\mathcal{U})$ equations (4) admit a unique weak solution $y\in W$.

In this context recall that $W\hookrightarrow C(H)$, e.g. [8].

The cost function in **(P)** can be rewritten as

$$J(y,u) = \frac12\,|y-z|^2_{L^2(H)} + \frac{\alpha}{2}\,|u|^2_{L^2(\mathcal{U})},$$

where $z\in L^2(H)$ is a desired state and $\alpha>0$ is fixed. The minimization of the first term involving $y-z$ is the real goal. The second term weighs the control costs and may be interpreted as Tychonov regularization.

To shorten notation set

$$X = W \times L^2(\mathcal{U}) \text{ and } Y = L^2(V') \times H.$$

Introducing the operator $e : X \to Y$ by

$$e(y,u) = (e_1(y,u), e_2(y,u)) = \big(y_t - \nu\nabla y - b(y) - \mathcal{B}u, y(0) - \phi\big)$$

the optimal control problem **(P)** can be regarded as minimization problem with equality constraints:

$$\text{minimize } J(y,u) \quad \text{s.t.} \quad e(y,u) = 0.$$

Both J and e are infinitely continuously Fréchet-differentiable. In particular, the second derivatives are constant and thus, Lipschitz continuous [16]. In the following Fréchet-derivatives with respect to the variable $x = (y,u)$ are denoted by primes.

The Lagrangian corresponding to the above minimization problems is given by

$$L(y,u,\lambda,\mu) = J(y,u) + \langle e_1(y,u), \lambda\rangle_{L^2(V'),L^2(V)} + (e_2(y,u),\mu)_H.$$

For problem **(P)** the following results are proved in [13, 16].

Theorem 2.3. There exists a solution $(y_*, u_*, \lambda_*, \mu_*)$ to the first-order necessary optimality conditions

$$L'(y_*, u_*, \lambda_*, \mu_*) = 0.$$

If in addition $|y_* - z|_{L^2(H)}$ is sufficiently small, then the second-order sufficient optimality condition holds, i.e., there exists a constant $\gamma > 0$ such that

$$\langle L''(y_*, u_*, \lambda_*, \mu_*)v, v\rangle_{X',X} \geq \gamma\ |v|_X^2 \quad \text{for all } v \in X \text{ satisfying } e'(y_*,u_*)v = 0.$$

Hence, (y_*, u_*) is a local solution to **(P)**.

3 Instantaneous control strategy

Next the moving horizon approach is described which is the main subject of the investigations. It works as follows. The uncontrolled Navier-Stokes equations is discretized with respect to time. At every time step a now stationary tracking type control problem is approximately solved by applying exactly one gradient-step with step-size $\rho > 0$. The control obtained is used to steer state to the next time slice and, in the long run, hopefully towards the desired state. By patching together the controls one obtains a piece-wise in time constant control function defined over the time interval $(0,T)$. Clearly, it can not be expected that the control function obtained by the instantaneous control strategy is optimal, i.e., a solution to problem **(P)**. However, as several applications of the method to large control problems show, it provides good controls at low numerical costs [5, 6, 28] and, which is even more important, allows for an interpretation as closed-loop control law [5, 12, 14]. Of course, the instantaneous control approach depends on the time integration scheme used.

In the following a semi-implicit (implicit with respect to the linear, explicit with respect to the non-linear part) Euler scheme is investigated. It is also compared to fully implicit time integration. To anticipate the discussion note, that from the applications point of view semi-implicit time integration is advantageous.

For notational purposes let $P : L^2(\Omega)^2 \to H$ denote the Leray projector [7, Remark 1.10]. Then, the Stokes operator S is given by

$$S : \mathcal{D}(S) \subset H \to H, \quad S := -P\Delta, \quad \mathcal{D}(S) = H^2(\Omega)^2 \cap V.$$

Now define

$$A \ := \ \nu S.$$

In this setting the Navier-Stokes equations (4) for $u = 0$ in variational formulation may be rewritten as Burgers equation in the space V,

$$\begin{aligned} y_t + Ay &= b(y), \\ y(0) &= \phi, \end{aligned}$$

where the nonlinearity $b(y)$ is defined in (3). For $m \in \mathbb{N}$ an equidistant discretization of the time interval $(0,T)$ is defined by $h = \frac{T}{m}$ and $t_k = kh$, $k = 0, 1, \ldots, m$. As instantaneous versions of the cost functional J now take

$$J^k : V \times \mathcal{U} \to \mathbb{R}, \quad (y,u) \mapsto \frac{1}{2}\ |y - z^k|_H^2 + \frac{\alpha}{2}\ |u|_{\mathcal{U}}^2,$$

where

$$z^k = \frac{1}{h}\int_{t_k - \frac{h}{2}}^{t_k + \frac{h}{2}} z(s,\cdot)\, ds \tag{5}$$

and $z(t,\cdot) = 0$ for $t > T$. Finally, for $k = 1, \ldots, m$ and $i = 1, 2$ introduce the operators $e_i^k : V \times \mathcal{U} \to V'$ by

$$e_1^k(y,u) = (I + hA)y \ - \ hb(y^{k-1}) - y^{k-1} - \mathcal{B}u,$$

and

$$e_2^k(y,u) = (I + hA)y \ - \ hb(y) - y^{k-1} - \mathcal{B}u$$

and where y^{k-1} denotes the state at the previous time slice.

The instantaneous optimal control problem for the semi-implicit time integration is given by

$$(\mathbf{P}^k) \qquad \text{minimize } J^k(y,u) \text{ subject to } e_1^k(y,u) = 0 \text{ in } V',$$

where $y^0 = \phi$. The initial value ϕ now is required to be an element of the space V. For given y^{k-1} a pair (y^k, u^k) satisfies the subsidiary condition $e_1^k(y,u) = 0$ in V' if and only if

$$\nu h\ (y^k, \varphi)_V + (y^k, \varphi)_H = (y^{k-1}, \varphi)_H + \langle \mathcal{B}u^k + hb(y^{k-1}), \varphi \rangle_{V',V} \quad \forall\, \varphi \in V. \tag{6}$$

Since $\phi \in V$ holds, the right-hand side in this linear equation defines a bounded linear functional on V. Thus, for every $u^k \in \mathcal{U}$ Eq. (6) admits a unique solution $y^k \in V$ which satisfies the a-priori estimate

$$|y^k|_V \le \frac{C}{\nu h}\left(|y^{k-1}|_H + h|y^{k-1}|^2{}_V + |u^k|_{\mathcal{U}}\right).$$

Since J^k is quadratic and e_1^k is linear every problem $(\mathbf{P}^k)$, $k = 1, \ldots, m$, admits a unique solution $(y_*^k, u_*^k) \in V \times \mathcal{U}$. Furthermore, the unique Lagrange multiplier $\lambda_*^k \in V$ together with the solution (y_*^k, u_*^k) satisfies the first-order necessary optimality conditions (note that A is selfadjoint)

$$(I + hA)y = \mathcal{B}u + y^{k-1} + hb(y^{k-1}), \tag{7a}$$

$$(I + hA)\lambda = -(y - z^k), \tag{7b}$$

$$\alpha u - \mathcal{B}^\star\lambda = 0. \tag{7c}$$

Furthermore, the second-order sufficient optimality condition holds on the whole space $V \times \mathcal{U} \times V$. Hence, the solution (y_*^k, u_*^k) of (7) is the minimum for $(\mathbf{P}^k)$.

The optimal control problem $(\mathbf{P}^k)$ with respect to existence is equivalent to the unconstrained minimization of the functional

$$\hat{J}^k(u) \;=\; J^k(y(u), u)$$

over $\mathcal{U}$, where for a control $u \in \mathcal{U}$ the state $y(u) \in V$ is given as the unique solution to (6). The gradient of $\hat{J}^k$ at u is given by

$$\nabla\hat{J}^k(u) = \alpha u \;-\; \mathcal{B}^\star\lambda,$$

where for given u the function λ is obtained by first solving the linear quasi-Stokes problem (7a) for the state y, and then solving (7b) for λ.

Remark 3.1. If one would use implicit time integration in problem $(\mathbf{P}^k)$, i.e. in the subsidiary condition the operator e_1^k is replaced by e_2^k, the adjoint equation (7b) alters to

$$(I + hA)\lambda - b'(y)^\star\lambda = -(y - z^k). \tag{8}$$

Thus, in this case the gradient $\nabla\hat{J}^k(u)$ at a control u depends on the observation $y^k - z^k$, which occurs as right-hand side in the adjoint equation for the computation of the auxiliary variable λ, and also on the whole state y^k in terms of a coefficient which enters into the adjoint equation for λ in terms of the derivative of b. This structure remains valid even in the case of boundary observation, where the observation enters as boundary condition into the adjoint equation, but the whole state y^k again as a coefficient function. As a consequence, computation of gradient information for $\hat{J}^k$ can not be based on observations alone.

On the other hand the adjoint equation (7b) only depends on the observation $y^k - z^k$. Therefore, gradient information for the functional $\hat{J}^k$ is available utilizing the observations only. In the particular case of boundary observation no information of the state in the whole computational domain is needed at all.

3.1 The algorithm

In order to stabilize the Navier-Stokes equations at every time slice one gradient step to the solution of the control problem $(\mathbf{P}^k)$ is applied. The control obtained in this way is used to steer the system to the next time slice. With the gradient of the functional $\hat{J}^k$ available this procedure in algorithmical form can be formulated as follows.

Algorithm 3.2 (Instantaneous control).

1.) Set $y^0 = \phi$, $k = 0$ and $t^0 = 0$.

2.) Given an initial control u_0^k, solve

$$\begin{aligned} (I + hA)y &= y^k + hb(y^k) + \mathcal{B}u_0^k, \\ (I + hA)\lambda &= -(y - z^k). \end{aligned}$$

3.) Set $\nabla \hat{J}(u_0^k) = \alpha u_0^k - \mathcal{B}^\star \lambda$.

4.) Given $\rho > 0$, set $u^{k+1} = u_0^k - \rho \nabla \hat{J}(u_0^k)$.

5.) Solve

$$(I + hA)y^{k+1} = y^k + hb(y^k) + \mathcal{B}u^{k+1}.$$

6.) Set $t_{k+1} = t_k + h$, $k = k + 1$. If $t_k < T$ goto 2.

Here, $\hat{J} = \hat{J}^k$. The choice of the stepsize ρ in step 4.) of the algorithm is crucial. Since $(\mathbf{P}^k)$ is quadratic with linear constraints, the optimal choice ρ^* can be computed exactly and is given by

$$\rho^* = \frac{|\nabla \hat{J}(u)|_{\mathcal{U}}^2}{(D^2 \hat{J}(u) \nabla \hat{J}(u), \nabla \hat{J}(u))_{\mathcal{U}}}, \tag{9}$$

where

$$\begin{aligned} &\Big(D^2 \hat{J}(u) \nabla \hat{J}(u), \nabla \hat{J}(u)\Big)_{L^2(\Omega_c^2)} \\ &\quad = \alpha\, |\nabla \hat{J}(u)|_{\mathcal{U}}^2 \\ &\qquad + \Big(\mathcal{B}^\star (D_y e_2(y(u), u))^{-\star} I_H (D_y e_2(y(u), u))^{-1} \mathcal{B} \nabla \hat{J}(u), \nabla \hat{J}(u)\Big)_{\mathcal{U}}. \end{aligned}$$

Since $D^2 \hat{J}(u)$ is positive definite, the denominator in (9) never vanishes. Note, that the computation of ρ^* requires the solution of two auxiliary linear problems, and ρ^* is bounded from above by $1/\alpha$.

Remark 3.3. The value of ρ^* can also be computed by utilizing only the solution of one additional auxiliary problem. To see this set $d = -\nabla \hat{J}(u)$. Decompose the function $y(u+\rho d)$ into its affine part $y(u)$ and its homogeneous part $y(d)$, i.e., write $y(u+\rho d) = y(u) + \rho y(d)$ and set

$$h(\rho) = J(y(u+\rho d), u+\rho d).$$

Then h is a quadratic polynomial in ρ which takes its minimum value at

$$\rho_1^* = -\frac{(y(u)-z, y(d))_H + \alpha(u,d)_{\mathcal{U}}}{|y(d)|_H^2 + \alpha\ |d|_{\mathcal{U}}^2}. \tag{10}$$

The computation of ρ_1^* requires only the computation of the auxiliary function $y(d)$. In the present situation for $d = -\nabla \hat{J}(u)$ the value of ρ_1^* coincides with that of ρ^*.

4 The control laws

Instantaneous control allows for an interpretation as a non-linear discrete-in-time suboptimal closed loop control method, which turns out to be the stable time discretization of some continuous closed loop controller. In what follows the reconstruction of the continuous controller from the discrete-in-time control approach given by Algorithm 3.2 is outlined. Furthermore, existence and uniqueness results for the continuous controller are presented, and finally the asymptotic behavior of both, the continuous and the discrete control law is quantified.

4.1 Derivation of the feedback controllers

To further simplify notations denote the solution operator in (6) by B, i.e., for given $f \in V'$ the function $v = Bf \in V$ solves

$$v - hAv = f \text{ in } V'. \tag{11}$$

The operator B is linear, bounded and self-adjoint and there holds $B^{-1} = I + hA$.

In many applications of distributed control applied to systems governed by parabolic equations, the operator $\mathcal{B}$ defined in (1) plays the role of an extension operator. For this reason and in order to simplify the exposition we from now on assume that $\mathcal{U} = V'$ and thus, $\mathcal{B} = I$.

Algorithm 3.2 allows the following interpretation.

Theorem 4.1. For $u_0^k = 0$ Algorithm 3.2 is equivalent to the semi-implicit time discretization with discretization step size h

$$(I+hA)y^{k+1} = y^k + hb(y^k) - \rho BB(y^k - z^k) - h\rho BB(b(y^k) - Az^k), \quad y^k = \phi, \tag{12}$$

of the dynamical system

$$\dot{y} + Ay = b(y) - \frac{\rho}{h} BB(y-z) - \rho BB(b(y) - Az), \quad y(0) = \phi. \tag{13}$$

Proof. For a proof it remains to show that for u^k in step 4.) of Algorithm 3.2

$$u^k = -\rho BB(y^k - z^k) - h\rho BB(b(y^k) - Az^k) \tag{14}$$

holds true. Note that

$$Bz^k = BB(I + hA)z^k = BBz^k + hBBAz^k.$$

Using this in steps 2.)–5.) of the algorithm yields the desired result. □

Due to Theorem 4.1 the term

$$Ky = -\frac{\rho}{h}BB(y - z) - \rho BB(b(y) - Az) \tag{15}$$

in (13) can be interpreted as a non-linear closed-loop control policy for the Navier-Stokes equations. It is important to note that the discretization step-size h and the descent parameter ρ in the gradient step of Algorithm 3.2 in the continuous case may now be regarded as parameters defining the controller.

In order to further improve the controller derived in Theorem 4.1 observe that if K defined in (15) steers y to z, eq. (13) necessarily implies that the desired state z would have to satisfy

$$z_t + Az - b(z) = -\rho BB(b(z) - Az), \quad z(0) = \phi.$$

This suggests to generalize the control law (15) to

$$Ky = -\frac{\rho}{h}BB(y - z) - \rho BB(b(y) - b(z)) + z_t + Az - b(z). \tag{16}$$

With this control law the controlled Navier-Stokes equations become the form

$$y_t + Ay - b(y) = Ky \quad \text{in } L^2(V') \text{ and } \quad y(0) = \phi. \tag{17}$$

The controller (16) can be derived from Algorithm (3.2) with a particular choice of the initial control u_0^k.

Lemma 4.2. Choosing the initial control u_0^k in Algorithm 3.2 as solution of

$$\left(I - \frac{\rho}{1-\rho\alpha}BB\right)u_0^k = \frac{1}{1-\rho\alpha}\left(z^{k+1} - z^k + Az^{k+1} - b(z^k) + \rho BB(b(z^k) - Az^k)\right)$$

in Theorem 4.1 one would end up with control law (16) instead of control law (15). The analogon to (12) with $w = y - z$ is given by

$$(I + hA)w^{j+1} = w^j + h\left(b(y^j) - b(z^j)\right) - \rho BBw^j - \rho hBB\left(b(y^j) - b(z^j)\right), \quad w^0 = \phi - z(0). \tag{18}$$

The related discrete controller is given by

$$K^D y = -\frac{\rho}{h}BB(y^j - z^j) - \rho BB(b(y^j) - b(z^j)) + \frac{z^{j+1} - z^j}{h} + Az^{j+1} - b(z^j), \tag{19}$$

Unless otherwise stipulated, from now onwards

Assumption 4.3. $0 \neq \phi \in V$ and $z \in H^{2,1}(Q)$.

Note that this assumption on the desired state z in particular implies that $z(0)$ is meaningful. Moreover, $z(0) \in V$.

5 Existence and uniqueness of solutions

In this section existence and uniqueness of a solution to (17) is proven and its regularity is investigated. Since the desired state z in this context plays the role of an inhomogeneity whose regularity properties could be adapted to the necessities of the proof, for the moment $z \equiv 0$ is assumed. Moreover, $\partial\Omega$ is assumed to be as smooth as required by the existence and regularity results for the Stokes and quasi-Stokes problems considered in the proofs of the Theorems below, see [26, Chap. I, Prop. 2.3]. Let

$$\rho_0 = \frac{\nu^2}{8\nu^2 + 4|\phi|_H^2}, \quad d_H = 2|\phi - z(0)|_H^2. \tag{20}$$

Theorem 5.1. Let $\phi \in H$ be a given initial state and let $h > 0$ be fix. Then for every $0 < \rho \leq \rho_0$ the equation

$$\begin{cases} y_t + Ay - b(y) = -\frac{\rho}{h} BBy - \rho BB(b(y)) \\ y(0) = \phi \end{cases} \tag{21}$$

for every $T > 0$ admits a unique weak solution $y \in W$ which satisfies the a-priori estimates

$$\begin{cases} |y|_H^2 & \leq 2\,\mathrm{e}^{-\frac{\rho}{h}t}\,|\phi|_H^2 & \text{for all } t \in [0,T] \\ |y|_{L^\infty(H)}^2 & \leq 2\,|\phi|_H^2, & \\ |y|_{L^2(V)}^2 & \leq \frac{4}{\nu}\,|\phi|_H^2 & \text{and} \\ |y_t|_{L^2(V')}^2 & \leq C(\nu)\,\left\{1 + \frac{\rho}{h}\right\}\,|\phi|_H^2. & \end{cases} \tag{22}$$

Proof. The proof is given in the appendix. □

The decay rate for the $L^\infty(H)$-estimate in (22) holds for all $h > 0$ provided $0 < \rho \leq \rho_0$. This means that the controller (16) enhances the stability properties of the system up to exponential decay of order $\frac{1}{h}$. As is shown next a similar result holds for the $L^\infty(V)$ norm of the solution y.

Theorem 5.2. Let ρ_0 be given by (20) and let $0 < \rho \leq \rho_0$. Then the solution y of problem (21) is an element of $H^{2,1}(Q)$ and satisfies the a-priori estimates

$$\begin{cases} |y|_V^2 & \leq C\,\mathrm{e}^{-\frac{\rho}{h}t}\,|\phi|_V^2 & \text{for all } t \in [0,T], \\ |y|_{L^\infty(V)}^2 & \leq C(\nu)\,|\phi|_V^2, & \\ |y|_{L^2(H^2(\Omega)^2 \cap V)}^2 & \leq C(\nu)\,|\phi|_V^2 & \text{and} \\ |y_t|_{L^2(H)}^2 & \leq C(\nu)\,\left\{1 + \frac{\rho}{h}\right\}\,|\phi|_V^2. & \end{cases} \tag{23}$$

Proof. The proof is given in the appendix. □

6 Stability of the continuous controller

In this section the stability properties of the control law (17) are investigated. It follows from the a-priori estimates to be derived that this feedback law admits a unique solution provided Assumption 4.3 is satisfied. Moreover, the H and V norms of difference $w = y - z$ decay exponentially with rate $-\frac{\rho}{h}$, which coincides with the results for $z = 0$. To achieve these results the range of ρ has to be adapted to the size of $|z|^2_{L^\infty(H)}$. Let the range of ρ be implicitly defined by

$$0 \; < \; \rho \; \le \rho_1 := \min\left(\rho_0, \frac{\nu^2}{2\nu^2 + e^{\frac{4+\rho}{\nu}|z|^2_{L^2(V)}} d_H + |z|^2_{L^\infty(H)}}\right). \tag{24}$$

Theorem 6.1. Let $0 < \rho \le \rho_1$ with ρ_1 from (24) and let y be the unique solution of (17). Then $w = y - z$ satisfies

$$\begin{cases} |w|^2_H & \le C\,(\nu, |z|_W)\; \mathrm{e}^{-\frac{\rho}{h}t} \qquad \text{for all } t \in [0,T], \\ |w|^2_{L^\infty(H)} & \le C\,(\nu, |z|_W)\,, \\ |w|^2_{L^2(V)} & \le C\,(\nu, |z|_W) \qquad \text{and} \\ |w_t|^2_{L^2(V')} & \le C\,(\nu, |z|_W)\; \left\{1 + \frac{\rho}{h}\right\}, \end{cases} \tag{25}$$

where $C\,(\nu, |z|_W)$ is a positive constant independent of ρ and h.

Proof. Using w as test function in the variational formulation of (17) leads to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}|w_t|^2_H + \nu|w|^2_V + \frac{\rho}{h}|Bw|^2_H = & \\ & \int_\Omega ((w\nabla)w + (z\nabla)w + (w\nabla)z)\; w\,dx \\ & - \rho \int_\Omega (B((w\nabla)w + (z\nabla)w + (w\nabla)z))\,Bw\,dx \;=\; (1) + \cdots + (6). \end{aligned}$$

Since (1) = (2) = 0, and addend (4) can be estimated according to (37) and (38) for w replaced by y, it remains to estimate the terms (3), (5) and (6) to derive a differential inequality for $y - z$ which is similar to that given in (40) for y. To begin with, estimate

$$(3) \;\le\; \frac{\nu}{2}|w|^2_V \;+\; \frac{2}{\nu}|z|^2_V|w|^2_H,$$

$$(4) \;\le\; \rho h|B((w\nabla)w)|^2_H + \frac{\rho}{4h}|Bw|^2_H \;\le\; \frac{\rho}{2\nu}|w|^2_H|w|^2_V + \frac{\rho}{4h}|Bw|^2_H,$$

$$\begin{aligned} (5) + (6) \;\le\; & \rho h\left\{|B((w\nabla)z)|^2_H + |B((z\nabla)w)|^2_H\right\} + \frac{\rho}{4h}\;|Bw|^2_H \\ & \le\; \frac{\rho}{4h}\;|Bw|^2_H \;+\; \frac{\rho}{2\nu}\left\{|w|^2_V|z|^2_H + |z|^2_V|w|^2_H\right\} \end{aligned}$$

and proceed as for the derivation of (40). This leads to

$$\frac{1}{2}\frac{d}{dt}|w|_H^2 + \left(\nu\left(\frac{1}{2}-\rho\right) - \frac{\rho}{2\nu}\left(|w|_H^2 + |z|_{L^\infty(H)}^2\right)\right)|w|_V^2 + \frac{\rho}{2h}|w|_H^2 \ \leq\ \frac{4+\rho}{2\nu}|z|_V^2|w|_H^2.$$

Since ρ satisfies (24) $\nu\left(\frac{1}{2}-\rho\right) - \frac{\rho}{2\nu}\left(e^{\frac{4+\rho}{\nu}|z|_{L^2(V)}^2} d_H + |z|_{L^\infty(H)}^2\right) > 0$. For ρ in this range arguments similar to those applied to (40) give

(26)

$$\frac{1}{2}\frac{d}{dt}|w|_H^2 + \left(\nu\left(\frac{1}{2}-\rho\right) - \frac{\rho}{2\nu}\left(e^{\frac{4+\rho}{\nu}|z|_{L^2(V)}^2} d_H + |z|_{L^\infty(H)}^2\right)\right)|w|_V^2 + \frac{\rho}{2h}|w|_H^2 \ \leq\ \frac{4+\rho}{2\nu}|z|_V^2|w|_H^2.$$

Since the right-hand-side in (26) is integrable a further Gronwall argument gives the desired result. Note that the estimate for w_t is a direct consequence of the second and third estimate in (25). $\square$

Remark 6.2. For the result of the previous theorem only $\phi \in H$ and $z \in W$ is required.

A similar result holds for the decay w.r.t. the V-norm of w. Let now the range of ρ implicitly defined by the relation

$$0 \ <\ \rho\ \leq \rho_2 := \min\left(\rho_0, \frac{1}{2}\frac{\nu^2}{2\nu^2 + 3e^{\frac{4+\rho}{\nu}|z|_{L^2(V)}^2} d_H + 3|z|_{L^\infty(H)}^2}\right). \tag{27}$$

Theorem 6.3. Let ρ satisfy (27) and let y be the unique solution of (17). Then $w = y - z$ satisfies

$$\begin{array}{lll} |w|_V^2 & \leq C\left(\nu, |z|_{H^{2,1}(Q)}\right) \mathrm{e}^{-\frac{\rho}{h}t} & \text{for all } t \in [0,T], \\ |w|_{L^\infty(V)}^2 & \leq C\left(\nu, |z|_{H^{2,1}(Q)}\right), & \\ |w|_{L^2(H^2(\Omega)^2\cap V)}^2 & \leq C\left(\nu, |z|_{H^{2,1}(Q)}\right) & \text{and} \\ |w_t|_{L^2(H)}^2 & \leq C\left(\nu, |z|_{H^{2,1}(Q)}\right)\left\{1 + \frac{\rho}{h}\right\}, & \end{array}$$

where $C\left(\nu, |z|_{H^{2,1}(Q)}\right)$ is a positive constant independent of ρ and h.

Proof. Use Sw as test function in (17). This leads to

$$\frac{1}{2}\frac{d}{dt}\ |w|_V^2 + \nu\ |Sw|_H^2 + \int_\Omega ((y\nabla)y - (z\nabla)z)\ Sw\, dx =$$

$$-\frac{\rho}{h}\int_\Omega BwBSw\, dx - \rho\int_\Omega B((y\nabla)y - (z\nabla)z)BSw\, dx.$$

Relation (27) implies $1 - \rho - \frac{3\rho}{\nu^2}\left(e^{\frac{4+\rho}{\nu}|z|_{L^2(V)}^2} d_H + |z|_{L^\infty(H)}^2\right) \geq \frac{1}{2}$. Restricting ρ to this range, similar to the derivation of (26) after several applications of (2), (41), Lemma A.3

and Young's inequality one ends up with the inequality

$$
(28)\quad \frac{1}{2}\frac{d}{dt}|w|_V^2+\frac{\nu}{2}|Sw|_H^2+\frac{\rho}{h}\left(1-\rho-\frac{3\rho}{\nu^2}\left(e^{\frac{4+\rho}{\nu}|z|^2_{L^2(V)}}d_H+|z|^2_{L^\infty(H)}\right)\right)|w|_V^2
$$
$$
\leq\ C_\nu\ \left\{|w|_H^2|w|_V^2+|z|_H|Sz|_H+|z|_V^2+|z|_V^4\right\}|w|_V^2.
$$

Since the right-hand-side in (28) is integrable a Gronwall argument gives the desired result. □

7 Stability of discrete controllers

Here, first stability properties of the instantaneous control procedure (18) are investigated. As will be shown stability for a certain parameter range of h and ρ can only be ensured by requiring additionally either largeness of the viscosity parameter ν or smallness of the initial defect $\phi - z(0)$ between the state and desired state. As will be sketched the latter restriction can be dropped when the procedure is applied to the instationary Burgers equation, see also [17]. Finally, a slightly modified version of the control operator (19) is applied to stabilize the fully implicit Euler-discretization of the Navier-Stokes equations and it turns out that stability can be obtained for a considerable parameter range, with requiring neither smallness of the data nor largeness of the viscosity parameter. Fully implicit discretization of the state is a realistic situation, since the discrete controller is applied to stabilize a physical system which is described by the Navier-Stokes equations. Therfore, the choice of the discretization procedure for the uncontrolled state need not be linked to the discrete controller.

Throughout this section it is assumed that

Assumption 7.1. In addition to Assumption 4.3 let $z \in C([0,T];H^{1,\infty}(\Omega)^2 \cap V)$.

To begin the investigation for the instantaneous control procedure (18) set $w = y^{j+1} - z^{j+1}$, $v = y^j - z^j$ and abbreviate $z = z^j$, $j \in \mathbb{N}$. Firstly derive an appropriate inequality which relates the H-norms of v and w. For this purpose use w as test function in the variational formulation of (18). This gives

$$
(29)\quad \frac{1}{2}|w|_H^2-\frac{1}{2}|v|_H^2+\frac{1}{2}|w-v|_H^2+\nu h|w|_V^2=
$$
$$
h\int_\Omega\left((v\nabla)v+(z^j\nabla)v+(v\nabla)z^j\right)w\ dx-\rho\int_\Omega BvBw\,dx
$$
$$
-\rho h\int_\Omega B\left((v\nabla)v+(z^j\nabla)v+(v\nabla)z^j\right)Bw\,dx=(1)+\cdots+(7).
$$

Estimating the odds using (2), (38) and Young's inequality one obtains

$$
\begin{array}{lll}
(1) & \le & \frac{\nu h}{4}\,|w|_V^2 \;+\; \frac{2h}{\nu}\,|v|_H^2|v|_V^2, \\
(2) & \le & \frac{\nu h}{4}\,|w|_V^2 \;+\; \frac{|z|_\infty^2 h}{\nu}\,|v|_H^2, \\
(3) & \le & \frac{h}{2}\,|w|_H^2 \;+\; \frac{|z|_{1,\infty}^2 h}{2}\,|v|_H^2, \\
(4) & = & -\rho \int\limits_\Omega |Bw|^2\,dx \;-\rho\int\limits_\Omega B(v-w)Bw \;\le\; \frac{\rho}{2}(\rho-1)|Bw|_H^2 \;+\; \frac{1}{2}|v-w|_H^2, \\
(5) & = & -\rho h\int\limits_\Omega B\left((v\nabla)v\right)\,Bw\,dx \;\le\; \frac{3\rho h}{2\nu(1-\rho)}\,|v|_H^2|v|_V^2 \;+\; \frac{\rho}{12}(1-\rho)\,|Bw|_H^2,
\end{array}
$$

and finally

$$
\begin{aligned}
(6)+(7) \;&=\; -\rho h\int\limits_\Omega B\left((v\nabla)z^j + (z^j\nabla)v\right)\,Bw\,dx \\
&\le\; \left\{\frac{3\rho h^2|z|_{1,\infty}^2}{1-\rho} + \frac{3\rho h|z|_\infty^2}{4(1-\rho)\nu}\right\}\,|v|_H^2 + \frac{\rho(1-\rho)}{6}|Bw|_H^2.
\end{aligned}
$$

Now introduce

$$
c_1(\rho,h) \;:=\; \frac{1}{2} + \frac{\rho}{4}(1-\rho) \;-\; \frac{h}{2},
$$

and

$$
c_2^j(\rho,h,z) \;:=\; \frac{1}{2} + \left[\frac{2h}{\nu} + \frac{3h\rho}{2\nu(1-\rho)}\right]|v|_V^2 + \underbrace{\left[\frac{|z|_{1,\infty}^2 h}{2} + \frac{|z|_\infty^2 h}{\nu} + \frac{3\rho h^2|z|_{1,\infty}^2}{1-\rho} + \frac{3\rho h|z|_\infty^2}{(1-\rho)4\nu}\right]}_{=:\tilde{c}_2(h,\rho,z)}.
$$

With this notation the estimates above and Lemma A.1 together with (29) give

$$
c_1(\rho,h)\,|w|_H^2 \;-\; c_2^j(\rho,h,z)\,|v|_H^2 \;+\; \frac{\nu h}{2}(1-\rho+\rho^2)\,|w|_V^2 \;\le\; 0. \tag{30}
$$

From now onwards let $w^j := y^j - z^j$, where y^j denote the iterates obtained by (18).

Theorem 7.2. (Conditional H-norm stability of instantaneous control)

$$
\forall_{\rho\in(0,1)}\,\exists_{h^*(\rho),\;0<\kappa<1}\,\forall_{j\in\mathbb{N}}:\;|w^{j+1}|_H^2 \;\le\; \kappa^j\,|w^0|_H^2,
$$

provided $0 < h \le h^*(\rho)$ and

$$
crit \;:=\; \frac{4-\rho}{\nu^2(1-\rho)(1-\rho+\rho^2)}|w^0|_H^2
$$

is sufficiently small.

Proof. Fix $\rho\in(0,1)$, define

$$
\hat{c}_2(\rho,h,z) \;:=\; \frac{1}{2} + crit + \tilde{c}_2(\rho,h,z)
$$

and argue by induction as follows.

1. Set $j = 0$, choose $h_0 = h_0(\rho)$ and $crit$ so small that for all $0 < h \le h_0$

   (a) $\hat{c}_2(\rho, h, z) \le 1$ and $c_2^0(\rho, h, z) \le 1$,

   (b) $\dfrac{\hat{c}_2(\rho, h, z)}{c_1(\rho, h)} = \kappa_1 < 1$ and

   (c) $\dfrac{c_2^0(\rho, h, z)}{c_1(\rho, h)} = \kappa_2 < 1$

   holds. This is possible, since the for $\rho \in (0, 1)$ the term $\frac{\rho}{4}(1-\rho)$ in the definition of $c_1(\rho, h)$ is positive. Define $\kappa := \max(\kappa_1, \kappa_2)$. Then (30) implies

$$|w^1|_H^2 \le \kappa |w^0|_H^2$$

   and

$$|w^1|_V^2 \le \frac{2}{\nu h(1-\rho+\rho^2)} \, |w^0|_H^2 = \frac{2}{\nu h(1-\rho+\rho^2)} \, \kappa^0 |w^0|_H^2 .$$

2. Now assume that for $j \in \mathbb{N}$

   (a) $|w^j|_H^2 \le \kappa^j |w^0|_H^2$ and

   (b) $|w^j|_V^2 \le \dfrac{2}{\nu h(1-\rho+\rho^2)} \, \kappa^{j-1} |w^0|_H^2$

   holds true.

3. Then conclude from (30)

$$\begin{aligned} c_2^j(\rho, h, z) &= \tfrac{1}{2} + \left[\tfrac{2h}{\nu} + \tfrac{3h\rho}{2\nu(1-\rho)}\right] |w^j|_V^2 + \tilde{c}_2(\rho, h, z) \\ &\le \tfrac{1}{2} + \left[\tfrac{2h}{\nu} + \tfrac{3h\rho}{2\nu(1-\rho)}\right] \tfrac{2}{\nu h(1-\rho+\rho^2)} \, \kappa^{j-1} |w^0|_H^2 + \tilde{c}_2(\rho, h, z) \\ &\le \tfrac{1}{2} + crit + \tilde{c}_2(\rho, h, z) = \hat{c}_2(\rho, h, z) \end{aligned}$$

   Thus, a further application of (30) implies

$$|w^{j+1}|_H^2 \le \frac{c_2^j(\rho, h, z)}{c_1(\rho, h)} |w^j|_H^2 \le \frac{\hat{c}_2(\rho, h, z)}{c_1(\rho, h)} |w^j|_H^2 \le \kappa^{j+1} \, |w^0|_H^2$$

   and

$$|w^{j+1}|_V^2 \le \frac{2}{\nu h(1-\rho+\rho^2)} \, c_2^j(\rho, h, z) |w^j|_H^2 \le \frac{2}{\nu h(1-\rho+\rho^2)} \, \kappa^j \, |w^0|_H^2,$$

which completes the proof of Theorem 7.2. $\square$

From the previous proof also stability with respect to the V norm can be inferred.

Theorem 7.3. (Conditional V-norm stability of instantaneous control)

$$\forall \rho \in (0,1)\ \exists h^*(\rho),\ 0<\kappa<1\ \forall j \in \mathbb{N},\ 0<h\leq h^*:\ |w^j|_V \ \leq\ \frac{2}{\nu h(1-\rho+\rho^2)}\kappa^{j-1}\,|w^0|_H,$$

provided

$$crit\ :=\ \frac{4-\rho}{\nu^2(1-\rho)(1-\rho+\rho^2)}|w^0|_H^2$$

is sufficiently small.

Remark 7.4. The smallness of *crit* in Theorems 7.2 and 7.3 is a condition either on the smallness of the initial difference between state and desired state or on the smallness on the Reynolds number of the fluid. It has to be required since there are no better estimates available for the term (1) above. The term (5) could be estimated in a slightly different way to obtain a ρ^2 in front of $|v|_V^2$, see the proof of the next Theorem, and therefore could be reduced by decreasing ρ. However, for (1) there is no further knob to fix its size. For the Burgers equation the situation is much more comfortable at this stage. Due to the continuous embedding $H^1 \hookrightarrow L^\infty$ and the well known L^2-H^1 interpolation estimate for L^∞ functions in one spatial dimension one has

$$h\int_\Omega vv'w\,dx\ \leq\ \frac{\nu h}{4}|w|_V^2+h^{1-2\alpha}|w|_H^2+h^{1+2\alpha}\frac{1}{2\sqrt{\nu}}|w|_V^2|w|_H^2 \quad \text{for all } \alpha\in(0,1).$$

Following the lines of the proof of Theorem 7.2 one can now conclude that the smallness requirement on *crit* may be dropped provided ρ is sufficiently small, since the power of h in the last addend on the right-hand-side of this estimate is larger than one. For more details see [17].

Consider now the discrete controller

$$K^D y = -\frac{\rho}{h}BB(y^j-z^j)-\rho BB(b(y^j)-b(z^j))+\frac{z^{j+1}-z^j}{h}+Az^{j+1}-b(z^{j+1}), \tag{31}$$

which differs from the controller defined in (19) only in the last addend on the right-hand-side. In the next theorem the uncontrolled Navier-Stokes equations are discretized with time step size h using the fully implicit Euler scheme and the discrete controller (31) is applied to stabilize the discretized Navier-Stokes system. The modified controller stabilizes the Navier-Stokes equations for the parameters ρ and h in a considerable range, without any smallness or largeness requirements on the data of the Navier-Stokes system. The controlled equation has the form

$$\frac{y^{j+1}-y^j}{h}+Ay^{j+1}-b(y^{j+1})=K^Dy^j, \quad j=0,1,\dots \quad \text{and } y^0=\phi, \tag{32}$$

with K^D from (31).

Theorem 7.5. (H- and V-norm stability of (31))
There exists some $\rho^* \in (0,1)$ such that for every $0 < \rho \le \rho^*$ there exists a $h^*(\rho) > 0$ and a positive $\kappa < 1$ such that for all $j \in \mathbb{N}$

$$\begin{aligned} |w^j|_H^2 &\le \kappa^j \, |w^0|_H^2 \quad \text{and} \\ |w^j|_V^2 &\le \tfrac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)} \, \kappa^{j-1} \, |w^0|_H^2, \end{aligned}$$

provided $0 < h \le h^*(\rho)$.

Proof. Test (32) with w. Then one obtains

$$\begin{aligned} (33) \quad & \frac{1}{2}|w|_H^2 - \frac{1}{2}|v|_H^2 + \frac{1}{2}|w-v|_H^2 + h\nu|w|_V^2 = \\ & \qquad h \int_\Omega \left((w\nabla)w + (z^j\nabla)w + (w\nabla)z^j\right) w \; dx - \rho \int_\Omega BvBw\, dx \\ & - \rho h \int_\Omega B\left((v\nabla)v + (z^j\nabla)v + (v\nabla)z^j\right) Bw\, dx = (1)' + (2)' + (3)' + (4) + (5)' + (6) + (7). \end{aligned}$$

There holds $(1)' = (2)' = 0$ and $(3)'$ can be estimated as

$$(3)' \;\le\; \frac{\nu h}{2}|w|_V^2 \;+\; \frac{h}{\nu}|z|_V^2|w|_H^2.$$

Utilizing the estimate $|BBw|_V \le |w|_V$ one obtains

$$(5)' \;\le\; \frac{\nu h}{3}|w|_V^2 \;+\; \frac{3\rho^2 h}{2\nu}|v|_H^2|v|_V^2$$

The remaining addenda can be estimated as above. Now introduce

$$c_1(\rho,h,z) \;:=\; \frac{1}{2} + \frac{\rho}{3}(1-\rho) \;-\; \frac{2h}{\nu}|z|_{L^\infty(V)}^2 \;-\; \frac{h}{2},$$

and

$$c_2^j(\rho,h,z) \;:=\; \frac{1}{2} + \frac{3\rho^2 h}{2\nu}|v|_V^2 + \underbrace{\left[\frac{3\rho h^2|z|_{1,\infty}^2}{1-\rho} + \frac{3\rho h|z|_\infty^2}{4(1-\rho)\nu}\right]}_{=:\tilde{c}_2(h,\rho,z)}.$$

With this notation and the estimates above one concludes from (33)

$$(34) \qquad c_1(\rho,h,z)\,|w|_H^2 \;-\; c_2^j(\rho,h,z)\,|v|_H^2 \;+\; \frac{\nu h}{2}(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)\,|w|_V^2 \;\le\; 0.$$

Now define

$$\hat{c}_2(\rho,h,z) \;:=\; \frac{1}{2} + \frac{3\rho^2}{\nu^2(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}|w^0|_H^2 \;+\; \tilde{c}_2(\rho,h,z)$$

and proceed as follows.

1. Choose $\rho^* \in (0,1)$ such that

$$\frac{3\rho}{\nu^2(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}|w^0|_H^2 \ \leq\ \frac{1}{6}(1-\rho) \quad \text{for all } \rho \in (0,\rho^*]. \tag{35}$$

2. Fix $\rho \in (0,\rho^*]$ and choose $h^* = h^*(\rho) > 0$ such that

   (a) $\tilde{c}_2(h,\rho,z) \ \leq\ \frac{1}{4}$

   (b) $\frac{3\rho^2 h}{2\nu}|w^0|_V^2 \ \leq\ \frac{1}{4}$

   (c) $\frac{h}{2} + \frac{2h}{\nu}|z|^2_{L^\infty(V)} + \tilde{c}_2(h,\rho,z) < \frac{\rho}{6}(1-\rho)$

   (d) $\frac{c_2^0(\rho,h,z)}{c_1(\rho,h,z)} \ \leq\ \kappa_1 \ <\ 1$

   holds for all h in the interval $(0,h^*]$. Let $h \in (0,h^*]$.

3. Now conclude from (35) and $\rho \in (0,1)$ that

$$\frac{3\rho^2}{\nu^2(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}|w^0|_H^2 \ \leq\ \frac{\rho}{6}(1-\rho) \ <\ \frac{1}{4},$$

   which together with (c) implies that

$$\hat{c}_2(\rho,h,z) \ <\ 1 \quad \text{and} \quad \frac{\hat{c}_2(\rho,h,z)}{c_1(\rho,h,z)} \ \leq\ \kappa_2 < 1.$$

   Furthermore, (a) and (b) give $c_2^0(\rho,h,z) \leq 1$, so that with (34)

$$|w^1|_V^2 \leq \frac{2c_2^0(\rho,h,z)}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,|w^0|_H^2 \ \leq\ \frac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,\kappa^0|w^0|^2.$$

4. Now assume that for $j \in \mathbb{N}$

   (a) $|w^j|_H^2 \leq \kappa^j|w^0|_H^2$ and

   (b) $|w^j|_V^2 \leq \dfrac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,\kappa^{j-1}|w^0|_H^2$

   holds true, where $\kappa :=\max(\kappa_1,\kappa_2) < 1$.

5. Then conclude from (b)

$$\begin{aligned} c_2^j(\rho,h,z) \ &= \tfrac{1}{2} + \tfrac{3h\rho^2}{2\nu}|w^j|_V^2 + \tilde{c}_2(\rho,h,z) \\ &\leq \tfrac{1}{2} + \tfrac{3h\rho^2}{2\nu}\tfrac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,\kappa^{j-1}|w^0|_H^2 + \tilde{c}_2(\rho,h,z) \\ &\leq \hat{c}_2(\rho,h,z). \end{aligned}$$

Thus, utilizing (34) one more time gives

$$|w^{j+1}|_H^2 \;\leq\; \frac{c_2^j(\rho,h,z)}{c_1(\rho,h,z)}|w^j|_H^2 \;\leq\; \frac{\hat{c}_2(\rho,h,z)}{c_1(\rho,h,z)}|w^j|_H^2 \;\leq\; \kappa^{j+1}\,|w^0|_H^2$$

and

$$|w^{j+1}|_V^2 \;\leq\; \frac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,c_2^j(\rho,h,z)|w^j|_H^2 \;\leq\; \frac{2}{\nu h(1-\frac{2}{3}\rho+\frac{2}{3}\rho^2)}\,\kappa^j\,|w^0|_H^2,$$

which completes the proof of Theorem 7.5. □

8 Numerical validation

Here the results obtained in the previous sections are numerically validated. In order to value the performance of the feedback operators (16) and (19) the numerical example is taken from [19]. As is demonstrated below the instantaneous controller presented here steers the H-norm and the V-norm of the difference $y-z$ to zero with exponential decay. This seems to be a more stable performance than that reported by Hou and Yan in [19] for their (1,1)-receding horizon controller (i.e. control horizon length coincides with time step size). The instantaneous controls are compared to the optimal control and it turns out that instantaneous controls give a much better reduction of the control gain, but at significant higher overall costs.

The control problem considered here is of tracking type and is given by (**P**) with cost functional

$$J(y,u) := \frac{1}{2}\int\limits_{\Omega^T} |y-z|^2\,dxdt + \frac{\alpha}{2}\int\limits_{\Omega^T} |u|^2\,dxdt$$

and control space $U := L^2(\Omega)^2$, with $\mathcal{B}$ denoting the injection from U into V' and $\Omega^T := (0,T)\times\Omega$. The initial value of the uncontrolled flow is chosen as

$$y(x,0) \;=\; e\begin{bmatrix} (\cos 2\pi x_1 - 1)\sin 2\pi x_2 \\ -(\cos 2\pi x_2 - 1)\sin 2\pi x_1 \end{bmatrix}$$

with e denoting the Euler number, and the desired state is time dependent and given by

$$z(t,x) = \begin{bmatrix} \varphi_{x_2}(t,x_1,x_2) \\ -\varphi_{x_1}(t,x_1,x_2) \end{bmatrix},$$

where φ is defined through the stream function

$$\varphi(t,x_1,x_2) = \theta(t,x_1)\theta(t,x_2)$$

with

$$\theta(t,y) = (1-y)^2(1-\cos 2k\pi t), \quad y\in[0,1].$$

For the results presented $\alpha = 1.e-2$, $k = 1$ and the time interval is chosen as $[0, 2]$, i.e. $T = 2$. For the discretization in time a equidistant grid with width $\delta t = 0.01$ is used, for the spatial discretization the Taylor-Hood finite element [18] is used on a grid containing 1024 triangles with 2113 velocity and 545 pressure nodes. The number of unknowns in the discretized control problem therefore has the magnitude 1.65×10^6, including the primal, adjoint and control variables.

In Fig. 1 the desired flow at $T = 2$ is shown. It forms four cells with opposite flow directions near the cell boarders.

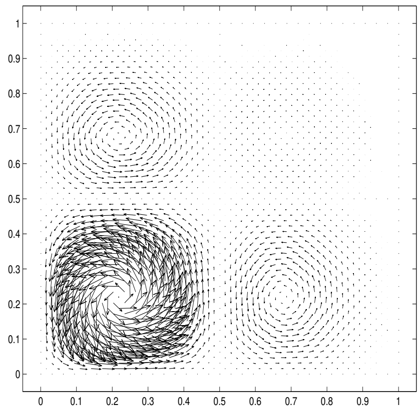

Figure 1: Desired flow at $T = 2$

In Figs. 2, 3 the evolution of the optimally controlled flow computed with Newton's method (see [13, 16] for computational details) and the instantaneously controlled flow is illustrated a selected time instances. The costs are compared in Fig. 4. For the instantaneous control strategy they become larger with increasing time. This is due to the increasing dynamics of the desired state. As is expected the optimal control strategy equidistributes the costs over the time horizon, whereas instantaneous control at every time instance intends to match the desired state. This is also illustrated by the evolution in Figs 2, 3.

For $\nu = 1/\mathrm{Re} = 1/10$ and $\gamma = 1.e-2$ the numerical computation of the optimal control takes about 45 minutes cpu-time on a DEC-ALPHA^{TM} station 500. The instantaneous feedback controller takes about 2 minutes to compute a control function on the time horizon $[0, 2]$.

In Fig. 5 the evolution of the L^2-cost for the instantaneous control law is shown for $\rho = 0.1$ and different values of h. In Fig. 6 $h = 0.1$ is fix and the evolution of the control gain in the L^2- and the H^1-norm for different values of ρ are shown. Exponential decay is observed and thus, the theoretical results of Theorems 6.1, 6.3, 7.2 and 7.3 are confirmed.

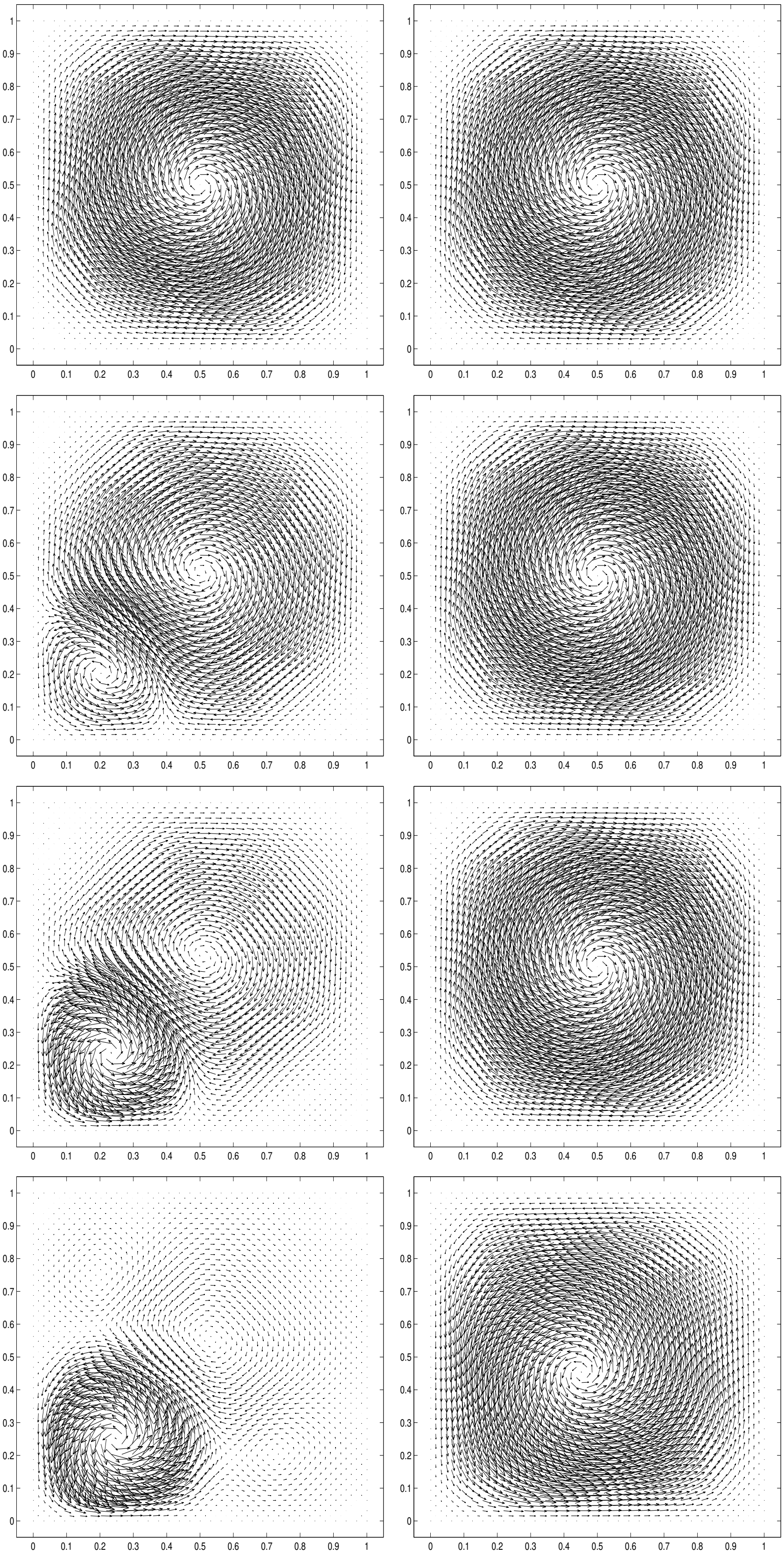

Figure 2: Optimally controlled (left) versus instantaneously controlled flows for $t = 0.01, 0.1, 0.2, 0.5$

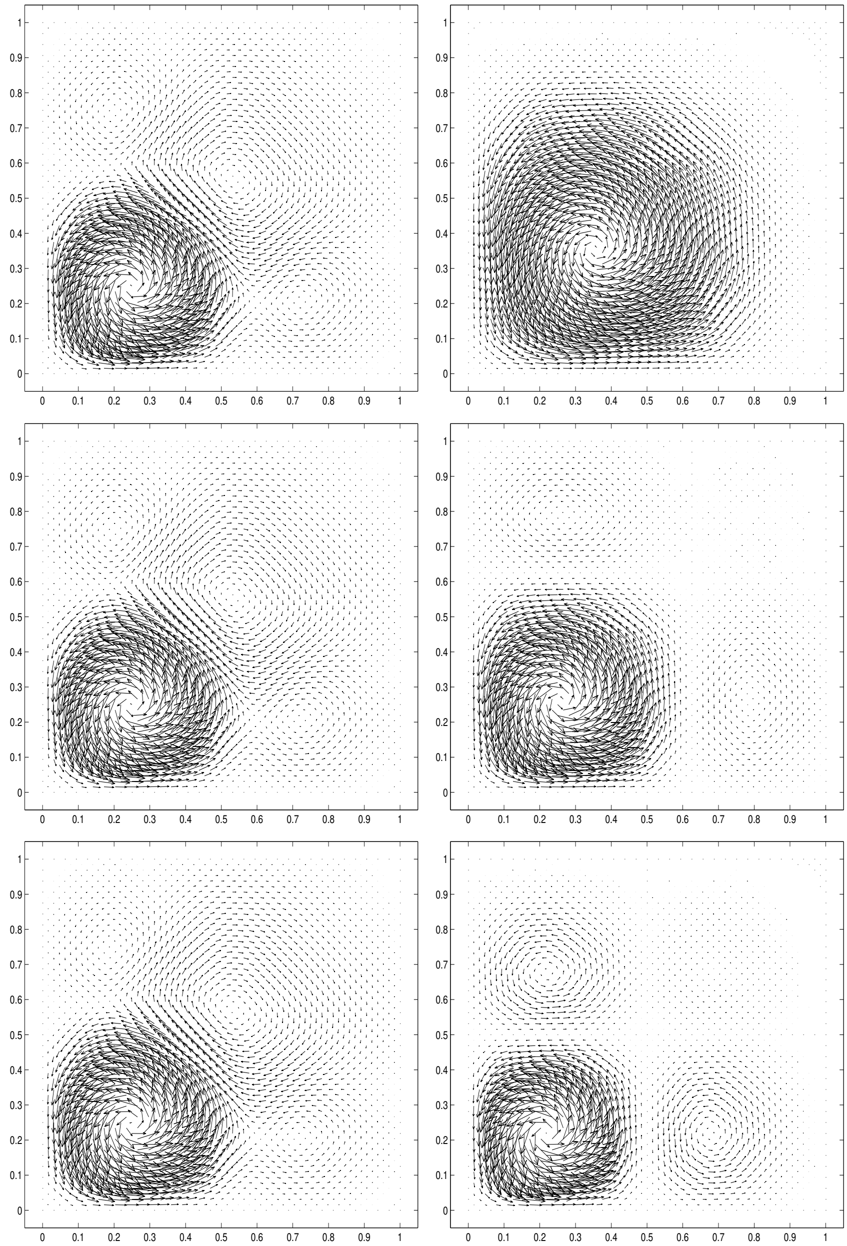

Figure 3: Optimally controlled (left) versus instantaneously controlled flows for $t = 1., 1.6, 2.$

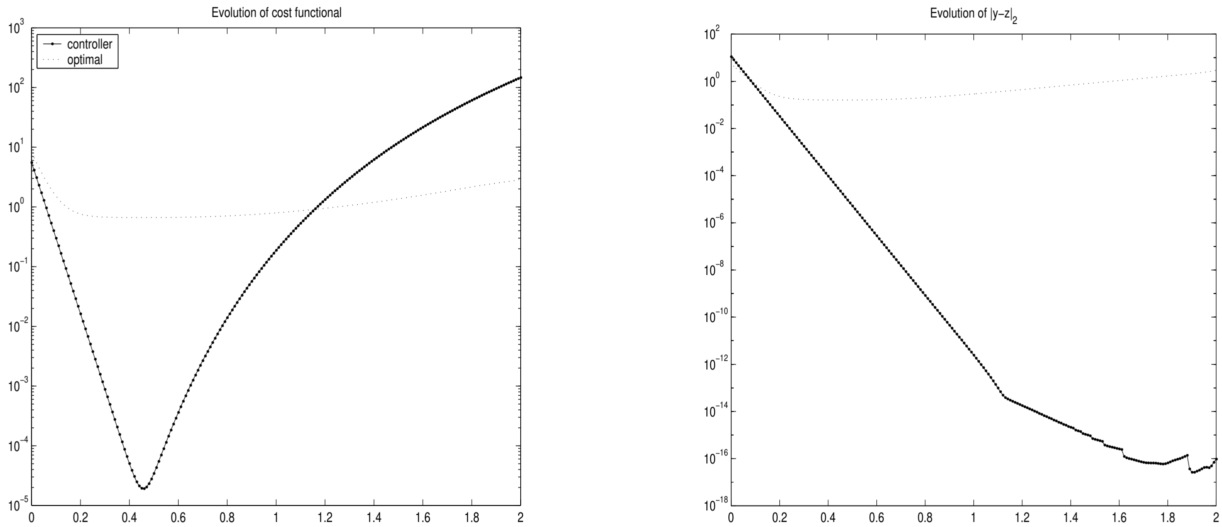

Figure 4: Evolution of cost (left) and control gain for $h = 0.01$ and $\rho = 0.1$

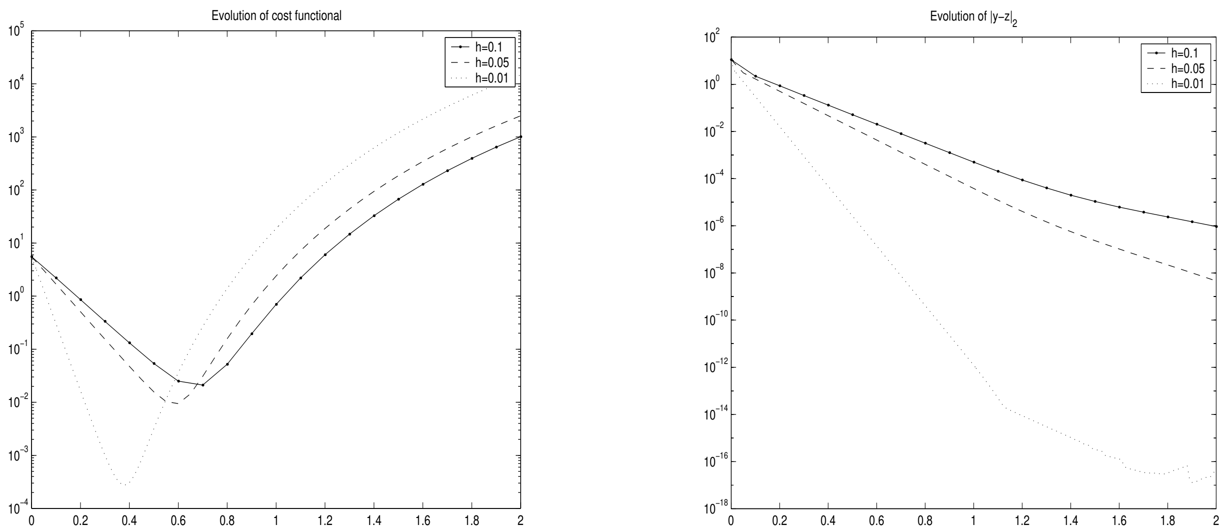

Figure 5: Evolution of cost (left) and control gain for $\rho = 0.1$ and $h = 0.01, 0.05, 0.1$

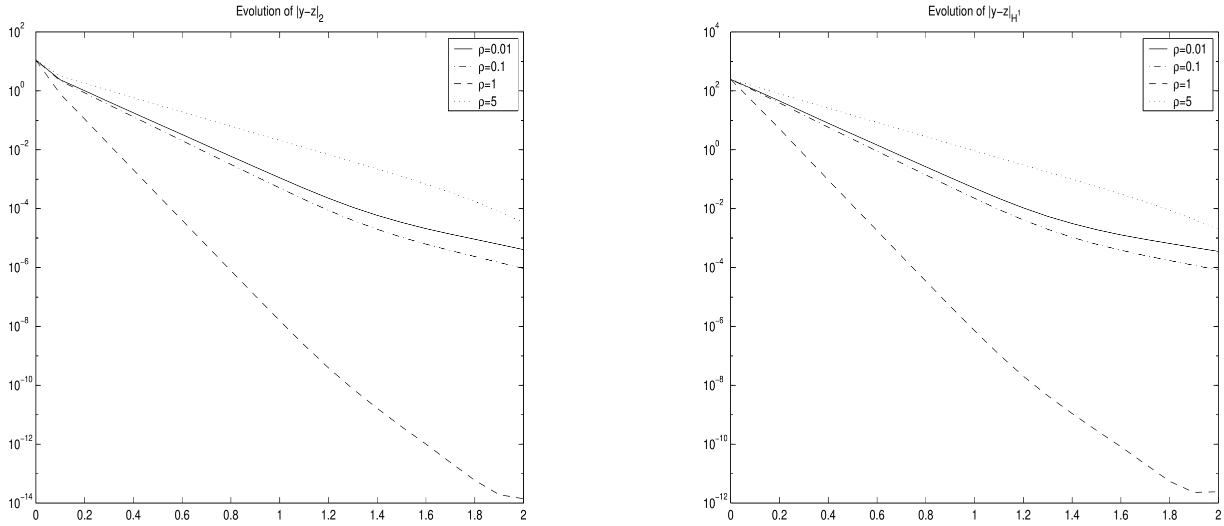

Figure 6: Evolution of control gain in L^2-norm (left) and H^1-norm for $h = 0.1$ and $\rho = 0.01, 0.1, 1., 5$.

A Proof of Theorem 5.1 and Theorem 5.2

A.1 Proof of Theorem 5.1

The proof follows the lines of the existence proof for the time-dependent Navier-Stokes equations given in [26, Chap. III] and is divided into two parts, namely the derivation of a-priori estimates and the passage to the limit. It is worth noting that in addition to the well understood analytical handling of the nonlinearity of the Navier-Stokes equations one in essence has to cope with the technical difficulties arising from non-local solution operator BB applied to the nonlinearity of the Navier-Stokes equations. However, as will be demonstrated the smoothing properties of this operator and its sign enhance the stability properties of the Navier-Stokes equations.

A.1.1 Part 1: A-priori estimates via Galerkin-ansatz

Let $\psi_1, \ldots, \psi_m, \ldots$ denote the eigenfunctions of the operator S^{-1} (which exist as elements of $\mathcal{D}(S)$ due to [7, Theorem 4.3,4.4, Proposition 4.7] and are total in V). For $m \in \mathbb{N}$ define the discrete Ansatz space by $V_m = \text{span}\ \{\psi_1, \ldots, \psi_m\} \subset V$ and choose for $m \in \mathbb{N}$ a vector $(y_1^0, \ldots, y_m^0) \in \mathbb{R}^m$ such that

$$y_{m_0} := \sum_{i=1}^{m} y_i^0 \chi_i \longrightarrow \phi \quad (m \longrightarrow \infty) \text{ in } V.$$

Set

$$y_m(t,x) := \sum_{i=1}^{m} y_i^m(t)\psi_i(x)$$

and note that by the linearity of the operator B

$$\begin{aligned}(By_m)(t,x) &= \sum_{i=1}^{m} y_i^m(t)(B\psi_i)(x) \text{ and}\\ (B((y_m\nabla)y_m))(t,x) &= \sum_{i,l=1}^{m} y_i^m(t)y_l^m(t)(B((\psi_i\nabla)\psi_l))(x)\end{aligned}$$

holds. Inserting y_m into the state equation and using ψ_j, $j=1,\ldots,m$, as test functions leads to the system of nonlinear differential equations for the vector $Y(t) := (y_1^m,\ldots,y_m^m)^t$ given next.

$$\begin{aligned}(36)\quad & M\dot{Y} + \nu DY + \left[\sum_{i,l=1}^{m} y_i^m(t)y_l^m(t)\int_\Omega (\psi_i\nabla)\psi_l\psi_j\,dx\right]_{j=1}^{m}\\ &+\frac{\rho}{h}\tilde{M}Y + \rho\left[\sum_{i,l=1}^{m} y_i^m(t)y_l^m(t)\int_\Omega B((\psi_i\nabla)\psi_l)B(\psi_j)\,dx\right]_{j=1}^{m} = 0,\\ &\qquad\qquad Y(0) = Y_0 = (y_1^0,\ldots,y_m^0)^t.\end{aligned}$$

Here, the matrix M denotes the mass matrix, D denotes the stiffness matrix and the matrix $\tilde{M}$ is defined by

$$\tilde{M} = (\tilde{m}_{il})_{i,l=1}^{m}, \quad \tilde{m}_{il} = \int_\Omega B\psi_i B\psi_l\,dx.$$

Since the mass matrix is regular, the nonlinear system (36) admits a solution in some maximal time interval $[0,t_m)$. It will follow from the a-priori estimates of the solution that this interval may be chosen as large as desired.

To obtain a-priori estimates for y_m scalar-multiply (36) with Y. Application of Young's inequality then leads to the differential inequality

$$(37)\qquad \frac{1}{2}\frac{d}{dt}|y_m|_H^2 + \nu\,|y_m|_V^2 + \frac{\rho}{h}|By_m|_H^2 \le \frac{\rho h}{2}|B((y_m\nabla)y_m)|_H^2 + \frac{\rho}{2h}|By_m|_H^2,$$

for the function $t\mapsto |y_m(t)|_H^2$. Note, that $w_m := B((y_m\nabla)y_m)$ exists in V and is unique, since $(y_m\nabla)y_m$ is an element of V'. Next, estimate $|w_m|_H^2$. By the definition of B the function w_m satisfies

$$\nu h\,|w_m|_V^2 + |w_m|_H^2 = \int_\Omega (y_m\nabla)y_m w_m\,dx \le \frac{1}{4\nu h}|(y_m\nabla)y_m|_{V^*}^2 + \nu h|w_m|_V^2.$$

Estimation of the term $|(y_m\nabla)y_m|_{V^*}^2$ utilizing the first estimate in (2) leads to

$$(38)\qquad |w_m|_H^2 \le \frac{1}{2\nu h}|y_m|_H^2|y_m|_V^2.$$

Note that in the first estimate in (2) $C = \sqrt{2}$ holds. Estimate (38) together with the differential inequality (37) leads to

$$\frac{1}{2}\frac{d}{dt}|y_m|_H^2 + \left(\nu - \frac{\rho}{4\nu}|y_m|_H^2\right)|y_m|_V^2 + \frac{\rho}{2h}|By_m|_H^2 \leq 0. \tag{39}$$

The outermost addend on the left-hand-side can be estimated with

Lemma A.1. For $y \in V$ let $w := By$. Then $w \in V \cap H^3(\Omega)^2$ and $Sw \in V$. Moreover,

$$|w|_H^2 \;\geq\; |y|_H^2 \;-\; 2\nu h|y|_V^2.$$

Proof of Lemma A.1. By the definition of the operator B the regularity claim for w follows from [26, Chap. I, Prop. 2.3]. Thus, $Sw \in H^1(\Omega)^2$ [7, Remark 1.10] and since $y \in V$ even an element of V. Therefore, Sw can be utilized as test function in the equation

$$w \,+\, hAw \;=\; y.$$

This gives

$$|w|_V^2 \,+\nu h|Sw|_H^2 \;=\; \int_\Omega \nabla w \nabla y \,dx \;\Rightarrow\; |w|_V^2 + 2\nu h|Sw|_H^2 \;\leq\; |y|_V^2.$$

Furthermore, using y as test function the latter estimate leads to

$$|y|_H^2 \;=\; \int_\Omega yw + \nu h\nabla y\nabla w \,dx \;\leq\; \frac{1}{2}|y|_H^2 + \frac{1}{2}|w|_H^2 + h\nu|y|_V|w|_V \leq\; \frac{1}{2}|y|_H^2 + \frac{1}{2}|w|_H^2 + h\nu|y|_V^2,$$

which gives the claim. $\square$

From (39) it now follows that

$$\frac{1}{2}\frac{d}{dt}|y_m|_H^2 + \left(\nu(1-\rho) - \frac{\rho}{4\nu}|y_m|_H^2\right)|y_m|_V^2 + \frac{\rho}{2h}|y_m|_H^2 \leq 0. \tag{40}$$

Since $y_m(0) \to \phi$ in H one may fix some $m_0 \in \mathbb{N}$ such that

$$|y_m(0)|_H^2 \;\leq\; 2|\phi|_H^2 = d_H \quad \text{for all } m \geq m_0.$$

Then, for all $0 < \rho \leq \rho_0$ with ρ_0 from (20) $\nu(1-\rho) - \frac{\rho}{4\nu}d_H > \frac{\nu}{2}$ holds true.

Since (36) describes a nonlinear system of ordinary differential equations there exists some t_m such that a solution exists and is continuous in the interval $[0, t_m)$. It shall be shown that $t_m = T$. To begin with assume that $|y_m(t)|_H \to \infty (t \to t_m)$. Since

$$\nu(1-\rho) - \frac{\rho}{4\nu}|y_m(0)|_H^2 \;\geq\; \nu(1-\rho) - \frac{\rho}{4\nu}d_H \;\geq\; \frac{7\nu}{8} > \;\frac{\nu}{2},$$

there exists a time $t_m^* \leq t_m$ and $t_m^* > 0$ such that

$$\nu(1-\rho) - \frac{\rho}{4\nu}|y_m(t_m^*)|_H^2 \;=\; \frac{\nu}{2}$$

and

$$\nu(1-\rho) - \frac{\rho}{4\nu}|y_m(t)|_H^2 \;>\; \frac{\nu}{2} \quad \text{for all } t \in [0, t_m^*).$$

Now (40) implies

$$|y_m(t)|_H^2 \;\leq\; \mathrm{e}^{-\frac{\rho}{h}t}\,|y_m(0)|_H^2 \;\leq\; \mathrm{e}^{-\frac{\rho}{h}t}\,d_H \quad \text{for all } t \in [0, t_m^*],$$

and this in turn together with a continuity argument gives

$$\frac{\nu}{2} \;=\; \nu(1-\rho) - \frac{\rho}{4\nu}|y_m(t_m^*)|_H^2 \;\geq\; \nu(1-\rho) - \frac{\rho}{4\nu}d_H\,\mathrm{e}^{-\frac{\rho}{h}t_m^*} \;>\; \frac{\nu}{2},$$

which is a contradiction. Therefore, $t_m = t_m^* = T$ and

$$|y_m(t)|_H^2 \;\leq\; \mathrm{e}^{-\frac{\rho}{h}t}\,d_H \quad \text{for all } t \in [0, T].$$

Integration of (40) with respect to time then yields

$$|y_m|_{L^2(V)}^2 \;\leq\; \frac{2}{\nu}\,d_H \quad \text{and} \quad |y_m|_{L^2(H)}^2 \;\leq \frac{2h}{\rho}\,d_H.$$

Utilizing these estimates one obtains for the time derivative the estimate

$$|y_{m_t}|_{L^2(V')}^2 \;\leq\; C(\nu)\,\left\{1 + \frac{\rho}{h}\right\}\,d_H.$$

Altogether,

a) The sequences $\{y_m\}_{m\in\mathbb{N}} \subset L^\infty(H)$ and $\{y_m\}_{m\in\mathbb{N}} \subset L^2(V)$, are bounded independent of the parameter h and the dimension m of the Ansatz space, for $T > 0$ arbitrary, but fixed .

b) The sequence $\{y_{m_t}\}_{m\in\mathbb{N}} \subset L^2(V')$ is bounded by a constant of size $1 + \frac{\rho}{h}$ and independent of the dimension m of the Ansatz space, for $T > 0$ arbitrary, but fixed.

A.1.2 Part 2: Passage to the limit

Now a) and b) from above are equivalent to $\{y_m\}_{m\in\mathbb{N}} \subset W$ bounded and since $W \hookrightarrow\hookrightarrow L^2(H)$ and $W \hookrightarrow C([0,T];H)$ one concludes convergence of a subsequence, again denoted by $\{y_m\}_{m\in\mathbb{N}}$, weakly in W, weak* in $L^\infty(H)$ and strong in $L^2(H)$ to a function $y \in W$. Thus, passing to the limit in (36) in the linear terms is straightforward, as is for the nonlinear term of the Navier-Stokes equations due to the strong convergence of the subsequence in $L^2(H)$, see [26, Lemma 3.2]. It finally remains to establish

$$\int_0^T \int_\Omega B(y_m - y)Bw_j\chi(t)\,dxdt \to 0 \quad (m \to \infty)$$

and

$$\int_0^T \int_\Omega B((y_m\nabla)y_m - (y\nabla)y)Bw_j\chi(t)\,dxdt$$
$$= \int_0^T \int_\Omega ((y_m\nabla)y_m - (y\nabla)y)BBw_j\chi(t)\,dxdt \to 0 \quad (m\to\infty),$$

where $\chi \in C_0^\infty(0,T)$. For the first term the boundedness of B and the strong convergence of $\{y_m\}_{m\in\mathbb{N}}$ in $L^2(H)$ gives the result. Since for $\chi \in C_0^\infty(0,T)$ the functions $BB\chi w_j \in C^1(\bar{Q})$ a further application of [26, Lemma 3.2] gives the convergence in the second term. Uniqueness of the solution is an immediate consequence of inequality (40). This completes the proof of Theorem 5.1. □

Remark A.2. In the proof of Theorem 5.1 only $\phi \in H$ is required.

A.2 Proof of Theorem 5.2

Proof. Choose the sequence $\{y_m\}_{m\in\mathbb{N}}$ such that $|y_m(0)|_V^2 \le 2|y(0)|_V^2$. This is possible since $y_m(0)\to y(0)$ in V. Proceeding as in the proof of Theorem 5.1 it is now sufficient to establish a-priori estimates for the solution of (21) in appropriate norms. To begin with denote by $y = y_m$ the solution of (36).

Lemma A.3. Let $\kappa := By$ and $\tau := B\kappa$. Then

$$\int_\Omega By\,BSy\,dx \;=\; |y|_V^2 \;-\; \nu h\int_\Omega (S\kappa \,+\, S\tau)\;Sy\,dx$$

and

$$|S\tau|_H^2,\;|S\kappa|_H^2 \;\le\; \frac{1}{4\nu h}\,|y|_V^2. \tag{41}$$

Proof of Lemma A.3. The definition of κ and τ implies $\kappa \in H^3(\Omega)^2\cap V$, $\tau \in H^5(\Omega)^2\cap V$. Moreover, $S\kappa$ and $S\tau$ are elements of V. Integration by parts gives the first part of the claim. To obtain the second claim test the equation for κ with $S\kappa$. This gives

$$|\kappa|_V^2 \;\le |y|_V^2 \quad\text{and}\quad |S\kappa|_H^2 \;\le\; \frac{1}{4\nu h}\,|y|_V^2.$$

Since the same estimate holds with κ replaced by τ and y replaced by κ the lemma is proved. □

Now test (36) with Sy to obtain

$$\frac{1}{2}\frac{d}{dt}\;|y|_V^2 + \nu\;|Sy|_H^2 + \int_\Omega (y\nabla)y\,Sy\,dx = -\frac{\rho}{h}\int_\Omega ByBSy\,dx - \rho\int_\Omega B((y\nabla)y)BSy\,dx.$$

Straightforward estimation utilizing the fourth inequality in (2), (22), (41), Young's inequality and the estimate

$$\rho \int_\Omega B((y\nabla)y)BSy\,dx \;\le\; \frac{\rho^2}{2\nu^2 h}|y|_H^2|y|_V^2 \;+\; \frac{\nu}{4}|Sy|_H^2$$

leads to

$$\frac{1}{2}\frac{d}{dt}|y|_V^2 + \frac{\nu}{4}|Sy|_H^2 + \frac{\rho}{h}\left(1-\rho-\frac{\rho}{2\nu^2}d_H\right)|y|_V^2 \;\le\; C_\nu\,|y|_V^4|y|_H^2. \tag{42}$$

By (20) $1-4\rho-\frac{\rho}{2\nu^2}d_H > \frac{1}{2}$ so that a Gronwall argument applied to the function $v(t) = \mathrm{e}^{\frac{\rho}{h}t}|y|_V^2$ gives

$$|y|_V^2 \;\le\; 2|\phi|_V^2\,\mathrm{e}^{-\frac{\rho}{h}t}\,\mathrm{e}^{2C_\nu|y|^2_{L^\infty(H)}|y|^2_{L^2(V)}} \;\overset{(22)}{\le}\; 2C(\nu)\,\mathrm{e}^{-\frac{\rho}{h}t}|\phi|_V^2.$$

This is the first inequality in (23) and it implies the second one. The third one follows by integrating (42). Finally the fourth inequality in (23) is obtained by testing (36) with y_t and straightforward estimation, utilizing (2). As a consequence of the a-priori estimates for $\{y_m\}_{m\in\mathbb{N}}$ the unique solution of (21) is an element of $H^{2,1}(Q)$, which completes the proof of Theorem 5.2. □

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