Als Manuskript gedruckt

Technische Universität Dresden
Herausgeber: Der Rektor

Fast closed loop control of the Navier-Stokes system

Michael Hinze and Daniel Wachsmuth

MATH-NM-08-2003
Juli 2003

Contents

Fast closed loop control of the Navier-Stokes system

Michael Hinze
Institut für Numerische Mathematik,
Technische Universität Dresden
D–01069 Dresden, Germany,
email: hinze@math.tu-dresden.de

Daniel Wachsmuth
Institut für Mathematik,
Technische Universität Berlin
D-10623 Berlin, Germany,
email: wachsmut@math.tu-berlin.de

Abstract

We present a construction for closed-loop feedback control of the time-dependent Navier-Stokes system. Its basic idea consists in approximately solving certain instantaneous optimization problems for the discrete-in-time dynamical system. Easy incorporation of control constraints is one key feature of the recipe. We state stabilizing properties of the controllers which we confirm by numerical tests.

1 Introduction

This research is devoted to the construction, numerical validation and of nonlinear feedback control policies for the instationary Navier-Stokes system. The governing equations in the primitive setting are given by

$$(\mathbf{P}) \qquad \begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p = \mathcal{B}u & \text{in } (0,T)\times\Omega, \\ -\text{div } y = 0 & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = \phi & \text{in } \Omega. \end{cases}$$

The control target is to match the given desired state z in the L^2-sense by adjusting the body force $\mathcal{B}u$. In this context $\mathcal{B}$ denotes an abstract control extension operator and $\Omega \subset \mathbb{R}^2$ denotes a bounded domain.

In this work we present a recipe for the construction of nonlinear for the time-dependent Navier-Stokes system of the form

$$\mathcal{B}u = K(y)$$

and numerically illustrate their performance.

The construction principle works as follows. The uncontrolled Navier-Stokes system is discretized with respect to time. Then, at selected time instances an appropriate cost functional is approximately minimized with respect to a stationary quasi-(Navier-)Stokes system, whose structure depends on the chosen time discretization method. The obtained control is used to steer the system to the next time instance, where the procedure is repeated. We note that this approach is related to model prediction control techniques, see [6].

Main result: Given a sufficiently smooth desired state z, and a time discretization scheme for the Navier-Stokes system, the above described construction process can be regarded as time discretization of a closed loop feedback policy K, i.e. with S denoting the Stokes operator and $b(y)$ the nonlinearity of the Navier-Stokes equations we get the controlled system

$$y_t + \nu Sy + b(y) = K(y).$$

Under certain assumptions on the initial states the controller K steers the Navier-Stokes system exponentially fast to z. To be more precise, the solution of this system satisfies $|y(t) - z(t)|_{H^1} \leq ce^{-\kappa t}$ with some positive constants c and κ.

It turns out that instantaneous control [9, 11] is a special case of our approach. For applications of instantaneous control we refer to [1, 2, 3, 8, 10, 16, 17, 19], stability analysis of the method is presented in [9, 11]. Further contributions to closed loop control of the Navier-Stokes system can be found in [7], where linear body force feedback control was applied to control the system. The analysis of special case of modelpredictive control of the Navier-Stokes equations can be found in [14, 15].

The paper is organized as follows. Section 2 contains the analytical preliminaries. In Section 3 we introduce the basic construction recipe which lead to certain discretized closed-loop control laws. These discrete laws are related to continuous closed-loop control laws, whose stability properties are also stated. Finally, in Section 5 numerical examples are presented, which confirm the theoretical results.

Throughout this work c and C denote global generic constants whose dependencies are mentioned when necessary.

2 Analytical preliminaries and time discretization

For given $T > 0$ let $Q = (0,T) \times \Omega$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain. We set $V = \{v \in H_0^1(\Omega)^2,\ \text{div}\ v = 0\}$, $H = \text{clos}_{L^2(\Omega)^2}\{v \in C_0^\infty(\Omega)^2,\ \text{div}\ v = 0\}$ and identify the Hilbert space H with its dual H'. The dual space of V is defined to get a Gelfand-triple $V \hookrightarrow H \hookrightarrow V'$. On H the common inner product is used, and V is endowed with the inner product

$$(\varphi, \psi)_V = (\varphi', \psi')_H \text{ for } \varphi, \psi \in V.$$

Moreover, with Z denoting a Hilbert space, $L^p(Z)$ $(1 \leq p \leq \infty)$ denotes the space of measurable abstract functions $\varphi : (0,T) \to Z$, which are p-integrable $(1 \leq p < \infty)$, or essentially bounded on $(0,T)$ $(p = \infty)$, respectively.

As control space $L^2(\mathcal{U})$ is taken, where $\mathcal{U}$ denotes the Hilbert space of abstract controls. The space $\mathcal{U}$ also is identified with its dual. Furthermore,

$$\mathcal{B} : \mathcal{U} \to V' \qquad (1)$$

denotes the control extension operator which is assumed to be bounded. The set of admissible controls is denoted by $\mathcal{U}_{ad} \subseteq \mathcal{U}$ and is required to be closed, convex and bounded. In order to formulate the weak form of the instationary Navier-Stokes equations let

$$W := W(V) = \{\varphi \in L^2(V) : \varphi_t \in L^2(V')\}$$

supplied with the common inner product. Further, we define

$$H^{2,1}(Q) := \{\varphi \in L^2(V \cap H^2(\Omega)), \varphi_t \in L^2(H)\}.$$

For convenience we introduce the tri-linear form

$$b(u, v, w) \; := \; \int_\Omega (u \cdot \nabla) v \, w \, dx.$$

Now, for $y \in L^2(V)$ the function $b(y)$ defined by

$$\langle b(y), v \rangle_{V',V} := -b(y, y, v) \quad \text{for all } v \in V \tag{2}$$

is an element of V' for almost all $t \in (0, T)$ and $b(y) \in L^1(V)$ [18, Lemma 3.1]. If in addition $y \in L^\infty(H)$ holds then $b(y)$ is an element of $L^2(V')$. This statement is true especially for functions $y \in W$, since W is continuously imbedded in $L^\infty(H)$, confer [5].

For controls $u \in L^2(\mathcal{U})$ the solenoidal form of the read: Find the state $y \in W$ such that

$$\begin{aligned} &\frac{d}{dt}(y(t), \varphi)_H + \nu(y(t), \varphi)_V \\ &\qquad = \langle b(y) + \mathcal{B}u(t), \varphi \rangle_{V',V} \text{ for all } \varphi \in V \text{ and a.e. } t \in [0, T] \end{aligned} \tag{3a}$$

and

$$(y(0), \chi)_H = (\phi, \chi)_H \text{ for all } \chi \in H. \tag{3b}$$

With Re denoting the Reynolds number, 1/Re=: $\nu > 0$ is the viscosity parameter. The proof of the following well-known existence theorem can be found in [18].

Theorem 2.1. *For any $\phi \in H$ and for every control $u \in L^2(\mathcal{U})$ equations (3) admit a unique weak solution $y \in W$.*

2.1 Time discretization

For notational purposes define the Stokes operator $A : V \mapsto V'$ by

$$\langle Ay, \, v \rangle_{V',V} \; := \; \nu(y, \, v)_V.$$

In this setting the Navier-Stokes equations (3) for $u = 0$ in variational formulation may be rewritten as Burgers equation in the space V',

$$\begin{aligned} y_t + Ay &= b(y), \\ y(0) &= \phi, \end{aligned}$$

where the nonlinearity $b(y)$ is defined in (2). For $m \in \mathbb{N}$ an equidistant discretization of the time interval $(0, T)$ is defined by $h = \frac{T}{m}$ and $t_k = kh$, $k = 0, 1, \dots, m$. Now let $z \in W \hookrightarrow C([0, T], H)$ the desired state. We define

$$J^k : V \times \mathcal{U} \to \mathbb{R}, \quad (y, u) \mapsto \frac{1}{2}\,|y - z^k|_H^2 + \frac{\gamma}{2}\,|u|_{\mathcal{U}}^2,$$

where

$$z^k = \frac{1}{h} \int_{t_k - \frac{h}{2}}^{t_k + \frac{h}{2}} z(s, \cdot)\, ds \tag{4}$$

and $z(t, \cdot) = 0$ for $t > T$. Finally, for $k = 1, \dots, m$ and $i = 1, 2$ introduce the operator $e^k : V \times \mathcal{U} \to V'$ by

$$e^k(y, u) = (I + hA)y \; - \; hb(y^{k-1}) - y^{k-1} - \mathcal{B}u,$$

where y^{k-1} denotes the state at the previous time slice.

The instantaneous optimal control problem for the semi-implicit time integration is given by

$$\text{minimize } J^k(y, u) \text{ subject to } e^k(y, u) = 0 \text{ in } V', \quad u \in \mathcal{U}_{ad}, \tag{$\mathbf{P}^k$}$$

where $y^0 = \phi$. The initial value ϕ now is required to be an element of the space V. For given y^{k-1} a pair (y^k, u^k) satisfies the subsidiary condition $e^k(y, u) = 0$ in V' if and only if

$$(y^k, v)_H + \nu h\, (y^k, v)_V = (y^{k-1}, v)_H + \langle \mathcal{B}u^k + hb(y^{k-1}), v \rangle_{V', V} \quad \forall\, v \in V. \tag{5}$$

Since $\phi \in V$ holds, the right-hand side in this linear equation defines a bounded linear functional on V. Thus, for every $u^k \in \mathcal{U}$ Eq. (5) admits a unique solution $y^k \in V$ which satisfies the a-priori estimate

$$|y^k|_V \le \frac{C}{\nu h} \left(|y^{k-1}|_H + h|y^{k-1}|^2{}_V + |u^k|_{\mathcal{U}} \right).$$

Since J^k is quadratic, e^k is linear and $\mathcal{U}_{ad}$ is closed and convex every problem $(\mathbf{P}^k)$, $k = 1, \dots, m$, admits a unique solution $(y_*^k, u_*^k) \in V \times \mathcal{U}$. Furthermore, the unique Lagrange multiplier $\lambda_*^k \in V$ together with the solution (y_*^k, u_*^k) satisfies the first-order necessary optimality conditions (note that A is selfadjoint)

$$(I + hA)y = \mathcal{B}u + y^{k-1} + hb(y^{k-1}), \tag{6a}$$

$$(I + hA)\lambda = -(y - z^k), \tag{6b}$$

$$(\gamma u - \mathcal{B}^\star \lambda, v - u) \ge 0 \text{ for all } v \in \mathcal{U}_{ad}, \tag{6c}$$

where we have set $(y, u, \lambda) = (y_*^k, u_*^k, \lambda_*^k)$. Furthermore, the second-order sufficient optimality condition holds on the whole space $V \times \mathcal{U} \times V$. Hence, the solution (y_*^k, u_*^k) of (6) is the minimum for $(\mathbf{P}^k)$.

The optimal control problem $(\mathbf{P}^k)$ is equivalent with respect to existence to the control-constrained minimization of the functional

$$\hat{J}^k(u) \;=\; J^k(y(u), u) \tag{7}$$

over $\mathcal{U}_{ad}$, where for a control $u \in \mathcal{U}$ the state $y(u) \in V$ is given as the unique solution to (5) (indexes dropped). The gradient of $\hat{J}^k$ at u is given by

$$\nabla \hat{J}^k(u) = \gamma u \;-\; \mathcal{B}^\star \lambda,$$

where for given u the function λ is obtained by first solving the linear quasi-Stokes problem (6a) for the state y, and then solving (6b) for λ.

From now onwards let $B := (I + hA)^{-1}$ denote the solution operator of the time-discrete equation (6a), i.e. $e^k(y, u) = 0$ implies $y = B(y^{k-1} + hb(y^{k-1}) + \mathcal{B}u)$.

Observe that the adjoint equation (6b) only depends on the observation $y^k - z^k$. Therefore, gradient information for the functional $\hat{J}^k$ is available utilizing the observations only. In the particular case of boundary observation no information of the state in the whole computational domain is needed at all.

We will now apply the instantaneous control strategy to derive a feedback controller.

3 The closed-loop feedback recipe

The feedback recipe is formulated in terms of a pseudo-algorithm or oracle. The particular form of the feedback strategy depends on the oracle $RECIPE$ called in step 2.) of the following algorithm.

Algorithm 3.1. Feedback recipe.

1.) Set $y^0 = \phi$, $k = 0$ and $t_0 = 0$.

2.) Given an initial control u_0^k, set

$$u^{k+1} = RECIPE(u_0^k, y^k, z^k, t_k)$$

3.) Solve

$$(I + hA)y^{k+1} = y^k + hb(y^k) + \mathcal{B}u^{k+1}.$$

4.) Set $t_{k+1} = t_k + h$, $k = k + 1$. If $t_k < T$ goto 2.

Next we discuss the $RECIPE$s which are investigated in the present work. We use the instantaneous control problem $(\mathbf{P})$ to define the first feedback law. For a given initial control u_0^k one can use a gradient step in direction $-\nabla \hat{J}(u_0^k)$ given by (7). Then one gets the following recipe, see also [9].

RECIPE 1. ()
The oracle RECIPE in case of the instantaneous control strategy, cf. [9], is defined by $u = RECIPE(v, y^k, z, t)$ iff

(i) Solve $(I + hA)y = y^k + hb(y^k) + \mathcal{B}v$,

(ii) solve $(I + hA)\lambda = -(y - z)$,

(iii) set $d = \gamma v - \mathcal{B}^\star\lambda$.

(iv) determine $\rho > 0$,

(v) set $RECIPE = v - \rho d$.

Let us a further look at the previous RECIPE. In [9] the following interpretation of RECIPE 1 is given.

Theorem 3.2. *For $u_0^k = 0$ RECIPE 1 is equivalent to the semi-implicit time discretization with discretization step size h*

$$(I + hA)y^{k+1} = y^k + hb(y^k) - \rho BB(y^k - z^k) - h\rho BB(b(y^k) - Az^k), \quad y^0 = \phi, \tag{8}$$

of the dynamical system

$$\dot{y} + Ay = b(y) - \frac{\rho}{h}BB(y - z) - \rho BB(b(y) - Az), \quad y(0) = \phi. \tag{9}$$

Due to Theorem 3.2 the term

$$Ky = -\frac{\rho}{h}BB(y - z) - \rho BB(b(y) - Az) \tag{10}$$

in (9) can be interpreted as a non-linear closed-loop control policy for the Navier-Stokes equations. It is important to note that the discretization step-size h and the descent parameter ρ of RECIPE 1 in the continuous case (9) may now be regarded as parameters defining the controller.

In order to further improve the controller derived in Theorem 3.2, suppose$K$ steers y to z, eq. (9) necessarily implies that the desired state z would have to satisfy

$$z_t + Az - b(z) = -\rho BB(b(z) - Az), \quad z(0) = \phi.$$

This suggests to generalize the control law (10) to

$$Ky = -\frac{\rho}{h}BB(y - z) - \rho BB(b(y) - b(z)) + z_t + Az - b(z). \tag{11}$$

With this control law the controlled Navier-Stokes equations become the form

$$y_t + Ay - b(y) = Ky \quad \text{in } L^2(V') \text{ and } \quad y(0) = \phi. \tag{12}$$

Concerning stability of this system we have from [9, Theorem 6.1,6.2]

$$|w(t)|^2_{H,V} \;\leq\; Ce^{-\frac{\rho}{h}t} \quad \forall t \in [0,T],$$

where C is a positive constant and $w := y - z$. It is worth noting that control law (12) can also be obtained by a special choice of u_0^k in Algorithm 3.1, see [12]. Its discrete in time version is given by

$$\begin{array}{rcl}(I+hA)w^{j+1} &=& w^j + h\left(b(y^j) - b(z^j)\right) - \rho BBw^j - \rho hBB\left(b(y^j) - b(z^j)\right), \\ w^0 &=& \phi - z(0).\end{array} \tag{13}$$

It is easy to extend RECIPE 1 for control constraints where $u \in \mathcal{U}_{ad}$ is required. A simple way to guarantee $u^{k+1} \in \mathcal{U}_{ad}$ is to project the gradient step onto $\mathcal{U}_{ad}$.

RECIPE 2. (Instantaneous control with)
For the instantaneous control strategy with control constraints the oracle RECIPE is defined as in RECIPE 1 except that (v) is replaced by

(v)′ set $RECIPE = P_{\mathcal{U}_{ad}}(v - \rho d)$.

The stability analysis of RECIPE 1 is given in [9], for the result see also the next section.

We now construct a recipe which applied in Algorithm 3.1 realizes a full optimization step for problem $(\mathbf{P}^k)$. For this purpose first let $\mathcal{U} = L^2(\Omega)^2$. Then the operator $\mathcal{B}$ is defined by

$$\langle \mathcal{B}u, v\rangle_{V',V} = (u,v).$$

We choose the control u^k to be the solution of $(\mathbf{P}^k)$ for the unconstrained case, i.e. $\mathcal{U}_{ad} = \mathcal{U}$. Then u^k solves the optimality system, compare (6),

$$\begin{array}{lcl}(I+hA)y^{k+1} &=& y^k + hb(y^k) + u^k \\ (I+hA)\lambda^k &=& z^k - y^{k+1} \\ \gamma u^k - \lambda^k &=& 0.\end{array} \tag{14}$$

It is easy to see that

$$u^k = -(BB + \gamma I)^{-1} B(B(y^k + hb(y^k)) - z^k)$$

holds. Now let

$$S = \gamma(BB + \gamma I)^{-1}BB, \tag{15}$$

which defines a continuous linear operator in $L(H,H)$. Further properties of S are investigated in [13]. Exploiting the relation $Bz^k = BB(z^k + hAz^k)$ we get

$$(I+hA)y^{k+1} = y^k + hb(y^k) - \frac{1}{\gamma}S(y^k - z^k + hb(y^k) - hAz^k), \quad y^0 = \phi, \tag{16}$$

Now it is easy to see that equation (16) in y is the semidiscretization with stepsize h of the dynamical system

$$y_t + Ay - b(y) = -\frac{1}{\gamma h} S(y - z + hb(y) - hAz), \quad y(0) = \phi. \tag{17}$$

In order to further improve this control law we proceed as in the derivation of (11) and obtain the modified control law

$$y_t + Ay - b(y) = -\frac{1}{\gamma h} S(y - z + hb(y) - hb(z)) + z_t + Az - b(z), \quad y(0) = \phi, \tag{18}$$

whose semidiscretization is given by

$$\begin{aligned}(I + hA)y^{k+1} = y^k + hb(y^k) - \frac{1}{\gamma} S(y^k - z^k) - \frac{h}{\gamma} S(b(y^k) - b(z^k)) \\ + z^{k+1} - z^k + hAz^k - hb(z^k).\end{aligned} \tag{19}$$

If we define the feedback control operator K by

$$u = K(y) = -\frac{1}{\gamma h} S(y - z + hb(y) - hb(z)) + z_t + Az - b(z) \tag{20}$$

we end up with a interpretation. Concerning stability we state from [12]

$$|w(t)|_H^2 \;\; \leq \;\; C\, e^{-\frac{\alpha(\gamma)}{h} t} |w(0)|_H^2 \quad \forall t \in [0, T], \text{ where } \alpha(\gamma) = \tfrac{\gamma}{(1+\gamma)^2}.$$

In order to motivate the related recipe we note that we would obtain the feedback operator K in (20) also by investigating the optimal control problem

$$\min J(v^k) = \frac{1}{2} \int_\Omega |w^{k+1}| + \frac{\gamma}{2} |v^k|^2, \tag{$\tilde{\mathbf{P}}^k$}$$

subject to

$$(I + hA)w^{k+1} = w^k + hb(w^k + z^k) - hb(z^k) + v^k.$$

instead of $(\mathbf{P}^k)$. Here $w = y - z$ denotes the difference of the state and the desired state. The related recipe then is given by

RECIPE 3. (Suboptimal control or extended $(\delta t, 1)$ model predictive control)
The oracle RECIPE is defined by $u = RECIPE(v, y^k, z^k, z^{k+1}, t)$ iff

- Solve the optimality system for u

$$\begin{aligned}(I + hA)y &= y^k - z^k + hb(y^k) - hb(z^k) + (I + hA)z^{k+1} + u \\ (I + hA)\lambda &= z^{k+1} - y \\ \gamma u - \lambda &= 0.\end{aligned}$$

- set $RECIPE = u$

For details we refer to [12].

4 Numerical results

In this section, we numerically investigate the stabilizing properties of Algorithm 3.1 with RECIPEs 1, 2, 3. Further numerical results for RECIPE 1 can be found in [9].

In the control problem considered here we intend to track a desired time dependent state z in the L^2-sense, i.e. our instantaneous cost functional has the form

$$J(y,u) = \frac{1}{2}\int_{\Omega_o} |y(x,t)) - z(x,t)|^2 dx + \frac{\gamma}{2}\int_{\Omega_c} |u(x,t)|^2 dx,$$

where t denotes the time instance, Ω_o denotes the observation domain, and Ω_c the control domain, respectively. State and control are coupled by the instationary Navier-Stokes system ($\mathbf{P}$). We realize the control u by the feedback control laws derived in the RECIPEs 1, 2 and 3.

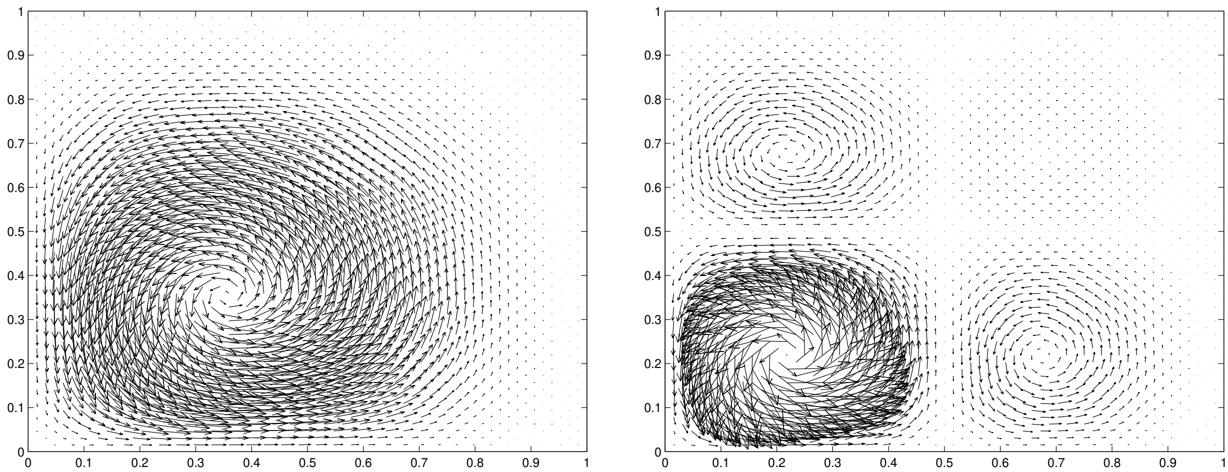

Figure 1: Desired flow at $T = 1$ and $T = 2$

Let us specify the problem setting. The computation domain is the unit square $\Omega = [0,1]^2$. The initial value is chosen as

$$y(x,0) = \phi(x) = e \begin{pmatrix} (\cos 2\pi x_1 - 1)\sin 2\pi x_2 \\ -(\cos 2\pi x_2 - 1)\sin 2\pi x_1 \end{pmatrix},$$

where e is the Euler number, and the desired flow is time-dependent and defined by

$$z(t,x) = \begin{pmatrix} \psi_{x_2}(t,x_1,x_2) \\ -\psi_{x_1}(t,x_1,x_2) \end{pmatrix},$$

where ψ is given through the stream function

$$\psi(t,x_1,x_2) = \theta(t,x_1)\theta(t,x_2)$$

with

$$\theta(t, y) = (1 - y)^2(1 - \cos 2\pi y t).$$

The Reynolds number is set to 10, which results in a viscosity coefficient of $\nu = 1/10$. The final time is chosen as $T = 2$. For the discretization in time a equidistant grid with stepsize $\delta t = 0.01$ is used, whereas for the spatial discretization the Taylor-Hood finite element is applied on a grid containing 1024 triangles with 2113 velocity and 545 pressure nodes.

The time discretization of the continuous controlled systems is always chosen in such a way that the related discrete-in-time controlled systems are obtained, i.e. the discretization of (12) gives (13). We note that this discretization leads to exponential stable discrete-in-time controlled systems. This can be shown utilizing the techniques of [9], where exponential stability of the discrete-in-time controlled system is proven for Algorithm 1 with RECIPE 1. At first, we compare the results of the RECIPEs 1 and 2. The step-size of the gradient iteration was set to $\rho = 0.1$. The evolution of the costs is shown in Figure 2 for the unconstrained and the box-constrained case, where $|u(x, t)| \leq 1e - 3$ is required. As one can see, both controllers give the same output if the control constraint is not active. Moreover, we observe exponential decay of $|y(t) - z(t)|_H$ in the unconstrained case, see Fig. 2, left.

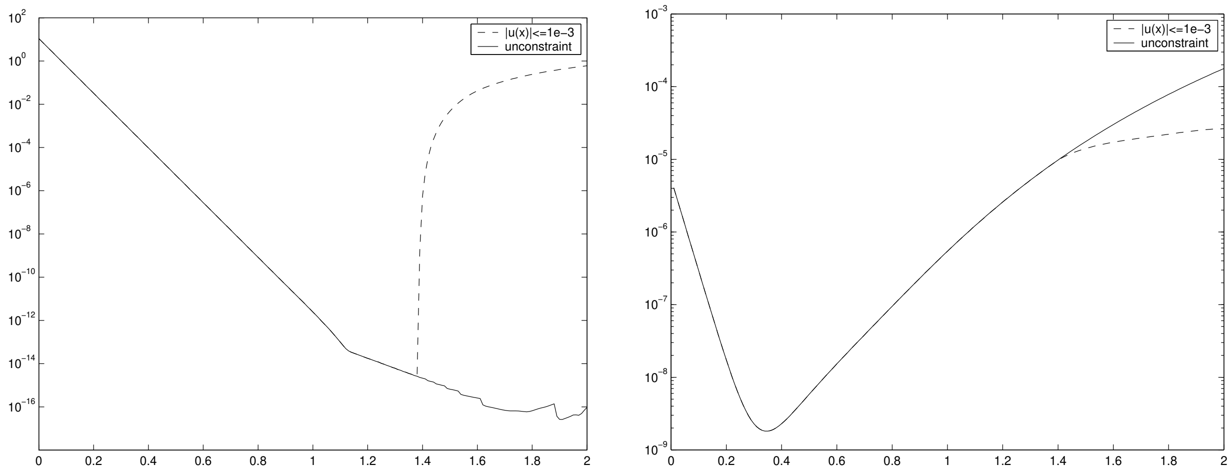

Figure 2: Evolution of $|y(t) - z(t)|_H^2$ and of $|u(t)|_{L^2(\Omega)}^2$ for unconstrained and constrained control

Since the desired state becomes more and more dynamic, the constrained controller can not adapt to this situation, and the distance between state and desired state grows for $t \geq 1.4$. On the other hand, the control costs per time step remain constant, see the right-hand diagram in Fig. 2. This fact is also illustrated byFigure 3.

Secondly, we present some results for the feedback controller given by RECIPE 3. In Fig. 4 the evolution of the L^2-Cost $|y(t) - z(t)|_H$ is shown for different values of γ. Exponential decay for both large and small values of the regularization parameter

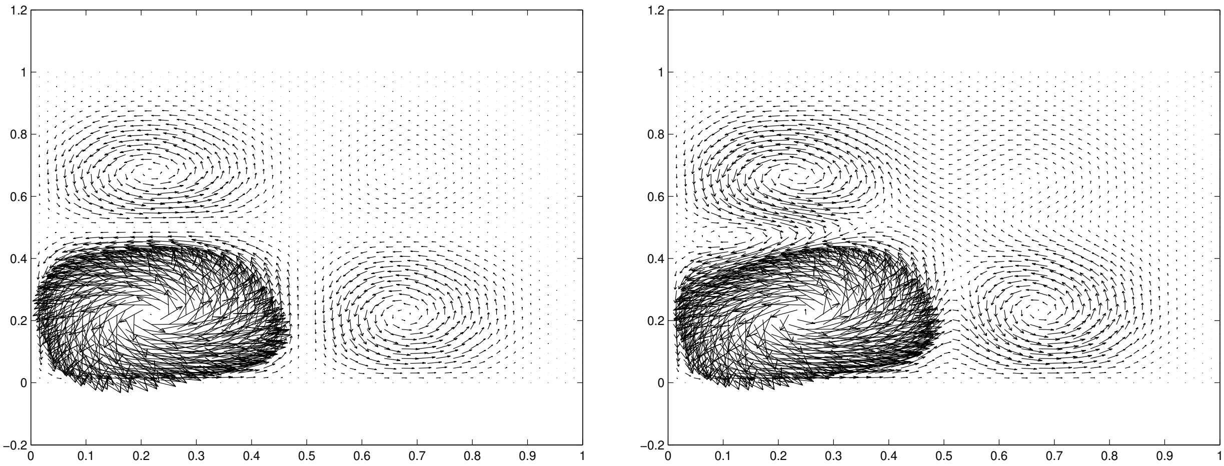

Figure 3: Flow at $T = 2$ for RECIPEs 1 and 2

is observed numerically, although the theory is only satisfying for large values. In each application of the control RECIPE 3, an optimization problem has to be solved. It turns out, that with the solution from the previous time-step as initial guess in the cg-method, very few conjugate gradient steps are needed. The optimization process is stopped if either the relative residual norm is less than $1e-4$ or the absolute residual norm is less than $1e-8$.

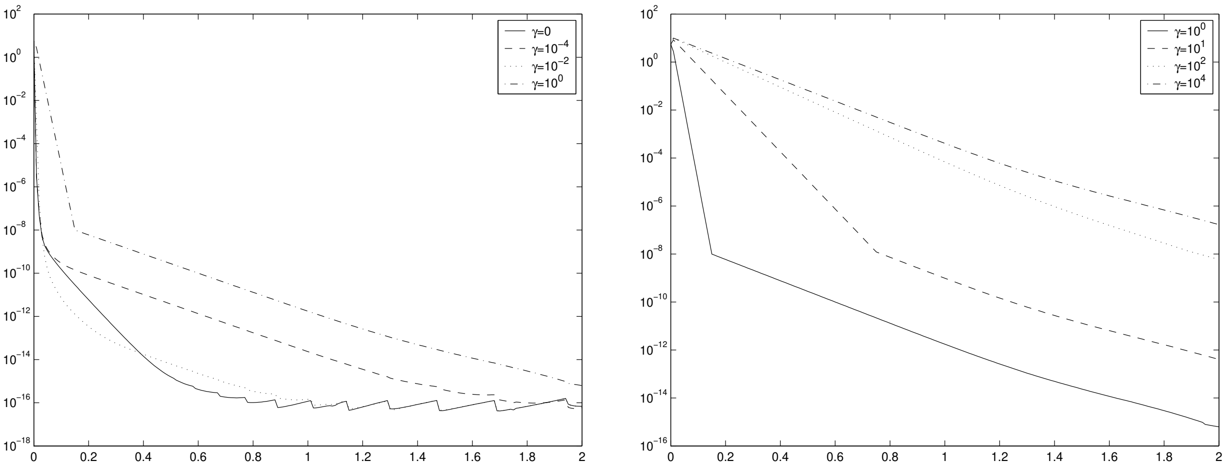

Figure 4: Evolution of $|y(t) - \bar{y}(t)|_H^2$ for different values of γ

At last, we note that the computational time required for the application of the recipe discussed in this research is in the range of 2.5 to 5 times the computational time required for one forward solve of the uncontrolled time dependent Navier-Stokes system, which in fact is *fast* compared to optimal control, say, see [10].

References

[1] Choi, H: Suboptimal Control of Turbulent Flow Using Control Theory. In: Proceedings of the International Symposium. on Mathematical Modelling of Turbulent Flows (1995), Tokyo, Japan.

[2] Choi, H., Hinze, M., Kunisch, K: Instantaneous control of backward-facing-step flows. Applied Numerical Mathematics, **31**:133–158 (1999)

[3] Choi, H, Temam, R., Moin, P., Kim, J: Feedback control for unsteady flow and its application to the stochastic Burgers equation. J. Fluid Mech., **253**:509–543 (1993)

[4] Constantin, P., Foias, C: Navier-Stokes Equations. The University of Chicago Press (1988)

[5] Dautray, R., Lions, J. L: Evolution problems I, volume 5 of Mathematical analysis and numerical methods for science and technology. Springer, Berlin (1992)

[6] García, C.E., Prett, D.M., Morari, M: Model predictive control: Theory and practice - a survey. Automatica, **25**:335–348 (1989)

[7] Gunzburger, M.D., Manservisi, S: Analysis and approximation for linear feedback control for tracking the velocity in Navier-Stokes flows. Comput. Methods Appl. Mech. Eng., **189**:803–823 (2000)

[8] Hill, D.C: Drag reduction strategies. CTR Annual Research Briefs (1993) Center for Turbulence Research, Stanford University/NASA Ames Research Center, 3-14.

[9] Hinze, M: Instantaneous closed loop control of the Navier-Stokes system. Preprint MATH-NM-09-2002. Institut für Numerische Mathematik, Technische Universität Dresden, Germany.

[10] Hinze, M., Kunisch, K: Control strategies for fluid flows - optimal versus suboptimal control. In: H.G.Bock et al., editor, ENUMATH 97, pages 351–358. World Scientific (1997), Singapore.

[11] Hinze, M., Volkwein, S: Instantaneous control for the Burgers equation: Convergence analysis and numerical implementation. Nonlinear Analysis T.M.A., **50**:1–26 (2002)

[12] Hinze, M., Wachsmuth, D: Controller design for the Navier-Stokes system. Part I: Concept, Numerical Results. Report 11-2003, Institut für Mathematik, Technische Universität Berlin (2003)

[13] Hinze, M., Wachsmuth, D: Controller design for the Navier-Stokes system. Part II: Stability Analysis. in preparation.

[14] Hou, L.S., Yan, Y: Dynamics and approximations of a velocity tracking problem for the Navier-Stokes flows with piecewise distributed controls. SIAM J. Control Optim., **35**:1847–1185 (1997)

[15] Hou, L.S., Yan, Y: Dynamics for controlled Navier-Stokes Systems with distributed controls. SIAM J. Control Optim., **35**:654–677 (1997)

[16] Lee, C., Kim, J., Choi, H: Suboptimal control of turbulent channel flow for drag reduction. J. Fluid Mech., **358**:245–258 (1998)

[17] Min, C., Choi, H: Suboptimal feedback control of vortex shedding at low Reynolds nunmbers. J. Fluid Mech., **401**:123 – 156 (1999)

[18] Temam, R: Navier-Stokes Equations. North-Holland (1979)

[19] Tröltzsch, F., Unger, A: Fast solution of optimal control problems in selective cooling of steel. ZAMM, **81**:447–456 (2001)