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Semidiscretization and error estimates for distributed control of the instationary Navier-Stokes equations

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Contents

1 Introduction **3**

2 The optimal control problem **5**

3 The discrete problem **9**

4 Proof of Theorem 3.1 **11**

5 Numerical results **18**

Semidiscretization and error estimates for distributed control of the instationary Navier-Stokes equations

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Abstract

Optimal control problems governed by the two-dimensional instationary Navier-Stokes equations and their spatial discretizations with finite elements are investigated. A concept of semi–discrete solutions to the control problem is introduced which is utilized to prove existence and uniqueness of discrete controls in neighbourhoods of regular continuous solutions. Furthermore, an optimal error estimate in terms of the spatial discretization parameter is given. Finally a numerical example is presented that illustrates the potential of the discrete approach.

Key words. Optimal control, instationary Navier-Stokes equations, semi-discretization, error estimates, Newton's method.
AMS(MOS) subject classification. 49M15,49M29,65K10,76D55.

1 Introduction

This research is devoted to the analysis and error control of finite element approximations of optimal control problems involving the time dependent Navier–Stokes equations. Thus we consider

$$\min \ J(y,u) \text{ over } (y,u) \tag{1.1}$$

subject to

$$\left\{\begin{array}{rcll} y_t-\nu\Delta y+(y\cdot\nabla)y+\nabla p &=& Bu(t) & \text{in } \Omega\times(0,T)\\ -\mathrm{div}\, y &=& 0 & \text{in } \Omega\times(0,T)\\ y(.,t) &=& 0 & \text{on } \partial\Omega\times(0,T)\\ y(.,0) &=& y_0 & \text{in } \Omega, \end{array}\right. \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^2$ and $T>0$. The vector valued variable y and the scalar valued variable p represent the velocity and the pressure of the fluid. Furthermore, u denotes the control variable and B the control operator. The precise setting of problem

(1.1), (1.2) will be given in Section 2. The class of functionals J, which we allow, will include tracking type functionals

$$J(y,u) = \tfrac{1}{2}\int_0^T\int_\Omega |y-z|^2 dx\,dt + \tfrac{\gamma}{2}\int_\Omega |y(T)-z(T)|^2 dx + \tfrac{\alpha}{2}\int_0^T\int_\Omega |u|^2 dx\,dt \tag{1.3}$$

and functionals involving the vorticity of the fluid

$$J(y,u) = \tfrac{1}{2}\int_0^T\int_\Omega |\text{curl } y|^2 dx\,dt + \tfrac{\alpha}{2}\int_0^T\int_\Omega |u|^2 dx\,dt \tag{1.4}$$

where $\alpha, \gamma > 0$ and z are given. For the following discussion it is convenient to formally represent all the equality constraints involved in (1.2) by $\hat{e}(y,p,u) = 0$ so that (1.1), (1.2) can be expressed in the form

$$\text{(P)} \qquad \begin{cases} \min \ J(y,u) \text{ over } (y,u) \\ \text{subject to} \quad \hat{e}(y,p,u) = 0. \end{cases}$$

For tackling optimal control problems governed by the instationary Navier-Stokes equations two approaches can be distinguished depending on whether (y,p) in (1.1), (1.2) are considered as independent variables or as functions of the control variable u. In the former case $\hat{e}(y,p,u) = 0$ represents an explicit constraint for the optimization problem whereas in the latter case $\hat{e}(y(u),p(u),u) = 0$ serves the purpose of describing the evaluation of (y,p) as a function of u. In fact (P) can be expressed as the reduced problem

$$(\hat{P}) \qquad \min \ \hat{J}(u) = J(y(u),u) \text{ over } u,$$

where $y(u)$ is implicitly defined via $\hat{e}(y(u),p(u),u) = 0$.

Candidates u^* for optimal solutions of problem $\hat{P}$ are characterized by the first order necessary optimality condition

$$\hat{J}'(u^*) = 0 \tag{1.5}$$

which for example can be solved by Newton's method [10].

The following discussion is related to problem $(\hat{P})$. In general two approaches can be distinguished to compute discrete solutions to this problem; the first approach is called *first discretize then optimize* and works as follows: The terms appearing in the functional $\hat{J}$ and the state equations together with the controls are discretized. The solutions of the resulting finite dimensional optimization problem then are considered as discrete approximations to the continuous optimization problem. In this research we will utilize a second method which is called *first optimize then discretize.* To apply it, the state and adjoint variables which appear in equation (1.5) are replaced by their discrete counterparts. This procedure gives rise to an approximation F_h of the gradient $\hat{J}'$ and we are left with the equation

$$F_h(u_h^*) = 0. \tag{1.6}$$

The corresponding solutions u_h^* are considered as approximations of critical points of the functional $\hat{J}$ in $(\hat{P})$. We emphasize that in this approach the control space U is not discretized explicitly but the discrete solutions u_h^* happen to be discretized implicitly in terms of the finite element approximation λ_h of the adjoint variable λ, compare (2.12) and (3.11).

To prove the existence of solutions to (1.6) and to estimate the error between u^* and u_h^* we shall use a quantified version of Newton's method. Roughly speaking, our main result is, that under suitable assumptions on the functional J and on u^*, there exists $h_0 > 0$ and a neighborhood $\mathcal{U}(u^*)$ such that for every $0 < h \leq h_0$ (1.6) admits a unique solution u_h^*. Moreover we have the error estimate

$$\int_0^T \|u^* - u_h^*\|_U^2 \, dt \ \leq Ch^4 \quad \forall\, 0 < h \leq h_0. \tag{1.7}$$

Let us briefly comment on some related contributions. In [6] Gunzburger and Manservisi investigate the *first discretize then optimize* approach for tracking type control of the instationary Navier-Stokes equations. Among other things they prove that discrete solutions converge to solutions of the continuous problem. However they did not provide error estimates and also did not prove that in a small enough neighborhood of every regular continuous solution there exists a unique discrete solution. Similar investigations for the boundary control case are provided by the same authors in [7]. Contributions for the stationary Navier-Stokes equations are given in [5, 12, 11],and semidiscrete Ritz-Galerkin approximations of nonlinear parabolic control problems are investigated in [14, 15].

The paper is organized as follows. Section 2 contains the necessary functional analytic prerequisites. The numerical preliminaries and the concept of discrete solutions are introduced in Section 3. Section 4 contains the convergence proof and in Section 5 we present a numerical example.

2 The optimal control problem

The purpose of this section is to describe the distributed control problem, which we are going to approximate. Let $\Omega \subset \mathbb{R}^2$ be a convex polygon and H and V the closure of $\{v \in C_0^\infty(\Omega) \,|\, \operatorname{div} v = 0\}$ in $L^2(\Omega)^2$ and $H^1(\Omega)^2$ respectively. Furthermore, we denote by $P : L^2(\Omega)^2 \to H$ the Leray–operator.
For $T > 0$ we define $\Omega^T := \Omega \times (0,T)$ and $(\partial\Omega)^T := \partial\Omega \times (0,T)$ as well as the space of time dependent functions $W := L^2(0,T; H_0^1(\Omega)^2) \cap C^0([0,T]; L^2(\Omega)^2)$ endowed with the norm

$$\|v\|_W := \sup_{0 \leq t \leq T} \|v(t)\| + \Big(\int_0^T \|\nabla v\|^2 dt \Big)^{\frac{1}{2}}.$$

Here and in what follows we use $\|\cdot\|$ to denote the L^2–norm. Our control space is a Hilbert space $\big(U, (.,.)_U\big)$ and $B \in \mathcal{L}(U, L^2(\Omega)^2)$ is the control operator. Now, let us fix $y_0 \in V \cap H^2(\Omega)^2$ and consider for $u \in L^2(0,T;U)$ the Navier–Stokes equations (1.2).

Since $Bu \in L^2(0,T;L^2(\Omega)^2)$ and Ω is convex, standard results imply that (1.2) has a unique solution (y,p) satisfying $y \in L^2(0,T;H^2(\Omega)^2) \cap C^0([0,T];V)$ with $y_t, \nabla p \in L^2(\Omega^T)$. Furthermore

$$\sup_{0\le t\le T} \|y(t)\|_{H^1} + \int_0^T \big(\|y\|_{H^2}^2 + \|y_t\|^2 + \|\nabla p\|^2\big)dt \le C = C(\|y_0\|_{H^1}, M) \tag{2.1}$$

as long as $\|u\|_{L^2(0,T;U)} \le M$. In particular, we can consider $u \mapsto y = y(u)$ as a mapping from $L^2(0,T;U)$ to W.
We shall study the following cost functional $\hat{J} : L^2(0,T;U) \to \mathbb{R}$:

$$\hat{J}(u) := \phi_0(y(T)) + \int_0^T \Theta(y(t),t)dt + \frac{\alpha}{2}\int_0^T \|u(t)\|_U^2 dt$$

where $y = y(u)$ satisfies (1.2). The functions ϕ_0 and Θ appearing in the definition of $\hat{J}$ have the following properties:

(C1) $\phi_0 \in C^2(L^2(\Omega)^2, \mathbb{R}^+)$ and $\phi_0'(\zeta), \phi_0''(\zeta)v \in H$ if $\zeta, v \in H$;

(i) $\|\big(\phi_0''(\zeta) - \phi_0''(\tilde{\zeta})\big)v\| \le C(L)\|\zeta - \tilde{\zeta}\|\,\|v\|, \quad$ for all $\zeta, \tilde{\zeta}, v \in L^2(\Omega)^2$, $\|\zeta\|, \|\tilde{\zeta}\| \le L$.

(ii) If $\zeta, v \in V$, then $\phi_0'(\zeta), \phi_0''(\zeta)v \in V$ and

$$\|\phi_0'(\zeta)\|_{H^1} \le C(L), \quad \|\phi_0''(\zeta)v\|_{H^1} \le C(L)\|v\|_{H^1} \quad \text{for } \|\zeta\|_{H^1} \le L.$$

(C2) $\Theta : H_0^1(\Omega)^2 \times [0,T] \to \mathbb{R}^+$, $D_y\Theta$ and $D_{yy}\Theta$ are continuous and

(i) $\|\big(D_{yy}\Theta(y,t) - D_{yy}\Theta(\tilde{y},t)\big)v\|_{H^{-1}} \le C(L)\|y - \tilde{y}\|_{H^1}\|v\|_{H^1}$
for all $y, \tilde{y}, v \in H_0^1(\Omega)^2$, $\|y\|_{H^1}, \|\tilde{y}\|_{H^1} \le L$ and $t \in [0,T]$.

(ii) If $y \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2$, $t \in [0,T]$, then $D_y\Theta(y,t) \in L^2(\Omega)^2$ and

$$\|D_y\Theta(y,t)\| \le C(L)(1 + \|y\|_{H^2}), \quad \text{for } \|y\|_{H^1} \le L.$$

(iii) If $v \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2$, $t \in [0,T]$, then $D_{yy}\Theta(y,t)v \in L^2(\Omega)^2$ and

$$\|D_{yy}\Theta(y,t)v\| \le C(L)\|v\|_{H^2}, \quad \text{for } \|y\|_{H^1} \le L.$$

Remark 2.1. We note that

a) the above hypotheses also imply for $\|\zeta\|, \|\tilde{\zeta}\| \le L$, $\|y\|_{H^1}, \|\tilde{y}\|_{H^1} \le L$

$$\|\phi_0'(\zeta) - \phi_0'(\tilde{\zeta})\| \le C(L)\|\zeta - \tilde{\zeta}\| \tag{2.2}$$
$$\|D_y\Theta(y,t) - D_y\Theta(\tilde{y},t)\|_{H^{-1}} \le C(L)\|y - \tilde{y}\|_{H^1}, \tag{2.3}$$

b) for a given $z \in C^0([0,T];V)$ and $\gamma > 0$ the choices

$$\phi_0(\zeta) := \frac{\gamma}{2}\int_\Omega |\zeta - z(T)|^2 dx, \quad \Theta(y,t) := \frac{1}{2}\int_\Omega |y - z(t)|^2 dx$$

lead to tracking–type functionals (1.3), and

c) functionals (1.4) regulating the square of the vorticity are included via the choices

$$\phi_0 \equiv 0, \quad \Theta(y,t) := \frac{1}{2}\int_\Omega |\mathrm{curl}\, y|^2 dx.$$

The proof of the following theorem can for example be found in [1, 9].

Theorem 2.1. The problem

$$\min \hat{J}(u) \quad \text{s.t.} \quad u \in L^2(0,T;U) \text{ and } y = y(u) \text{ satisfies (1.2)} \tag{2.4}$$

has at least one solution u^*.

A possible way to find a solution of (2.4) is to analyze the necessary first order optimality condition, which requires the first derivative of $\hat{J}$. From the arguments in [1] it follows that the derivative of $u \mapsto y(u)$ in the direction $\phi \in L^2(0,T;U)$ is given by $\langle y'(u), \phi\rangle = v$, where $v \in W$ is the solution the linearized Navier–Stokes equations

$$\left\{ \begin{array}{rcll} v_t - \nu\Delta v + (y\cdot\nabla)v + (v\cdot\nabla)y + \nabla r &=& B\phi & \text{in } \Omega^T \\ -\mathrm{div}\, v &=& 0 & \text{in } \Omega^T \\ v(.,t) &=& 0 & \text{on } (\partial\Omega)^T \\ v(.,0) &=& 0 & \text{in } \Omega. \end{array} \right. \tag{2.5}$$

It is not difficult to verify that (v,r) satisfies

$$\sup_{0\le t\le T} \|v(t)\|^2_{H^1} + \int_0^T \big(\|v\|^2_{H^2} + \|v_t\|^2 + \|\nabla r\|^2\big)dt \le C(M)\|\phi\|^2_{L^2(0,T;U)} \tag{2.6}$$

as long as $\|u\|_{L^2(0,T;U)} \le M$.

Thus, we obtain the following formula for the derivative of $\hat{J}$:

$$\langle \hat{J}'(u), \phi\rangle = (\phi_0'(y(T)), v(T)) + \int_0^T \langle D_y\Theta(y(t),t), v(t)\rangle dt + \alpha\int_0^T (u(t),\phi(t))_U dt.$$

In order to rewrite this expression and for later calculations it is convenient to introduce the operator $R : W \times L^2(0,T;H^{-1}(\Omega)^2) \times L^2(\Omega)^2 \to W$ as follows: $w = R(z,f,\zeta)$ if w solves the linear backward problem

$$\left\{ \begin{array}{rcll} -w_t - \nu\Delta w - (z\cdot\nabla)w + (\nabla z)^t w + \nabla\xi &=& f & \text{in } \Omega^T \\ -\mathrm{div}\, w &=& 0 & \text{in } \Omega^T \\ w(.,t) &=& 0 & \text{on } (\partial\Omega)^T \\ w(.,T) &=& P\zeta & \text{in } \Omega. \end{array} \right. \tag{2.7}$$

An energy argument combined with the well–known inequality $\|v\|_{L^4} \le \sqrt[4]{2}\|v\|^{\frac{1}{2}}\|\nabla v\|^{\frac{1}{2}}$ for $v \in H^1_0(\Omega)$ implies that

$$\|w\|_W \le C(K)\big(\|\zeta\| + \|f\|_{L^2(0,T;H^{-1})}\big) \quad \text{provided that } \|z\|_W \le K.$$

Note that under the above conditions on (z, f, ζ) the pressure in (2.7) only exists as a distribution on $\Omega\times(0,T)$ (cf. e.g. [13], Chapter III). If in addition $(z, f, \zeta) \in C^0([0,T]; H^1_0(\Omega)^2)\times L^2(\Omega^T)\times V$ then it is possible to prove

$$(2.8) \qquad \sup_{0\le t\le T} \|w(t)\|^2_{H^1} + \int_0^T \big(\|w\|^2_{H^2} + \|w_t\|^2 + \|\nabla\xi\|^2\big)dt \le C(K)\big(\|\zeta\|^2_{H^1} + \int_0^T \|f\|^2 dt\big)$$

as long as $\sup_{0\le t\le T}\|z(t)\|_{H^1} \le K$.
If we now define

$$(2.9) \qquad \lambda := R\big(y(u), D_y\Theta(y(u), .), \phi_0'(y(T))\big)$$

it is not difficult to verify using integration by parts together with the fact that $P\phi_0'(y(T)) = \phi_0'(y(T))$ (by (C1)) that (cf. [1])

$$\langle \hat{J}'(u), \phi\rangle = \int_0^T \big((\lambda(t), B\phi(t)) + \alpha(u(t), \phi(t))_U\big)dt = \int_0^T \big(B^*\lambda(t) + \alpha u(t), \phi(t)\big)_U dt$$

for all $\phi \in L^2(0,T;U)$. Thus, it is natural to introduce the mapping $F : L^2(0,T;U) \to L^2(0,T;U)$ by

$$(2.10) \qquad F(u) := B^*\lambda + \alpha u,$$

where λ is given by (2.9). The functions u that satisfy the first order optimality conditions then are precisely the zeros of F.

In what follows we shall assume that $u^* \in L^2(0,T;U)$ satisfies $F(u^*) = 0$ and that this solution is nonsingular in the sense that

$$(2.11) \qquad F'(u^*) \text{ is an isomorphism from } L^2(0,T;U) \text{ onto } L^2(0,T;U).$$

It is straightforward to derive the following expression for $F'(u)$:

$$(2.12) \quad \langle F'(u), \phi\rangle = B^*\, R\Big(y(u), (v\cdot\nabla)\lambda - (\nabla v)^t\lambda + D_{yy}\Theta(y(u), .)v, \phi_0''(y(T))v(T)\Big) + \alpha\phi,$$

where v and λ are given by (2.5) and (2.9) respectively.

Remark 2.2. With $\hat{e}$ from (P) the action of the reduced Hessian $\hat{J}''(u)$ associated to the control problem $(\hat{P})$ at $u \in U$ is given by

$$\begin{aligned}(2.13) \quad \hat{J}''(u)\bullet = {}& \hat{e}_u^*(y,p,u)\hat{e}_y^{-*}(y,p,u)J_{yy}(y,u)\hat{e}_y^{-1}(y,p,u)\hat{e}_u(y,p,u)\bullet + \\ & \hat{e}_u^*(y,p,u)\hat{e}_y^{-*}(y,p,u)\langle \hat{e}_{yy}(y,p,u)\left(\hat{e}_y^{-1}(y,p,u)\hat{e}_u(y,p,u)\bullet, \circ\right), \lambda\rangle + J_{uu}(y,p,u)\bullet,\end{aligned}$$

with λ from (2.9) and $\langle\cdot,\cdot\rangle$ denoting the duality pairing between $L^2(0,T;H^1_0(\Omega)^2)$ and $L^2(0,T;H^{-1}(\Omega)^2)$. Clearly, $\hat{J}''(u) = F'(u)$ with $F'(u)$ from (2.12). Second order sufficiency conditions for control problem (P) which ensure (2.11) are discussed in [9, 10].

3 The discrete problem

Let $\mathcal{T}_h$ be a triangulation of Ω with maximum mesh size $h := \max_{S\in\mathcal{T}_h} \operatorname{diam}(S)$. We assume that $\mathcal{T}_h$ is uniformly regular, i.e. that each triangle $S \in \mathcal{T}_h$ contains a ball of radius $\kappa_1 h$ and is contained in a ball of radius $\kappa_2 h$, where the constants κ_1 and κ_2 are independent of h. Our discretization of the Navier–Stokes equations uses the Taylor–Hood element, so that the velocity space $H_0^1(\Omega)^2$ is approximated by

$$X_h = \{v_h \in C^0(\bar{\Omega})^2 \,|\, v_{h|S} \in P^2(S)^2 \text{ for all } S \in \mathcal{T}_h, v_{h|\partial\Omega} = 0\},$$

while the pressure space $L_0^2(\Omega) = \{q \in L^2(\Omega) \,|\, \int_\Omega q\,dx = 0\}$ is approximated by

$$M_h = \{q_h \in C^0(\bar{\Omega}) \,|\, q_{h|S} \in P^1(S) \text{ for all } S \in \mathcal{T}_h, \int_\Omega q_h dx = 0\}.$$

The discrete analogue of V is given by

$$V_h := \{v_h \in X_h \,|\, (\operatorname{div} v_h, q_h) = 0 \text{ for all } q_h \in M_h\}.$$

The L^2–projection $P_h : L^2(\Omega)^2 \to V_h$ is defined by

$$\int_\Omega (v, \phi_h) dx = \int_\Omega (P_h v, \phi_h) dx, \qquad \forall \phi_h \in V_h$$

and satisfies (cf. [8, I,Lemma 4.3])

$$\|v - P_h v\| + h\|\nabla(v - P_h v)\| \le ch^2 \|D^2 v\| \qquad \forall v \in V \cap H^2(\Omega)^2. \tag{3.1}$$

Furthermore, there exists an interpolation operator $i_h : L_0^2(\Omega) \to M_h$ with the property that

$$\|p - i_h p\| \le ch\|\nabla p\| \qquad \forall p \in L_0^2(\Omega) \cap H^1(\Omega). \tag{3.2}$$

It is convenient to introduce the following trilinear form

$$N_h(v, w, z) := \frac{1}{2} \int_\Omega \Big(\big((v \cdot \nabla) w\big) z - \big(v \cdot \nabla) z\big) w \Big), \qquad v, w, z \in H_0^1(\Omega)^2,$$

which satisfies $N_h(v, z, z) = 0$ for all $v, z \in H_0^1(\Omega)^2$.

The discrete analogue of (1.2) then takes the form: given $u \in L^2(0, T; U)$, find $(y_h(t), p_h(t)) \in X_h \times M_h$, $0 \le t \le T$ such that $y_h(0) = P_h y_0$ and

$$(y_{h,t}, \phi_h) + \nu(\nabla y_h, \nabla \phi_h) + N_h(y_h, y_h, \phi_h) - (\operatorname{div} \phi_h, p_h) = (Bu(t), \phi_h) \tag{3.3}$$
$$-(\operatorname{div} y_h, q_h) = 0 \tag{3.4}$$

for all $\phi_h \in X_h$ and $q_h \in M_h$. This problem has a unique solution, which we denote by $y_h = y_h(u)$ in order to emphasize the dependence on u. Inserting $\phi_h = y_h$ into (3.3) and observing that $N_h(y_h, y_h, y_h) = 0$ we immediately obtain the estimate

$$\|y_h\|_W \le C\big(\|y_0\| + \|u\|_{L^2(0,T;U)}\big) \le C(M), \tag{3.5}$$

as long as $\|u\|_{L^2(0,T;U)} \le M$.
An error analysis for finite element solutions to the Navier–Stokes equations was carried out in [8, I]. A careful inspection of the proofs of Lemma 5.1 and Lemma 5.4 therein shows that the estimates (2.1) on the continuous solution (y,p) of (1.2) are sufficient to ensure that

$$\int_0^T \big(\|y-y_h\|^2 + h^2\|\nabla(y-y_h)\|^2\big)dt \;\le\; ch^4 \tag{3.6}$$

$$\sup_{0\le t\le T} \|(y-y_h)(t)\|^2 \;\le\; ch^2 \tag{3.7}$$

with a constant c which is uniform in u as long as $\|u\|_{L^2(0,T;U)} \le M$.

Next, we introduce a discrete approximation $R_h : W \times L^2(0,T;H^{-1}(\Omega)^2)\times L^2(\Omega)^2 \to W$ to R. Given (z,f,ζ) denote by $w_h = R_h(z,f,\zeta)$ the unique solution of the discrete problem:

$$-(w_{h,t},\phi_h) + \nu(\nabla w_h,\nabla\phi_h) - ((z\cdot\nabla)w_h,\phi_h) + ((\nabla z)^t w_h,\phi_h) - (\operatorname{div}\phi_h,\xi_h) \;=\; \langle f,\phi_h\rangle \tag{3.8}$$

$$-(\operatorname{div} w_h, q_h) \;=\; 0 \tag{3.9}$$

for all $\phi_h \in X_h$ and $q_h \in M_h$, which satisfies $w_h(T) = P_h\zeta$. Inserting $\phi_h = w_h$ into (3.8) one deduces the following a–priori estimate:

$$\|w_h\|_W \le C(K)\big(\|\zeta\| + \|f\|_{L^2(0,T;H^{-1}(\Omega)^2)}\big), \qquad \text{provided that } \|z\|_W \le K. \tag{3.10}$$

We now define an approximation $F_h : L^2(0,T;U) \to L^2(0,T;U)$ to F by

$$F_h(u) := B^*\lambda_h + \alpha u, \tag{3.11}$$

where $\lambda_h = R_h(y_h(u), D_y\Theta(y_h(u),.), \phi_0'(y_h(T)))$.

Definition 3.1. We call u_h^* a discrete solution of the optimality system, if $F_h(u_h^*) = 0$.

In other words, u_h^* is a discrete solution, provided the following relations hold for (y_h^*, p_h^*),
$(\lambda_h^*, \xi_h^*) \in X_h \times M_h$: $y_h^*(0) = P_h y_0$, $\lambda_h^*(T) = \phi_0'(y_h^*(T))$ and

$$(y_{h,t}^*,\phi_h) + \nu(\nabla y_h^*,\nabla\phi_h) + N_h(y_h^*,y_h^*,\phi_h) - (\operatorname{div}\phi_h, p_h^*) \;=\; (Bu_h^*(t),\phi_h) \tag{3.12}$$

$$-(\operatorname{div} y_h^*, q_h) \;=\; 0 \tag{3.13}$$

$$-(\lambda_{h,t}^*,\phi_h) + \nu(\nabla\lambda_h^*,\nabla\phi_h) - ((y_h^*\cdot\nabla)\lambda_h^*,\phi_h) + ((\nabla y_h^*)^t\lambda_h^*,\phi_h) - (\operatorname{div}\phi_h,\xi_h^*) \;=\; \langle D_y\Theta(y_h^*,.),\phi_h\rangle \tag{3.14}$$

$$-(\operatorname{div}\lambda_h^*, q_h) \;=\; 0 \tag{3.15}$$

for all $\phi_h \in X_h$, $q_h \in M_h$ with $B^*\lambda_h^* + \alpha u_h^* = 0$. Note that we have not actually discretized the space U. Our main result can now be formulated as follows:

Theorem 3.1. Suppose that u^* is a solution of $F(u^*) = 0$ and that (2.11) holds. Then there exist $h_0 > 0$, $\delta_0 > 0$ such that for all $0 < h \le h_0$ there is a unique discrete solution $u_h^* \in L^2(0,T;U)$ of the optimality system satisfying $\|u^* - u_h^*\|_{L^2(0,T;U)} \le \delta_0$. Furthermore, we have for $0 < h \le h_0$

$$\|u^* - u_h^*\|_{L^2(0,T;U)} \le ch^2. \tag{3.16}$$

The proof uses a quantified version of Newton's method and will be presented in the next section. It is worth mentioning at this point that Newton's method is also used in order to solve the discrete problem numerically as will be described in §5.

4 Proof of Theorem 3.1

Let us begin by stating the quantified version of Newton's method mentioned above.

Theorem 4.1. Let X, Y be Banach spaces, $u_0 \in X$, $R > 0$ and $G \in C^1(B_R(u_0), Y)$. Assume that $G'(u_0)$ is an isomorphism of X onto Y with $\|G'(u_0)^{-1}\|_{L(Y,X)} \le \gamma$, that $\|G'(u_0)^{-1}G(u_0)\|_Y \le \epsilon$ and that

$$\|G'(u) - G'(v)\|_{L(X,Y)} \le K\|u - v\|_X \quad \text{for all } u, v \in B_R(u_0).$$

If

$$2\epsilon < R, \quad 2K\epsilon\gamma < 1 \tag{4.1}$$

then the problem $G(u) = 0$ has a unique solution $\bar{u} \in \overline{B_{2\epsilon}(u_0)}$.

Proof. see [3, Theorem 15.6].

We shall apply the above result with the choices $X = Y = L^2(0,T;U)$, $u_0 = u^*$, $R = 1$ and $G = F_h$, where F_h is defined in (3.11). In order to verify the corresponding conditions on F_h, we shall exploit the fact that F_h is an approximation to F together with our assumptions on F. Since F and F_h are defined in terms of R and R_h, a first step is to compare these two operators.

Lemma 4.1. Let $(z, f, \zeta) \in C^0([0,T]; H_0^1(\Omega)^2) \times L^2(\Omega^T) \times V$, $(\tilde{z}, \tilde{f}, \tilde{\zeta}) \in W \times L^2(0,T;H^{-1}(\Omega)^2) \times L^2(\Omega)^2$, $w = R(z, f, \zeta)$, $w_h = R_h(\tilde{z}, \tilde{f}, \tilde{\zeta})$ and suppose that $\|z\|_W, \|\tilde{z}\|_W \le K$. Then there exists a constant $C = C(K)$ such that

$$\begin{aligned}\|w - w_h\|_W \;&\le\; C\Big(\|w - P_h w\|_W + \|\xi - i_h\xi\|_{L^2(\Omega^T)} \\ &\qquad + \|z - \tilde{z}\|_W\|w_h\|_W + \|P\zeta - \tilde{\zeta}\| + \|f - \tilde{f}\|_{L^2(0,T;H^{-1}(\Omega)^2)}\Big).\end{aligned}$$

Proof. Multiplying the first equation in (2.7) by $\phi_h \in X_h$ we obtain

$$-(w_t, \phi_h) + \nu(\nabla w, \nabla \phi_h) - ((z\cdot\nabla)w, \phi_h) + ((\nabla z)^t w, \phi_h) - (\operatorname{div}\phi_h, \xi) = \langle f, \phi_h\rangle.$$

Combining this equation with (3.8) we see that $e := w - w_h$ satisfies $e(T) = P\zeta - P_h\tilde{\zeta}$ and

$$\begin{aligned}
&(4.2)\quad -(e_t, \phi_h) + \nu(\nabla e, \nabla\phi_h) - ((z\cdot\nabla)e, \phi_h) + ((\nabla z)^t e, \phi_h) - (\operatorname{div}\phi_h, \xi - \xi_h) &=& \langle g, \phi_h\rangle \\
&(4.3)\quad -(\operatorname{div} e, q_h) &=& 0
\end{aligned}$$

for all $\phi_h \in X_h$, $q_h \in M_h$. Here, $g \in L^2(0,T;H^{-1}(\Omega)^2)$ is given by

$$\langle g, \phi\rangle = \langle f - \tilde{f}, \phi\rangle + (((z-\tilde{z})\cdot\nabla)w_h, \phi) - (((\nabla z)^t - (\nabla\tilde{z})^t)w_h, \phi), \qquad \phi \in H_0^1(\Omega)^2.$$

Inserting $\phi_h = P_h e =: e_h \in V_h$ into (4.2) and observing that $(e_t, P_h e) = (P_h e_t, P_h e)$ as well as $(\operatorname{div} e_h, q_h) = 0$ for all $q_h \in M_h$ we obtain

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|e_h\|^2 + \nu\|\nabla e_h\|^2 \\
&= \nu(\nabla e_h, \nabla(P_h w - w)) + ((z\cdot\nabla)e, e_h) - ((\nabla z)^t e, e_h) + (\operatorname{div} e_h, \xi - i_h\xi) + \langle g, e_h\rangle \\
&\le \nu\|\nabla e_h\|\|\nabla(w - P_h w)\| + \|z\|_{L^4}\big(\|\nabla e_h\| + \|\nabla(w - P_h w)\|\big)\|e_h\|_{L^4} \\
&\quad + \|\nabla z\|\big(\|e_h\|_{L^4} + \|w - P_h w\|_{L^4}\big)\|e_h\|_{L^4} + \|\nabla e_h\|\|\xi - i_h\xi\| \\
&\quad + \|f - \tilde{f}\|_{H^{-1}}\|\nabla e_h\| + \|z - \tilde{z}\|_{L^4}\|\nabla w_h\|\|e_h\|_{L^4} + \|\nabla(z-\tilde{z})\|\|w_h\|_{L^4}\|e_h\|_{L^4}.
\end{aligned}$$

Using the interpolation inequality $\|v\|_{L^4} \le \sqrt[4]{2}\|v\|^{\frac{1}{2}}\|\nabla v\|^{\frac{1}{2}}$ for $v \in H_0^1(\Omega)$ along with Young's inequality we derive for $\beta, \delta > 0$

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\|e_h\|^2 + \nu\|\nabla e_h\|^2 \quad &\le \quad \delta\|\nabla e_h\|^2 + C_\delta\|\nabla(w - P_h w)\|^2 \\
&+ C_\delta(1 + \|z\|^2)\|\nabla z\|^2\|e_h\|^2 + C_\delta\|\nabla z\|^2\|w - P_h w\|^2 + C_\delta\|\xi - i_h\xi\|^2 \\
&+ C_\delta\|f - \tilde{f}\|^2_{H^{-1}} + C_\delta\|z - \tilde{z}\|^2\|\nabla w_h\|^2 + C_\delta\|\nabla(z-\tilde{z})\|^2(\beta^2 + \|w_h\|^2) + \frac{1}{\beta^2}\|\nabla w_h\|^2\|e_h\|^2.
\end{aligned}$$

If we choose $\delta = \frac{\nu}{2}$ and integrate with respect to time the result is

$$\begin{aligned}
\frac{1}{2}\|e_h(t)\|^2 + \frac{\nu}{2}\int_t^T \|\nabla e_h\|^2 ds \quad &\le \quad \frac{1}{2}\|P_h(P\zeta - \tilde{\zeta})\|^2 \\
&+ C(1 + \|z\|_W^2)\|w - P_h w\|_W^2 + C\int_0^T \|\xi - i_h\xi\|^2 ds + C\int_0^T \|f - \tilde{f}\|^2_{H^{-1}} ds \\
&+ C(\beta^2 + \|w_h\|_W^2)\|z - \tilde{z}\|_W^2 + C(K)\int_t^T \Big(\|\nabla z\|^2 + \frac{1}{\beta^2}\|\nabla w_h\|^2\Big)\|e_h\|^2 ds.
\end{aligned}$$

Choosing $\beta = \|w_h\|_W$ and applying Gronwall's lemma we finally obtain

$$\begin{aligned}
\sup_{0\le t\le T}\|e_h(t)\|^2 + \int_0^T \|\nabla e_h\|^2 dt \le C(K)\Big(&\|w - P_h w\|_W^2 + \|\xi - i_h\xi\|^2_{L^2(\Omega^T)} \\
&+ \|P\zeta - \tilde{\zeta}\|^2 + \|f - \tilde{f}\|^2_{L^2(0,T;H^{-1}(\Omega)^2)} + \|w_h\|_W^2\|z - \tilde{z}\|_W^2\Big)
\end{aligned}$$

and the lemma follows since $w - w_h = (w - P_h w) + e_h$. $\square$

Before we apply this result to derive an estimate for $\lambda - \lambda_h$, it is convenient to prove a uniform bound on $\|y_h(t)\|_{H^1}$. Let us denote by J_h the Riesz projection onto X_h. Using an inverse estimate we get

$$\begin{aligned} \|\nabla y_h(t)\| &\leq \|\nabla(y_h(t) - J_h y(t))\| + \|\nabla(J_h y(t) - y(t))\| + \|\nabla y(t)\| \\ &\leq Ch^{-1}\|y_h(t) - J_h y(t)\| + 3\|\nabla y(t)\| \\ &\leq Ch^{-1}\|y_h(t) - y(t)\| + C\|\nabla y(t)\|. \end{aligned}$$

In view of (3.7) and (2.1) we obtain

$$\sup_{0\leq t\leq T} \|y_h(t)\|_{H^1} \leq C \tag{4.4}$$

for all $u \in B_1(u^*) = \{u \in L^2(0,T;U) \mid \|u - u^*\|_{L^2(0,T;U)} \leq 1\}$.

Lemma 4.2. Let $\lambda = R\big(y, D_y\Theta(y,.), \phi_0'(y(T))\big)$, $\lambda_h = R_h\big(y_h, D_y\Theta(y_h,.), \phi_0'(y_h(T))\big)$, where $y = y(u)$, $y_h = y_h(u)$, $u \in B_1(u^*)$. Then

$$\|\lambda - \lambda_h\|_W \leq Ch \qquad \text{for all } u \in B_1(u^*). \tag{4.5}$$

Proof. In view of the regularity of y and (C1), (C2) we have $(y, D_y\Theta(y,.), \phi_0'(y(T)) \in C^0([0,T]; H_0^1(\Omega)^2) \times L^2(\Omega^T) \times V$, so that Lemma 4.1 together with (2.1), (4.4) and Remark 2.1 implies

$$\begin{aligned} \|\lambda - \lambda_h\|_W^2 &\leq C\Big(\|\lambda - P_h\lambda\|_W^2 + \|\xi - i_h\xi\|_{L^2(\Omega^T)}^2 + \|y - y_h\|_W^2\|\lambda_h\|_W^2 \\ &\qquad + \|\phi_0'(y(T)) - \phi_0'(y_h(T))\|^2 + \int_0^T \|D_y\Theta(y,.) - D_y\Theta(y_h,.)\|_{H^{-1}}^2 dt\Big) \\ &\leq C\Big(\|\lambda - P_h\lambda\|_W^2 + \|\xi - i_h\xi\|_{L^2(\Omega^T)}^2 + \big(1 + \|\lambda_h\|_W^2\big)\|y - y_h\|_W^2\Big). \end{aligned} \tag{4.6}$$

Furthermore, we deduce from (C1)(ii), (C2)(ii) and (2.1) that $\phi_0'(y(T)) \in V$, $D_y\Theta(y,.) \in L^2(0,T;L^2(\Omega)^2)$ as well as

$$\|\phi_0'(y(T))\|_{H^1}^2 + \int_0^T \|D_y\Theta(y,t)\|^2 dt \leq C,$$

so that (2.8) yields

$$\sup_{0\leq t\leq T} \|\lambda(t)\|_{H^1}^2 + \int_0^T \big(\|\lambda\|_{H^2}^2 + \|\lambda_t\|^2 + \|\nabla\xi\|^2\big)dt \leq C. \tag{4.7}$$

In view of Remark 2.1a) (C1), (C2), (4.4), (3.10) and (3.5) we also have

$$\|\lambda_h\|_W^2 \leq C\Big(\|\phi_0'(y_h(T))\|^2 + \int_0^T \|D_y\Theta(y_h,t)\|_{H^{-1}}^2 dt\Big) \leq C. \tag{4.8}$$

Combining these estimates with (3.6), (3.7), (3.1) and (4.6) finally shows

$$\begin{aligned}\|\lambda-\lambda_h\|_W^2 &\le Ch^2\Big(\sup_{0\le t\le T}\|\lambda(t)\|_{H^1}^2+\int_0^T\big(\|\lambda\|_{H^2}^2+\|\nabla\xi\|^2\big)dt\Big)+C\|y-y_h\|_W^2\\ &\le Ch^2 \qquad (4.9)\end{aligned}$$

which concludes the proof of the lemma. □

For $u=u^*$ and corresponding λ^*,λ_h^* we also have the following

Lemma 4.3. Let u^* be a solution of $F(u^*)=0$. Then

$$\int_0^T\|\lambda^*-\lambda_h^*\|^2dt\le Ch^4, \qquad (4.10)$$

in particular, $\|F_h(u^*)\|_{L^2(0,T;U)}\le Ch^2$.

Proof. The proof of (4.10) will be given in the Appendix. With the same notation as above we have $F_h(u^*)=F_h(u^*)-F(u^*)=B^*(\lambda_h^*-\lambda^*)$. Therefore,

$$\|F_h(u^*)\|_{L^2(0,T;U)}\le C\|\lambda^*-\lambda_h^*\|_{L^2(\Omega^T)}\le Ch^2.$$

□

Our next aim is to compare $F'(u)$ and $F_h'(u)$. Recall from (2.12) that $\langle F'(u),\phi\rangle=B^*\eta+\alpha\phi$ where

$$\eta=R\big(y,(v\cdot\nabla)\lambda-(\nabla v)^t\lambda+D_{yy}\Theta(y,.)v,\phi_0''(y(T))v(T)\big). \qquad (4.11)$$

We infer from (C1)(ii), (C2)(iii), (2.1), (2.6) that $\phi_0''(y(T))v(T)\in V$, $D_{yy}\Theta(y,.)v\in L^2(\Omega^T)$ and

$$\begin{aligned}&\|\phi_0''(y(T))v(T)\|_{H^1}^2+\int_0^T\|D_{yy}\Theta(y,t)v\|^2dt\\ &\le C\Big(\sup_{0\le t\le T}\|v(t)\|_{H^1}^2+\int_0^T\|v\|_{H^2}^2dt\Big)\le C\|\phi\|_{L^2(0,T;U)}^2.\end{aligned}$$

Applying (2.8) to (4.11) and using (4.7) as well as the above estimate we arrive at

$$\begin{aligned}&\sup_{0\le t\le T}\|\eta(t)\|_{H^1}^2+\int_0^T\big(\|\eta\|_{H^2}^2+\|\eta_t\|^2+\|\nabla\rho\|^2\big)dt\\ &\le C\|\phi_0''(y(T))v(T)\|_{H^1}^2+C\int_0^T\big(\|(v\cdot\nabla)\lambda\|^2+\|(\nabla v)^t\lambda\|^2+\|D_{yy}\Theta(y,.)v\|^2\big)dt\\ &\le C\|\phi\|_{L^2(0,T;U)}^2+C\Big(\sup_{0\le t\le T}\|v(t)\|_{H^1}^2\int_0^T\|\lambda\|_{H^2}^2dt+\sup_{0\le t\le T}\|\lambda(t)\|_{H^1}^2\int_0^T\|v\|_{H^2}^2dt\Big) \qquad (4.12)\\ &\le C\|\phi\|_{L^2(0,T;U)}^2.\end{aligned}$$

Similar as above we can write $\langle F_h'(u), \phi\rangle = B^*\eta_h + \alpha\phi$, where

$$\eta_h = R_h\Big(y_h, (v_h\cdot\nabla)\lambda_h - (\nabla v_h)^t\lambda_h + D_{yy}\Theta(y_h,.)v_h, \phi_0''(y_h(T))v_h(T)\Big) \tag{4.13}$$

and v_h is the solution of the discrete linearized problem: $v_h(0)=0$ and

$$\begin{aligned}(v_{h,t},\phi_h) + \nu(\nabla v_h,\nabla\phi_h) + N_h(v_h,y_h,\phi_h) + N_h(y_h,v_h,\phi_h) - (\operatorname{div}\phi_h, r_h) &= (B\phi(t),\phi_h) &(4.14)\\ -(\operatorname{div} v_h, q_h) &= 0 &(4.15)\end{aligned}$$

for all $\phi_h \in X_h, q_h\in M_h$.
In view of the regularity (2.6) of (v,r) it is possible to prove the following error estimate

$$\begin{aligned}\|v-v_h\|_W^2 &\le C\Big(\sup_{0\le t\le T}\|\nabla v\|^2\int_0^T\|\nabla(y-y_h)\|^2dt + h^2\int_0^T\big(\|v\|_{H^2}^2+\|\nabla r\|^2\big)dt\Big)\\ &\le Ch^2\|\phi\|^2_{L^2(0,T;U)}. &(4.16)\end{aligned}$$

Lemma 4.4. We have for $u\in B_1(u^*)$

$$\|F'(u)-F_h'(u)\|_{\mathcal{L}(L^2(0,T;U))} \le Ch, \tag{4.17}$$

in particular there exists an $h_0>0$ so that for all $0<h\le h_0$ $F_h'(u^*)$ is an isomorphism and $\|F_h'(u^*)^{-1}\|_{\mathcal{L}(L^2(0,T;U))} \le 2\|F'(u^*)^{-1}\|_{\mathcal{L}(L^2(0,T;U))}$.

Proof. Combining (4.11), (4.13) with Lemma 4.1 (note that $\big(y,(v\cdot\nabla)\lambda - (\nabla v)^t\lambda + D_{yy}\Theta(y,.)v, \phi_0''(y(T))v(T)\big) \in C^0([0,T];H_0^1(\Omega)^2)\times L^2(\Omega^T)\times V$) and denoting the corresponding pressure by ρ we obtain

$$\begin{aligned}&\|\eta-\eta_h\|_W^2\\ &\le C\Big(\|\eta-P_h\eta\|_W^2 + \|\rho-i_h\rho\|^2_{L^2(\Omega^T)} + \|y-y_h\|_W^2\|\eta_h\|_W^2\\ &\quad+\|\phi_0''(y(T))v(T)-\phi_0''(y_h(T))v_h(T)\|^2 + \int_0^T\|D_{yy}\Theta(y,.)v - D_{yy}\Theta(y_h,.)v_h\|^2_{H^{-1}}dt\\ &\quad+\int_0^T\big(\|(v\cdot\nabla)\lambda-(v_h\cdot\nabla)\lambda_h)\|^2_{H^{-1}} + \|(\nabla v)^t\lambda - (\nabla v_h)^t\lambda_h\|^2_{H^{-1}}\big)dt\Big)\\ &\le Ch^2\Big(\sup_{0\le t\le T}\|\eta(t)\|^2_{H^1} + \int_0^T\big(\|\eta\|^2_{H^2}+\|\nabla\rho\|^2\big)dt\Big) &(4.18)\\ &\quad+C\|y-y_h\|_W^2\big(\|\eta_h\|_W^2 + \sup_{0\le t\le T}\|v(t)\|^2_{H^1}\big) + C\sup_{0\le t\le T}\|v_h(t)\|^2_{H^1}\|\lambda-\lambda_h\|_W^2\\ &\quad+C\|v-v_h\|_W^2\big(1+\sup_{0\le t\le T}\|\lambda(t)\|^2_{H^1}\big)\end{aligned}$$

by (3.1), (3.2), (C1)(i), (C2)(i) and (2.1), (4.4).
Let us next find a bound on $\|\eta_h\|_W$. Applying (3.10) to (4.13) and recalling (4.4) we deduce

$$\begin{aligned}
&\|\eta_h\|_W^2 \\
&\leq C\Big(\|\phi_0''(y_h(T))v_h(T)\|^2 + \int_0^T \|D_{yy}\Theta(y_h,.)v_h + (v_h\cdot\nabla)\lambda_h - (\nabla v_h)^t\lambda_h\|_{H^{-1}}^2 dt\Big) \\
&\leq C\Big(\|v_h\|_W^2 + \sup_{0\leq t\leq T}\|v_h(t)\|_{H^1}^2\|\lambda_h\|_W^2 + \sup_{0\leq t\leq T}\|\lambda_h(t)\|_{H^1}^2\|v_h\|_W^2\Big).
\end{aligned}$$

The same argument which led to (4.4) can be applied to (2.6), (4.16) and (4.7), (4.5) respectively to give

$$\sup_{0\leq t\leq T}\|v_h(t)\|_{H^1} \leq C\|\phi\|_{L^2(0,T;U)}, \quad \sup_{0\leq t\leq T}\|\lambda_h(t)\|_{H^1} \leq C \qquad \text{for all } u\in B_1(u^*). \tag{4.19}$$

Inserting (4.19) into our estimate for η_h we arrive at

$$\|\eta_h\|_W \leq C\|\phi\|_{L^2(0,T;U)} \qquad \text{for all } u\in B_1(u^*). \tag{4.20}$$

If we now return to (4.18) and use (4.12), (3.6), (4.20), (4.19), (4.5) and (4.16) the result is

$$\|\eta-\eta_h\|_W \leq Ch\|\phi\|_{L^2(0,T;U)} \tag{4.21}$$

which implies (4.17) in view of $\langle F'(u),\phi\rangle - \langle F_h'(u),\phi\rangle = B^*(\eta-\eta_h)$. The second part of the lemma is a consequence of (2.11) and (4.17). $\square$

Finally, it remains to prove the Lipschitz continuity of F_h' on $B_1(u^*)$.

Lemma 4.5. We have

$$\|F_h'(u) - F_h'(\tilde u)\|_{\mathcal{L}(L^2(0,T;U))} \leq C_2\|u-\tilde u\|_{L^2(0,T;U)} \qquad \text{for all } u,\tilde u\in B_1(u^*).$$

Proof. Let $\phi\in L^2(0,T;U)$. Clearly, $\langle F_h'(u) - F_h'(\tilde u),\phi\rangle = B^*(\eta_h-\tilde\eta_h)$, where

$$\begin{aligned}
\eta_h &= R_h\big(y_h, (v_h\cdot\nabla)\lambda_h - (\nabla v_h)^t\lambda_h + D_{yy}\Theta(y_h,.)v_h, \phi_0''(y_h(T))v_h(T)\big) \\
\tilde\eta_h &= R_h\big(\tilde y_h, (\tilde v_h\cdot\nabla)\tilde\lambda_h - (\nabla \tilde v_h)^t\tilde\lambda_h + D_{yy}\Theta(\tilde y_h,.)\tilde v_h, \phi_0''(\tilde y_h(T))\tilde v_h(T)\big).
\end{aligned}$$

Then

$$\eta_h - \tilde\eta_h = R_h\big(y_h, g, \phi_0''(y_h(T))v_h(T) - \phi_0''(\tilde y_h(T))\tilde v_h(T)\big)$$

where

$$\begin{aligned}
\langle g,\phi\rangle &= \langle D_{yy}\Theta(y_h,.)v_h - D_{yy}\Theta(\tilde y_h,.)\tilde v_h,\phi\rangle + \big((v_h\cdot\nabla)\lambda_h - (\tilde v_h\cdot\nabla)\tilde\lambda_h,\phi\big) \\
&\quad -\big((\nabla v_h)^t\lambda_h - (\nabla\tilde v_h)^t\tilde\lambda_h,\phi\big) + \big(((y_h-\tilde y_h)\cdot\nabla)\tilde\eta_h - ((\nabla y_h)^t - (\nabla\tilde y_h)^t)\tilde\eta_h,\phi\big)
\end{aligned}$$

for $\phi \in H_0^1(\Omega)^2$. We use (3.10) and (C1), (C2) in order to estimate

$$
\begin{aligned}
&\|\eta_h - \tilde{\eta}_h\|_W^2 \\
&\quad \le C\Big(\|\phi_0''(y_h(T))v_h(T) - \phi_0''(\tilde{y}_h(T))\tilde{v}_h(T)\|^2 + \int_0^T \|g\|_{H^{-1}}^2 dt\Big) \\
&\quad \le C\Big(\|\big(\phi_0''(y_h(T)) - \phi_0''(\tilde{y}_h(T))\big)v_h(T)\|^2 + \|\phi_0''(\tilde{y}_h(T))(v_h(T) - \tilde{v}_h(T))\|^2 \\
&\qquad + \int_0^T \big(\|\big(D_{yy}\Theta(y_h,.) - D_{yy}\Theta(\tilde{y}_h,.)\big)v_h\|_{H^{-1}}^2 + \|D_{yy}\Theta(\tilde{y}_h,.)(v_h - \tilde{v}_h)\|_{H^{-1}}^2\big)dt \\
&\qquad + \int_0^T \big(\|(v_h\cdot\nabla)\lambda_h - (\tilde{v}_h\cdot\nabla)\tilde{\lambda}_h\|_{H^{-1}}^2 + \|(\nabla v_h)^t\lambda_h - (\nabla\tilde{v}_h)^t\tilde{\lambda}_h\|_{H^{-1}}^2\big)dt \\
&\qquad + \int_0^T \big(\|((y_h - \tilde{y}_h)\cdot\nabla)\tilde{\eta}_h\|_{H^{-1}}^2 + \|((\nabla y_h)^t - (\nabla\tilde{y}_h)^t)\tilde{\eta}_h\|_{H^{-1}}^2\big)dt\Big) \\
&\quad \le C\Big(\|y_h - \tilde{y}_h\|_W^2 \sup_{0\le t\le T}\|v_h(t)\|_{H^1}^2 + \big(1 + \sup_{0\le t\le T}\|\lambda_h(t)\|_{H^1}^2\big)\|v_h - \tilde{v}_h\|_W^2 \\
&\qquad + \sup_{0\le t\le T}\|\tilde{v}_h(t)\|_{H^1}^2\|\lambda_h - \tilde{\lambda}_h\|_W^2 + \|y_h - \tilde{y}_h\|_W^2 \sup_{0\le t\le T}\|\tilde{\eta}_h(t)\|_{H^1}^2\Big).
\end{aligned}
$$

Using (4.4), (4.19) and deriving an analogous bound on $\sup_{0\le t\le T}\|\tilde{\eta}_h(t)\|_{H^1}$ (which follows from (4.12) and (4.21)) we deduce

$$
\|\eta_h - \tilde{\eta}_h\|_W \le C\|\phi\|_{L^2(0,T;U)}\big(\|y_h - \tilde{y}_h\|_W + \|\lambda_h - \tilde{\lambda}_h\|_W\big) + C\|v_h - \tilde{v}_h\|_W. \tag{4.22}
$$

It remains to estimate the above differences. The difference $\tilde{e}_h := y_h - \tilde{y}_h$ satisfies $\tilde{e}_h(0) = 0$ as well as

$$
\begin{aligned}
(\tilde{e}_{h,t},\phi_h) + \nu(\nabla\tilde{e}_h,\nabla\phi_h) + N_h(y_h,\tilde{e}_h,\phi_h) + N_h(\tilde{y}_h,\tilde{e}_h,\phi_h)& \\
-(\operatorname{div}\phi_h, r_h - \tilde{r}_h) &= (B(u-\tilde{u}),\phi_h) \\
-(\operatorname{div}\tilde{e}_h, q_h) &= 0
\end{aligned}
$$

for all $\phi_h \in X_h$ and all $q_h \in M_h$. Inserting $\phi_h = \tilde{e}_h$ and recalling that $N_h(y_h,\tilde{e}_h,\tilde{e}_h) = 0$ we can continue

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\tilde{e}_h\|^2 + \nu\|\nabla\tilde{e}_h\|^2 &\le C\|u-\tilde{u}\|_U\|\tilde{e}_h\| + C\|\tilde{e}_h\|_{L^4}\|\tilde{y}_h\|_{H^1}\|\tilde{e}_h\|_{H^1} \\
&\le C\|u-\tilde{u}\|_U\|\nabla\tilde{e}_h\| + C\|\tilde{e}_h\|^{\frac{1}{2}}\|\nabla\tilde{e}_h\|^{\frac{3}{2}} \\
&\le \frac{\nu}{2}\|\nabla\tilde{e}_h\|^2 + C\|u-\tilde{u}\|^2 + C\|\tilde{e}_h\|^2,
\end{aligned}
$$

where we again used (4.4). Gronwall's lemma implies

$$
\|y_h - \tilde{y}_h\|_W \le C\|u-\tilde{u}\|_{L^2(0,T;U)}. \tag{4.23}
$$

Arguing in a similar way we also obtain

$$
\|v_h - \tilde{v}_h\|_W \le C\|y_h - \tilde{y}_h\|_W\|\phi\|_{L^2(0,T;U)} \le C\|u-\tilde{u}\|_{L^2(0,T;U)}\|\phi\|_{L^2(0,T;U)}. \tag{4.24}
$$

Finally,

$$\begin{aligned}
&\lambda_h - \tilde{\lambda}_h \\
&\quad = \; R_h\big(y_h, D_y\Theta(y_h, .), \phi_0'(y_h(T))\big) - R_h\big(\tilde{y}_h, D_y\Theta(\tilde{y}_h, .), \phi_0'(\tilde{y}_h(T))\big) \\
&\quad = \; R_h\big(y_h, D_y\Theta(y_h, .) - D_y\Theta(\tilde{y}_h, .) + ((y_h - \tilde{y}_h)\cdot\nabla)\tilde{\lambda}_h - ((\nabla y_h)^t - (\nabla\tilde{y}_h)^t)\tilde{\lambda}_h, \\
&\qquad\qquad \phi_0'(y_h(T)) - \phi_0'(\tilde{y}_h(T))\big).
\end{aligned}$$

Thus, we infer from (3.10), Remark 2.1a), (4.19) and (4.23) that

$$\begin{aligned}
\|\lambda_h - \tilde{\lambda}_h\|_W^2 \;&\leq\; C\Big(\|\phi_0'(y_h(T)) - \phi_0'(\tilde{y}_h(T))\|^2 + \int_0^T \|D_y\Theta(y_h, .) - D_y\Theta(\tilde{y}_h, .)\|_{H^{-1}}^2 dt \\
&\qquad + \int_0^T \big(\|((y_h - \tilde{y}_h)\cdot\nabla)\tilde{\lambda}_h\|_{H^{-1}}^2 + \|((\nabla y_h)^t - (\nabla\tilde{y}_h)^t)\tilde{\lambda}_h\|_{H^{-1}}^2\big) dt\Big) \\
&\leq\; C\|y_h - \tilde{y}_h\|_W^2\big(1 + \sup_{0\leq t\leq T}\|\tilde{\lambda}_h(t)\|_{H^1}^2\big) \leq C\|u - \tilde{u}\|_{L^2(0,T;U)}^2.
\end{aligned}$$

If we insert the above estimates into (4.22) we finally infer

$$\begin{aligned}
\|\langle F_h'(u) - F_h'(\tilde{u}), \phi\rangle\|_{L^2(0,T;U)} \;&\leq\; \|B^*\|\,\|\eta_h - \tilde{\eta}_h\|_{L^2(\Omega^T)} \leq C\|\eta_h - \tilde{\eta}_h\|_W \\
&\leq\; C\|u - \tilde{u}\|_{L^2(0,T;U)}\|\phi\|_{L^2(0,T;U)}
\end{aligned}$$

for all $\phi \in L^2(0,T;U)$, which implies the result. □

We are now in position to prove Theorem 3.1. Lemma 4.2 and Lemma 4.3 imply that

$$\|F_h'(u^*)^{-1}\|_{\mathcal{L}(L^2(0,T;U))} \leq \gamma, \quad \|F_h'(u^*)^{-1}F_h(u^*)\|_{L^2(0,T;U)} \leq \gamma C_1 h^2$$

where $\gamma = 2\|F'(u^*)^{-1}\|_{\mathcal{L}(L^2(0,T;U))}$. If we choose $h_0 > 0$ so small that

$$2\gamma C_1 h_0^2 < 1, \quad 2C_2\gamma^2 C_1 h_0^2 < 1$$

(C_2 as in Lemma 4.5), then Theorem 4.1 yields the existence of a unique solution $u_h^* \in \overline{B_{\delta_0}(u^*)}$ of the equation $F_h(u) = 0$ for all $h \leq h_0$ with $\delta_0 = 2\gamma C_1 h_0^2$. Repeating this argument with $\epsilon = \gamma C_1 h^2$ we even infer that $u_h^* \in \overline{B_{2\gamma C_1 h^2}(u^*)}$ for $h \leq h_0$, which completes the proof of Theorem 3.1.

5 Numerical results

The control problem considered here is of tracking type and is given by $(\hat{P})$ with cost functional

$$\hat{J}(u) := \frac{1}{2}\int_{\Omega^T} |y(u) - z|^2\, dxdt + \frac{\alpha}{2}\int_{\Omega^T} |u|^2\, dxdt,$$

i.e. $\phi_0(\zeta) := 0$ and $\Theta(y,t) := \frac{1}{2}\int_\Omega |y - z(t)|^2 dx$, where $\Omega = (0,1)\times(0,1)$ and $T = 2$. Our control space is $U := L^2(\Omega)^2$ and the control operator is given by $B = \mathrm{id}_{L^2(\Omega)^2}$. The initial value of the uncontrolled flow is chosen as

$$y(x,0) \;=\; e\begin{bmatrix} (\cos 2\pi x_1 - 1)\sin 2\pi x_2 \\ -(\cos 2\pi x_2 - 1)\sin 2\pi x_1 \end{bmatrix}$$

with e denoting the Euler number, and the desired state z is time–dependent and given by

$$z(x,t) = \begin{bmatrix} \varphi_{x_2}(x_1,x_2,t) \\ -\varphi_{x_1}(x_1,x_2,t) \end{bmatrix},$$

where φ is defined through the stream function

$$\varphi(x_1,x_2,t) = \theta(x_1,t)\theta(x_2,t)$$

with

$$\theta(y,t) = (1-y)^2(1-\cos 2k\pi t), \quad y \in [0,1].$$

In the calculations we chose $\alpha = 1.e{-}2$, $k = 1$ and $\nu = 1/\mathrm{Re} = 1/10$. For the discretization in time an equidistant grid with step size $\delta t = 0.01$ is used, for the spatial discretization the Taylor-Hood finite element [4], [2] is used on a grid containing 1024 triangles with 2113 velocity and 545 pressure nodes. The number of unknowns in the discretized control problem therefore has the magnitude 1.65×10^6, including the primal, adjoint and control variables.

We applied Newton's method to the equation $F_h(u) = 0$ defined in (3.11) with initial control $u = 0$. In every iteration step the Newton system $F_h'(u^k)(\delta u^k) = -F_h(u^k)$ itself was solved iteratively using a conjugate gradient method. Note that due to the definition of the operator R_h in (3.8)-(3.9) every iteration of the conjugate gradient method amounts to solving one forward and one backward linearized Navier-Stokes system. For further computational details see [9, Chapter 6].

In Fig. 1 (left) the desired flow at $T = 2$ is shown. It forms four cells with opposite flow directions near the cell borders. In Fig. 2 a snapshot of the optimally controlled flow together with a snapshot of the control force at $t = T$ is shown. The evolution of the costs is illustrated in Fig. 1 (right). The performance of Newton's method applied to solve for an optimal solution is illustrated in Table 1. The method takes 2 iterations to compute an optimal solution. The numerical computation of the optimal control takes about 45 minutes cpu-time on a ORIGIN 200^{TM} work station.

Appendix

In this appendix we shall prove (4.10) with the help of a duality argument. Let $e := \lambda^* - \lambda_h^*$ and denote by μ the solution of the linear problem

$$\begin{aligned} \mu_t - \nu\Delta\mu + (y^*\cdot\nabla)\mu + (\mu\cdot\nabla)y^* + \nabla q &= e \quad \text{in } \Omega^T \\ -\mathrm{div}\,\mu &= 0 \quad \text{in } \Omega^T \\ \mu(.,t) &= 0 \quad \text{on } (\partial\Omega)^T \\ \mu(.,0) &= 0 \quad \text{in } \Omega. \end{aligned}$$

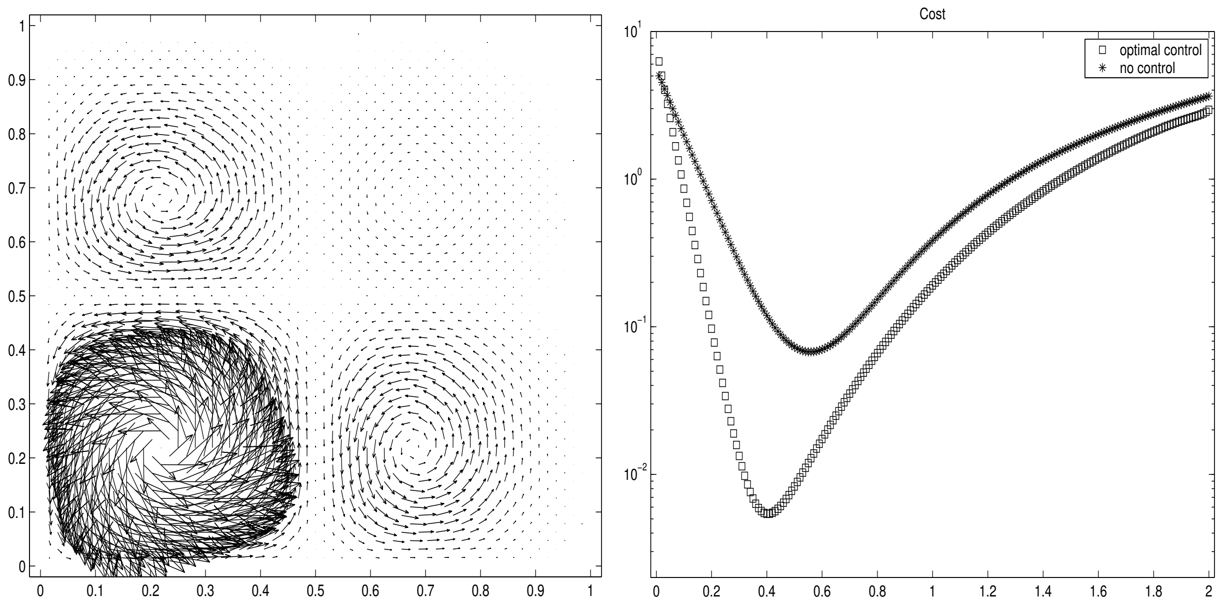

Figure 1: Desired flow at $t = T = 2$ (left) and Evolution of the cost functional for $t \in [0, T]$ (right)

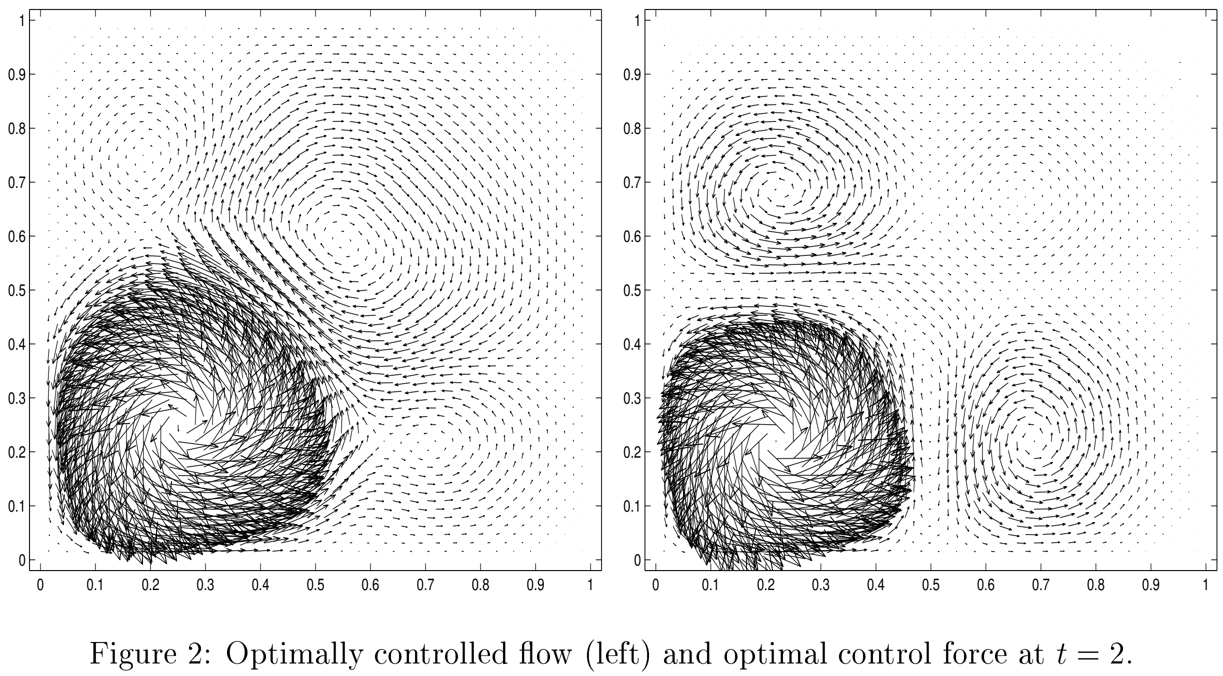

Figure 2: Optimally controlled flow (left) and optimal control force at $t = 2$.

Iteration	CG-steps	$\frac{\lvert \hat{J}'(u)\rvert}{\lvert \hat{J}'(u^0)\rvert}$	$\frac{\lvert \delta u^k\rvert_U}{\lvert \delta u^{k-1}\rvert_U}$	$\hat{J}(u^k)$
1	-	1.e0	-	2.385417
2	6	5.113960e-3	1.	1.612258
3	6	1.862481e-4	2.70e-2	1.611528

Table 1: Performance of Newton's method

We infer from (2.1) that

$$\sup_{0\le t\le T} \|\mu(t)\|^2_{H^1} + \int_0^T \big(\|\mu\|^2_{H^2} + \|\nabla q\|^2\big)dt \le C\int_0^T \|e\|^2 dt. \tag{A.1}$$

Using integration by parts we obtain

$$\int_0^T \|e\|^2 dt = \int_0^T \big(e, \mu_t - \nu\Delta\mu + (y^*\cdot\nabla)\mu + (\mu\cdot\nabla)y^* + \nabla q\big)dt \tag{A.2}$$

$$= \int_0^T \Big(-(e_t,\mu) + \nu(\nabla e,\nabla\mu) - ((y^*\cdot\nabla)e,\mu) + ((\nabla y^*)^t e,\mu) - (\operatorname{div} e, q)\Big)dt + (e(T),\mu(T)).$$

The relations (4.2), (4.3) applied to $w=\lambda^*$, $w_h = \lambda^*_h$ yield

$$\begin{aligned} -(e_t,\phi_h) + \nu(\nabla e,\nabla\phi_h) - ((y^*\cdot\nabla)e,\phi_h) + ((\nabla y^*)^t e,\phi_h) - (\operatorname{div}\phi_h, \xi^*-\xi^*_h) &= \langle g^*,\phi_h\rangle \\ -(\operatorname{div} e, q_h) &= 0 \end{aligned}$$

for all $\phi_h\in X_h$, $q_h\in M_h$, where $g^*\in L^2(0,T;H^{-1}(\Omega)^2)$ is given by

$$\langle g^*,\phi\rangle = \langle D_y\Theta(y^*,.) - D_y\Theta(y^*_h,.),\phi\rangle + (((y^*-y^*_h)\cdot\nabla)\lambda^*_h,\phi) - (((\nabla y^*)^t - (\nabla y^*_h)^t)\lambda^*_h,\phi).$$

Combining the above relations for $\phi_h = P_h\mu$ with (A.2) we deduce

$$\begin{aligned}
\int_0^T \|e\|^2 dt &= \int_0^T \Big(\langle g^*, P_h\mu\rangle - (\operatorname{div} e, q) + (\operatorname{div}(P_h\mu), \xi^*-\xi^*_h)\Big)dt + (e(T),\mu(T)) \\
&\quad + \int_0^T \Big(-(e_t,\mu-P_h\mu) + \nu\big(\nabla e,\nabla(\mu-P_h\mu)\big) - \big((y^*\cdot\nabla)e - (\nabla y^*)^t e, \mu-P_h\mu\big)\Big)dt \\
&= \int_0^T \Big(\langle g^*,\mu\rangle + \langle g^*, P_h\mu-\mu\rangle - (\operatorname{div} e,q) + (\operatorname{div}(P_h\mu-\mu), \xi^*-i_h\xi^*)\Big)dt + (e(T),\mu(T)) \\
&\quad + \int_0^T \Big(-(\lambda^*_t,\mu-P_h\mu) + \nu\big(\nabla e,\nabla(\mu-P_h\mu)\big) - \big((y^*\cdot\nabla)e - (\nabla y^*)^t e,\mu-P_h\mu\big)\Big)dt \qquad \text{(A.3)} \\
&= \int_0^T \Big(\langle g^*,\mu\rangle - (\operatorname{div} e, q) + \langle h^*,\mu-P_h\mu\rangle\Big)dt + (e(T),\mu(T)).
\end{aligned}$$

Note that we have used the facts that $(\operatorname{div} P_h\mu, q_h) = 0$ for all $q_h\in M_h$ and $(\lambda^*_{h,t},\mu-P_h\mu) = 0$. Also h^* is given by

$$\langle h^*,\phi\rangle = -\langle g^*,\phi\rangle - (\operatorname{div}\phi, \xi^*-i_h\xi^*) - (\lambda^*_t,\phi) + \nu(\nabla e,\nabla\phi) - \big((y^*\cdot\nabla)e - (\nabla y^*)^t e,\phi\big).$$

Let us estimate the terms on the right hand side of (A.2). In order to bound $\langle g^*, \mu\rangle$ we first use integration by parts to see

$$\int_\Omega ((\nabla y^*)^t - (\nabla y_h^*)^t)\lambda_h^* \cdot \phi = -\int_\Omega \operatorname{div} \lambda_h^* \, (y^* - y_h^*) \cdot \phi - \int_\Omega \big(\lambda_h^* \cdot \nabla)\phi \cdot (y^* - y_h^*)$$

so that we deduce from (C2)(iii) and the continuous embeddings $H^1(\Omega) \hookrightarrow L^4(\Omega)$, $H^2(\Omega) \hookrightarrow C^0(\bar\Omega)$ that

$$\begin{aligned}
& |\int_0^T \langle g^*, \mu\rangle dt| \\
& \quad \le \int_0^T |\langle D_y\Theta(y^*,.) - D_y\Theta(y_h^*,.), \mu\rangle| dt + \int_0^T \int_\Omega |y^* - y_h^*| \big(|\nabla \lambda_h^*| \, |\mu| + |\lambda_h^*| \, |\nabla \mu|\big) dt \\
\text{(A.4)} & \quad \le \int_0^T |\int_0^1 \langle D_{yy}\Theta(sy^* + (1-s)y_h^*,.)(y^* - y_h^*), \mu\rangle ds| dt \\
& \qquad + C \int_0^T \|y^* - y_h^*\| \big(\|\nabla \lambda_h^*\| \, \|\mu\|_{C^0} + \|\lambda_h^*\|_{L^4} \|\nabla \mu\|_{L^4}\big) dt \\
& \quad \le C \int_0^T \|y^* - y_h^*\| \, \|\mu\|_{H^2} dt \le \epsilon \int_0^T \|\mu\|_{H^2}^2 dt + C_\epsilon \int_0^T \|y^* - y_h^*\|^2 dt \\
& \quad \le \epsilon \int_0^T \|e\|^2 dt + C_\epsilon h^4
\end{aligned}$$

by (A.1) and (3.6). Next, recalling that $(\operatorname{div} e, q_h) = 0$ for all $q_h \in M_h$ we infer from (3.2), (4.5) and (A.1)

$$\begin{aligned}
|\int_0^T (\operatorname{div} e, q) dt| &= |\int_0^T (\operatorname{div} e, q - i_h q) dt| \le \int_0^T \|\nabla e\| \, \|q - i_h q\| dt \le Ch \int_0^T \|\nabla e\| \, \|\nabla q\| dt \\
\text{(A.5)} \qquad &\le \epsilon \int_0^T \|e\|^2 dt + C_\epsilon h^2 \|\lambda^* - \lambda_h^*\|_W^2 \le \epsilon \int_0^T \|e\|^2 dt + C_\epsilon h^4.
\end{aligned}$$

Next, we have for $\phi \in H_0^1(\Omega)^2$

$$\begin{aligned}
& |\langle h^*, \phi\rangle| \\
& \quad \le \|g^*\|_{H^{-1}} \|\nabla \phi\| + \|\xi^* - i_h \xi^*\| \, \|\nabla \phi\| + \|\lambda_t^*\| \, \|\phi\| + \nu \|\nabla e\| \, \|\nabla \phi\| + C \|\nabla y^*\| \, \|\nabla e\| \, \|\nabla \phi\| \\
& \quad \le C \|\nabla \phi\| \Big(\|\nabla (y^* - y_h^*)\| \big(1 + \|\nabla \lambda_h^*\|\big) + h \|\nabla \xi^*\| + \|\nabla e\| \Big) + \|\lambda_t^*\| \, \|\phi\| \\
& \quad \le C \|\nabla \phi\| \Big(\|\nabla (y^* - y_h^*)\| + h \|\nabla \xi^*\| + \|\nabla e\| \Big) + \|\lambda_t^*\| \, \|\phi\|
\end{aligned}$$

by (4.19) and (3.2). As a consequence, using (3.1), (A.1), (3.6), (4.7) and (4.5)

$$
\begin{aligned}
&|\int_0^T \langle h^*, \mu - P_h\mu\rangle dt| \\
&\le\ C\int_0^T \|\nabla(\mu - P_h\mu)\|\Big(\|\nabla(y^*-y_h^*)\| + h\|\nabla\xi^*\| + \|\nabla e\|\Big)dt + C\int_0^T \|\mu - P_h\mu\|\,\|\lambda_t^*\|dt \\
&\le\ Ch\int_0^T \|D^2\mu\|\Big(\|\nabla(y^*-y_h^*)\| + h\|\nabla\xi^*\| + \|\nabla e\| + h\|\lambda_t^*\|\Big)dt \qquad \text{(A.6)}\\
&\le\ \epsilon\int_0^T \|e\|^2dt + C_\epsilon h^4\int_0^T \big(\|\nabla\xi^*\|^2 + \|\lambda_t^*\|^2\big)dt \\
&\qquad + C_\epsilon h^2\int_0^T \big(\|\nabla(y^*-y_h^*)\|^2 + \|\nabla(\lambda^*-\lambda_h^*)\|^2\big)dt \\
&\le\ \epsilon\int_0^T \|e\|^2dt + C_\epsilon h^4.
\end{aligned}
$$

Finally, (C1)(i) and (A.1) imply

$$
\begin{aligned}
|(e(T),\mu(T))| &= |\big(\phi_0'(y^*(T)) - \phi_0'(y_h^*(T)), \mu(T)\big)| \\
&\le C\|y^*(T) - y_h^*(T)\|\,\|\mu(T)\| \le C\sup_{t\in[0,T]}\|y^*(t) - y_h^*(t)\|\sup_{t\in[0,T]}\|\mu(t)\|_{H^1} \\
&\le \epsilon\int_0^T \|e\|^2dt + C_\epsilon\sup_{t\in[0,T]}\|y^*(t) - y_h^*(t)\|^2. \qquad \text{(A.7)}
\end{aligned}
$$

In order to estimate the last term, we note that u^* is a solution of $B^*\lambda^* + \alpha u^* = 0$ so that the right hand side Bu^* in (1.2) and (3.3) respectively satisfies

$$
\int_0^T \|(Bu^*)_t\|^2dt = \int_0^T \|\frac{1}{\alpha}BB^*\lambda_t^*\|^2dt \le C\int_0^T \|\lambda_t^*\|^2dt \le C. \qquad \text{(A.8)}
$$

A careful inspection of the calculations in §2 of [8, I] shows that (A.8) together with the fact that $y_0 \in V\cap H^2(\Omega)^2$ implies

$$
\sup_{t\in[0,T]}\big(\|y^*(t)\|_{H^2} + \|\nabla p^*(t)\|\big) \le C, \qquad \int_0^T \sigma(t)\big(\|y_t^*\|_{H^2}^2 + \|\nabla p_t^*\|^2\big)dt \le C,
$$

where $\sigma(t) = \min(t,1)$. The error analysis in §5 of [8, I] then yields

$$
\sup_{t\in[0,T]}\|(y^*-y_h^*)(t)\| \le Ch^2. \qquad \text{(A.9)}
$$

Inserting (A.4)–(A.7), (A.9) into (A.3) and choosing ϵ sufficiently small, the result is

$$
\int_0^T \|\lambda^* - \lambda_h^*\|^2dt = \int_0^T \|e\|^2dt \le Ch^4,
$$

which proves (4.10).

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