Als Manuskript gedruckt

Technische Universität Dresden
Herausgeber: Der Rektor

Second Order Methods for Boundary Control of the Instationary Navier-Stokes System

Michael Hinze and Karl Kunisch

MATH-NM-07-2002

Juli 2002

Contents

Second Order Methods for Boundary Control of the Instationary Navier-Stokes System

Michael Hinze
Institut für Numerische Mathematik,
Technische Universität Dresden
D–01069 Dresden, Germany,
email: hinze@math.tu-dresden.de

Karl Kunisch
Institut für Mathematik,
Karl-Franzens-Universität Graz
A–8010 Graz, Austria,
email: karl.kunisch@uni-graz.at

Abstract

Second order methods for open loop optimal boundary control problems governed by the instationary Navier-Stokes system are investigated. A general analytic framework is developed which allows an elegant representation of first and second order derivatives of the objective functional and of the state equations. Moreover a second order sufficient optimality condition is proved which guarantees local quadratic convergence of Newton's method.

1 Introduction and Preliminaries

In this paper we study second order methods for solving boundary optimal control problems of the type

$$(\mathcal{P}) \qquad \text{minimize } \; J(y,u)$$

subject to

$$(NS) \qquad \begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p = 0 \;\text{ in }\; Q, \\ -\text{div }\; y = 0 \;\text{ on }\; \Sigma, \\ y = Bu \;\text{ on }\; \Sigma \\ y(0,\cdot) = y_0 \;\text{ in }\; \Omega, \end{cases}$$

where $\nu > 0, y = y(t,x) \in \mathbb{R}^2$ for $(t,x) \in Q = (0,T)\times\Omega,\ T > 0,\ \Sigma = (0,T)\times\partial\Omega$, and Ω is bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Further J denotes a real–valued cost functional and B the control operator acting from the space of controls U to functions of appropriate regularity defined on Σ. (NS) is short–hand notation for the instationary Navier–Stokes equation with boundary control, in particular $\Delta y = (\Delta y_1, \Delta y_2)^T$ and $((y\cdot\nabla)y)_j = (\sum_{i=1}^2 y_i\partial x_i)y_j$, for $j = 1,2$. Optimal control problems have received a significant amount of attention in the past years.

Below we comment on some contributions which are of direct relevance to our paper. Let us also mention a recent survey [Be], featuring primarily computational issues.

However, the analysis of second order methods has barely been touched, in spite of the fact that the quadratic nature of the Navier-Stokes equations naturally lends itself to such techniques. One of the causes for this fact are the potential difficulties related to storage requirements for the Hessian matrix. As we shall emphasize, there is no necessity to store or partially store the Hessian matrix when realizing second order methods. Rather, it suffices to be able to compute the action of the Hessian in the current search direction. With the solution of the Navier-Stokes equations and its adjoint available, the action of the Hessian can be numerically realized by two additional solves of linear equations: one forward sweep of the linearized Navier-Stokes equations and one backwards sweep of appropriately defined adjoint equations. For shape optimization problems an analogous procedure was followed in [L].

Let us briefly comment on some selected related research contributions. In [GB, H] sequential quadratic programming methods are applied to optimal control problems for stationary Navier-Stokes equations. It appears worth-while here to point out that the difference between one step of the Newton algorithm and one step of the sequential programming algorithm applied to $(\mathcal{P})$ is that the former utilizes the nonlinear equations (NS) to compute the state y corresponding to a current control u, whereas in the sequential quadratic programming technique this state is approximated by the state of the linearized equations. For instationary problems, and moderate Reynolds numbers the difference in the expense for the solution to the nonlinear, respectively the linearized equations is not so significant. In [HK] we studied second order methods for optimal control problem related to $(\mathcal{P})$ but with distributed rather than boundary control.

Boundary control add significant difficulties over distributed controls. From the point of view of analysis, Dirichlet boundary controls are non-variational and hence the choice of an appropriate analytic and function space setting is neither unique nor obvious. From the practical perspective the influence of boundary forces being much weaker than that of body forces, it is more challenging to reach the design objective. In [GHS] the authors derive the optimality system for stationary boundary optimal control problems and discuss their finite element approximations. The difficulty with Dirichlet controls is approached in [HR] by incorporating them in a Robin type condition containing the Dirichlet part with high weighting factors. Time-dependent boundary control problems are considered in [B], for example, where the design objective is vorticity reduction. Here the controls are taken in L^2 with respect to time while the states are in $L(0,1;H^1)$. This is a practically convenient choice but it appears that it cannot be supported by a function space framework. In [FGH] an elegant function space framework for Dirichlet-boundary optimal controls of exterior problems is developed and optimality conditions are derived. The "optimal" choice of function spaces for boundary control involves spaces of fractional powers, which do not readily lend themselves to numerical implementation. In [GM] the control space is chosen to be H^1 with respect to space and time. This choice, while it assumes more regularity than necessary for well-posedness of the optimal control problems, has the advantage of being more readily implementable. Our choice of function space for the controls is formulated such that we can focus on the development of the second order derivatives. It contains the choices made in [GM] and [FGH] as special cases.

The remainder of the paper is organized as follows. At the end of this section we summarize technical results which we shall draw upon in the remainder of the paper. The optimal control problems are formulated and existence is proved in Section 2. Section 3 contains the characterization of the first and second derivatives of the cost functionals with respect to the controls. These results directly lend themselves to formulate first order necessary and second order sufficient optimality conditions.

In the 2^{nd} part of this work we investigate the numerical implementation of 2^{nd} order methods for boundary control of the instationary Navier-Stokes system and provide numerical examples.

We now summarize some technical results that we shall refer to subsequently. Throughout Ω denotes a bounded domain in $\mathbb{R}^2$ with Lipschitzian boundary. We set

$$\mathbb{V} = \{v \in H^1(\Omega)^2 \colon \text{ div } v = 0\}$$

as subset of $H^1(\Omega)^2$, and

$$\mathbb{H} = \{v \in H^1(\Omega)^2 \colon \text{ div } = 0\}^{-L^2(\Omega)^2},$$

where the superscript denotes closure in $L^2(\Omega)^2$.

Lemma 1.1. There holds

(i) $\mathbb{H}^\perp = \{v \in L^2(\Omega)^2 \colon v = \nabla p$ for some $p \in H_0^1(\Omega)\}$ and

(ii) $\mathbb{H} = \{v \in L^2(\Omega)^2 \colon \text{ div } v = 0\}$.

Proof. We argue analogously as in [CF, p.7]. To verify (i) let Z denote the right hand side of (i). Choose $v \in Z$ so that $v = \nabla p$ for some $p \in H_0^1(\Omega)$. Then for every $u \in \mathbb{H}$ we have $\int_\Omega vu = \int_\Omega \nabla pu = 0$ and hence $v \in \mathbb{H}^\perp$ and $Z \subset \mathbb{H}^\perp$. Conversely, if $v \in \mathbb{H}^\perp$ then $\int_\Omega vu = 0$ for all $u \in \mathbb{H}$ and hence (cf. [CF, Proposition 1.8]) there exists $p \in H^1(\Omega)$ such that $v = \nabla p$ and $\int_\Omega \nabla p\, u = 0$ for all $u \in \mathbb{H}$. Consequently $\int_\Gamma pu \cdot n = 0$ for all $u \in H^1(\Omega)^2$ with div $u = 0$. Since $\{u \cdot n \colon u \in H^1(\Omega)^2, \text{div } u = 0\}$ is dense in $L^2(\Gamma)$, which can be argued for example, by considering the Stokes problem with inhomogeneous boundary conditions, it follows that $p \in H_0^1(\Omega)$ and $v \in Z$. - To verify (ii) let Z denote the right hand side of (ii). Clearly $\mathbb{H}$ is a closed (in $L^2(\Omega)^2$) subspace of Z. If it was a strict subspace then there would exist an element $v \in Z$ with $v \in \mathbb{H}^\perp$. Consequently there exists $p \in H_0^1(\Omega)$ such that $\nabla p = v$ and p is a variational solution to $\Delta p = 0$. Consequently $p = v = 0$ and $H = Z$. $\Box$

Recall that $\gamma_n \colon v \to v \cdot n|\partial\Omega$ is a bounded linear functional from $H(\text{div }, \Omega) = \{v \in L^2(\Omega)^2 \colon \text{ div } v \in L^2(\Omega)\}$ to $H^{-1/2}(\partial\Omega)$. In particular γ_n is well defined from $\mathbb{H}$ to $H^{-1/2}(\partial\Omega)$ and $\int_{\partial\Omega} v \cdot n = 0$ for all $v \in \mathbb{H}$. Clearly $\mathbb{V}$ is densely and continuously injected into $\mathbb{H}$ so that the Gelfand triple

$$\mathbb{V} \to \mathbb{H} \to \mathbb{V}^*$$

is well–defined. We shall also require the analogs to $\mathbb{V}$ and $\mathbb{H}$ for homogenous Dirichlet boundary conditions defined by

$$V = \{v \in H_0^1(\Omega)^2 \colon \text{div } v = 0\} \text{ as subset of } H^1(\Omega)^2, \text{ and}$$

$$H = \overline{\{v \in H_0^1(\Omega)^2 \colon \text{div } v = 0\}}^{L^2(\Omega)^2}.$$

From now onwards we abbreviate $L^2(\mathbb{V}) \equiv L^2(0,T;\mathbb{V})$, $L^2(V^*) \equiv L^2(0,T;V^*)$ and $W^{p,q}(X) \equiv W^{p,q}(0,T;X)$ where $T > 0$ denotes a fixed time. We further put

$$\mathbb{W} = \{\varphi \in L^2(\mathbb{V}) \colon \varphi_t \in L^2(V^*)\}, \text{ and}$$

$$W = \{\varphi \in L^2(V) \colon \varphi_t \in L^2(V^*)\},$$

endowed with the norms $|\varphi|_{\mathbb{W}}^2 = (|\varphi|_{L^2(\mathbb{V})}^2 + |\varphi_t|_{L^2(V^*)}^2)^{1/2}$ and $|\varphi|_W = (|\varphi|_{L^2(V)}^2 + |\varphi_t|_{L^2(V^*)}^2)^{1/2}$ respectively. Recall that W is continuously injected into $C([0,T];H)$, see e.g.[DL]. Similarly $\mathbb{W}$ is continuously embedded in $C([0,T];\mathbb{H})$, see ([FGH], Lemma 3.1). To characterize the admissible inhomogeneities for Dirichlet boundary conditions we introduce

$$W_\Sigma = \{g = \tau\hat{g} \colon \text{ with } \hat{g} \in \mathbb{W}\}.$$

Here $\tau \colon \mathbb{W} \to L^2(H^{1/2}(\partial\Omega)^2)$ is the trace operator onto the lateral boundary $\Sigma = (0,T) \times \partial\Omega$ of the cylinder $Q = (0,T) \times \Omega$ given by $(\tau\, v)(t) = v(t,\cdot)_{|}\partial\Omega$ for a.e. to $[0,T]$. Since τ is a bounded linear operator, ker τ is closed and the quotient space $\mathbb{W}/\text{ker } \tau$ is a Banach space. In particular, we can endow W_Σ with the norm

$$|g|_{W_\Sigma} = \inf\{|\hat{g}|_{\mathbb{W}} \colon \tau\hat{g} = g, \hat{g} \in \mathbb{W}\}, \tag{1}$$

the infimum is attained and $(W_\Sigma, |g|_{W_\Sigma})$ is a Banach space. For $g \in W_\Sigma$ we define by $\tilde{g}$ the unique element in $\mathbb{W}$ which achieves the infimum in (1). This "tilde" operation serves as an analytical tool which is not used for computations. Note that $\tilde{g}(0) \in \mathbb{H}$ and $\gamma_n \tilde{g}(0)$ is a well–defined element of $H^{-1/2}(\partial\Omega)$. - For future reference we also introduce

$$L_0^2(\Omega) = \left\{p \in L^2(\Omega) \colon \int_\Omega p\, dx = 0\right\}.$$

Theorem 1.2. For every $g \in W_\Sigma$ and $y_0 \in \mathbb{H}$ satisfying $y_0 - \tilde{g}(0) \in H$ there exists a unique solution $y \in \mathbb{W}$ to the nonhomogenous Stokes equations

$$\left\{ \begin{array}{l} \langle y_t, v\rangle_{L^2(V^*),L^2(V)} + \nu(\nabla y, \nabla v)_{L^2(L^2)} = 0, \text{ for all } v \in L^2(V) \\ \tau\, y = g \quad \text{in} \quad W_\Sigma \\ y(0,\cdot) = y_0 \quad \text{in} \quad \mathbb{H}. \end{array} \right. \tag{2}$$

Moreover there exists a constant K_1 independent of g and y_0 such that

$$|y|_{L^2(\mathbb{V})} + |y_t|_{L^2(V^*)} + |y|_{C(\mathbb{H})} \le K_1(|g|_{W_\Sigma} + |y_0|_{\mathbb{H}}). \tag{3}$$

Proof. It is simple to argue uniqueness. To verify existence let $\tilde{g} \in \mathbb{W}$ be chosen as described above. Note that $\tilde{g}(0) \in \mathbb{H}$ and that $\psi \to (\tilde{g}_t, \psi) + \nu(\nabla\tilde{g}, \nabla\psi)$, $\psi \in L^2(V)$, defines an element of $L^2(V^*)$. By assumption we have $y_0 - \tilde{g}(0) \in H$. We next consider the homogenous Stokes problem

$$
(4) \qquad \begin{cases} \langle \bar{y}_t, v\rangle_{L^2(V^*),L^2(V)} + \nu(\nabla\bar{y}, \nabla v)_{L^2(L^2)} = -\langle \tilde{g}_t, v\rangle_{L^2(V^*),L^2(V)} - \\ -\nu(\nabla\tilde{g}, \nabla v)_{L^2(L^2)} \ \text{ for all } \ v \in L^2(V) \\ \tau\,\bar{y} = 0 \quad \text{in} \quad W_\Sigma \\ \bar{y}(0,\cdot) = y_0 - \tilde{g}(0) \quad \text{in} \quad H. \end{cases}
$$

By well known techniques (see e.g.[DL, p.507 ff]) (4) admits a unique solution $\bar{y} \in W$ satisfying

$$|\bar{y}|_W \le C_1\,(|y_0|_{\mathbb{H}} + |\tilde{g}|_{\mathbb{W}}),$$

where C_1 is independent of y_0 and g.

Let us set $y = \bar{y} + \tilde{g}$. Then $y \in L^2(\mathbb{V})$, $y_t \in L^2(V^*)$ and

$$|y|_{L^2(\mathbb{V})} + |y_t|_{L^2(V^*)} \le C_2\,(|y_0|_{\mathbb{H}} + |g|_{W_\Sigma}),$$

for a constant C_2 independent of $(y_0, g) \in \mathbb{H} \times W_\Sigma$. Since both W and $\mathbb{W}$ embed continuously into $C([0,T];\mathbb{H})$ there exists K_1 with the specified properties, such that

$$|y|_{L^2(\mathbb{V})} + |y|_{L^2(V^*)} + |y|_{C(\mathbb{H})} \le K_1(|y_0|_{\mathbb{H}} + |g|_{W_\Sigma}).$$

Finally y satisfies (2). $\Box$

Now we turn to the nonlinear equations. Throughout the following trilinear form will be essential:

$$c(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_j \,\frac{\partial v_i}{\partial x_j} w_i \ \text{ for } \ u,v,w \in H^1(\Omega)^2.$$

The next lemma summarizes some frequently used properties of c.

Lemma 1.3. There exists a constant C such that

(i) $c(u,v,w) = -c(u,w,v)$ for all $u \in V$, $v,w \in H^1(\Omega)^2$,

(ii) $c(u,v,w) = -c(u,w,v)$ for all $u \in \mathbb{V}$, $v \in H^1(\Omega)^2, w \in H_0^1(\Omega)^2$

(iii) $|c(u,v,w)| \le \sqrt{2}|u|^{1/2}|\nabla u|^{1/2}|\nabla v||w|^{1/2}|\nabla w|^{1/2}$, for all $u,v,w \in H^1(\Omega)^2$,

(iv) $|c(u,v,w)| \le C|u|_{L^2}^{1/2}|u|_{H^1}^{1/2}|v|_{H^1}^{1/2}|v|_{H^2}^{1/2}|w|_{L^2}$, for all $u \in H^1(\Omega)^2, v \in H^2(\Omega)^2, w \in L^2(\Omega)^2$,

(v) $|c(u,v,w)| \le C|u|_{L^2}|v|_{H^1}|w|_{L^2}^{1/2}|w|_{H^2}^{1/2}$, for all $u \in L^2(\Omega)^2, v \in H^1(\Omega)^2, w \in H^2(\Omega)^2$, and

(vi) $|c(u,v,w)| \le C|u|_{L^2}^{1/2}|u|_{H^2}^{1/2}|v|_{H^1}|w|_{L^2}$, for all $u \in H^2(\Omega)^2, v \in H^1(\Omega)^2, w \in L^2(\Omega)^2$.

Proof. (i), (iii)–(vi) are standard results, see e.g. [T1] or [GM]. As for (ii) note that for $u, v, w \in H^1(\Omega)^2$ with div $u = 0$ we have

$$c(u,v,w) = \sum_{i,j} \int_\Omega u_i \frac{\partial}{\partial x_i} v_j w_j \, dx = \int_{\partial\Omega} (u \cdot n)(v \cdot w) ds -$$

$$- \sum_{i,j} \int_\Omega u_i \frac{\partial}{\partial x_i} w_j v_j \, dx = \int_{\partial\Omega} (u \cdot n)(v \cdot w) dx - c(u,w,v).$$

Here we use the fact that the trace operator on $H^1(\Omega)$ embeds continuously into $L^s(\partial\Omega)$ for every $s < \infty$, so that $\int_{\partial\Omega}(u \cdot n)(v \cdot w) ds$ is well defined. For $w \in H_0^1(\Omega)$ we therefore deduce that (ii) holds. □

Theorem 1.4. For every $g \in W_\Sigma$ and $y_0 \in \mathbb{H}$ with $y_0 - \tilde{g}(0) \in H$ there exists a unique solution $y \in \mathbb{W}$ to the nonhomogenous Navier–Stokes problem

$$(5) \qquad \begin{cases} \langle y_t, v\rangle_{L^2(V^*),L^2(V)} + \nu(\nabla y, \nabla v)_{L^2(L^2)} \quad + \langle (y \cdot \nabla) y, v\rangle = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } v \in L^2(V), \\ \tau\, y = g \quad \text{in} \quad W_\Sigma \\ y(0,\cdot) = y_0 \quad \text{in} \quad \mathbb{H}. \end{cases}$$

Moreover there exists a continuous function $K_2 \colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$(6) \qquad |y|_{L^2(\mathbb{V})} + |y_t|_{L^2(V^*)} + |y|_{C(\mathbb{H})} \le K_2\left(|g|_{W_\Sigma}, |y_0|_{\mathbb{H}}\right).$$

Proof. For $(g, y_0) \in W_\Sigma \times \mathbb{H}$ let y_L denote the solution to (2). Introducing a Navier–Stokes like problem for $\hat{y} \in L^2(V)$ by

$$(7) \qquad \begin{cases} \langle \hat{y}_t, v\rangle_{L^2(V^*),L^2(V)} + \nu(\nabla \hat{y}, \nabla v)_{L^2(L^2)} + \displaystyle\int_0^T (c(\hat{y},\hat{y},v) + c(\hat{y}, y_L, v) \\ +c(y_L, \hat{y}, v) + c(y_L, y_L, v))dt = 0, \quad \text{for all} \;\; v \in L^2(V) \\ \tau\, \hat{y} = 0 \quad \text{in} \quad W_\Sigma \\ \hat{y}(0,\cdot) = 0 \quad \text{in} \quad H, \end{cases}$$

the candidate for the solution to (5) with the required properties is given by $y = \hat{y} + y_L$. Existence of a unique solution $\hat{y} \in W$ to (7) can be argued with the techniques of [T, Chapter III], and we therefore restrict ourselves to the essential a–priori estimates. Choosing $v = \chi_{(0,t)}\hat{y}$ ($t \in (0,T)$ arbitrary) in (7) and utilizing the fact that

$$c(u,v,w) = -c(u,w,v) \;\; \text{for all} \;\; u \in V, \; v, w \in H^1(\Omega)$$

we obtain

$$\int_0^t \tfrac{1}{2}\tfrac{d}{ds}|\hat{y}(s)|^2 + \nu|\nabla\hat{y}(s)|^2 + c(\hat{y}(s), y_L(s), \hat{y}(s)) + c(y_L(s), y_L(s), \hat{y}(s))\, ds = 0.$$

Since $|c(u,v,u)| \le \sqrt{2}|u|\,|\nabla u|\,|\nabla v|$ for all $u, v \in H^1(\Omega)$ by Lemma 1.3 (iii) we have, utilizing Lemma 1.3 (ii),

$$\int_0^t \tfrac{1}{2}\tfrac{d}{ds}|\hat{y}(s)|^2 + \nu|\nabla\hat{y}(s)|^2\, ds \le \sqrt{2}\int_0^t |\hat{y}(s)|\,|\nabla\hat{y}(s)|\,|\nabla y_L(s)| + |y_L(s)|\,|\nabla y_L(s)|\,|\nabla\hat{y}(s)|\, ds$$

and consequently

$$\int_0^t \tfrac{d}{ds}|\hat{y}|^2 + \nu|\nabla\hat{y}|^2\, ds \le c\int_0^t |\hat{y}|^2|\nabla y_L|^2 + |y_L|^2|\nabla y_L|^2\, ds \tag{8}$$

for a constant c depending on ν but independent of t, g and y_0. An application of Poincaré's and of Young's inequality then implies

$$\int_0^t \tfrac{d}{ds}\left(e^{\alpha s}|\hat{y}(s)|^2\right)\, ds \le c\int_0^t e^{\alpha s}\left(|\hat{y}(s)|^2|\nabla y_L(s)|^2 + |y_L(s)|^2|\nabla y_L(s)|^2\right)\, ds \tag{9}$$

where $\alpha := \nu/(c_p^2)$ and c_p denotes the constant of Poincaré's inequality. Gronwall's lemma together with Theorem 1.2 and equation (8) then leads to the estimate

$$|\hat{y}|_{C(H)} + |\hat{y}|_{L^2(V)} \le c_1(|g|_{W_\Sigma}, |y_0|_{\mathbb{H}}) \tag{10}$$

with a continuous function $c_1\colon \mathbb{R}^2 \to \mathbb{R}$. Moreover, from (7) and Lemma 1.3 (iii) we have

$$\begin{aligned}
&|\hat{y}_t|_{L^2(V^*)} = \\
&\qquad \sup_{|v|_{L^2(V)}=1}\Big[\nu(\nabla\hat{y}, \nabla v)_{L^2(L^2)} + \int_0^T (c(\hat{y},\hat{y},v) + c(\hat{y}, y_L, v) + c(y_L, \hat{y}, v) + \\
&\quad c(y_L, y_L, v))dt\Big] \le \nu\,\bar{c}(|g|_{W_\Sigma}, |y_0|_{\mathbb{H}}) + \sup_{|v|_{L^2(V)}=1}\sqrt{2}\int_0^T (|\hat{y}(t)|_H|\hat{y}(t)|_V|v(t)|_V + \\
&\qquad + 2|\hat{y}(t)|_H^{1/2}|\hat{y}(t)_V^{1)2}|y_L(t)|_{\mathbb{H}}^{1/2}|y_L(t)|_{\mathbb{V}}^{1/2}|v(t)|_V + |y_L(t)|_{\mathbb{H}}|y_L(t)|_{\mathbb{V}}|v(t)|_V)dt.
\end{aligned}$$

Theorem 1.2 and (10) imply the existence of a continuous $c_2\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$|\hat{y}_t|_{L^2(V^*)} \le c_2(|g|_{W_\Sigma}, |y_0|_H). \tag{11}$$

Combining (10), (11), Theorem 1.4 and the continuous embedding of W into $C([0,T];\mathbb{H})$ there exists a continuous function $K_2\colon \mathbb{R}^2 \to \mathbb{R}$ such that (6) is satisfied for $y = y_L + \hat{y}$. In addition y satisfies (5). $\square$

We have chosen to work in solenoidal spaces, mainly to reduce notational burden. Pressure can be reintroduced into (5) by standard methods [T, p.14 and p. 307]. In fact, from (5) we have

$$\nu(\nabla Y(t), \nabla v)_{L^2} = (b(t), v) \text{ for all } v \in V \text{ and } t \in [0,T], \tag{12}$$

where $b(t) = y_0 - y(t) - \int_0^t (y\cdot\nabla)y\, ds,\ Y(t) = \int_0^t y\, ds$. From (12) it follows that for every $t \in [0,T]$ there exists $P(t) \in L_0^2(\Omega)$ such that

$$\nu(\nabla Y(t), \nabla v)_{L^2} + (\nabla P(t) - b(t), v)_{L^2} = 0, \text{ for all } v \in H_0^1(\Omega).$$

Recall that $y = \tilde{g} + \bar{y} + \hat{y}$ with $\tilde{g} \in \mathbb{W}$ and $\bar{y}, \hat{y} \in W$. Utilizing Lemma 1.3 one argues that $t \to \nu(\nabla Y(t), \nabla\cdot) - (b(t),\cdot) \in W^{1,2}(H^{-1}(\Omega))$. Consequently $t \to \nabla P(t) \in W^{1,2}(H^{-1}(\Omega))$ and since grad is an isomorphism from $L_0^2(\Omega)$ onto its range, we have $P(t) \in W^{1,2}(L_0^2(\Omega))$. Setting $p(t) = \frac{d}{dt}P(t)$ we have

$$\begin{aligned}\langle y_t, v\rangle_{L^2(H^{-1}),L^2(H_0^1)} + \nu(\nabla y, \nabla v)_{L^2(L^2)} + \\ + \langle (y\cdot\nabla)y, v\rangle + \langle \nabla p, v\rangle_{L^2(H^{-1}),L^2(H_0^1)} = 0\end{aligned} \tag{13}$$

for all $v \in L^2(H_0^1(\Omega))$.

2 The Optimal Control Problem

Let $y_0 \in \mathbb{H}$ be given and define

$$W_\Sigma^{y_0} = \{g \in W_\Sigma\colon y_0 - \tilde{g}(0) \in H\}.$$

Let U be a Hilbert space with inner product $(\cdot,\cdot)_U$ and norm $|\cdot|_U$, and $B \in \mathcal{L}(U, W_\Sigma)$. Further U_0 denotes a closed linear subspace in U and u_0 an element in U, with the property that

$$\widetilde{Bu_0}(0) - y_0 \in H, \text{ and } \widetilde{Bu_1}(0) \in H \text{ for all } u_1 \in U_0. \tag{14}$$

As a consequence $B(\overline{U}) \subset W_\Sigma^{y_0}$, where $\overline{U} = u_0 + U_0$. We consider the optimal control problem

$$(P) \quad \begin{cases} \min J(y, y(T), u) = J_1(y) + J_2(y(T)) + J_3(u) \\ \langle y_t, v\rangle_{L^2(V^*),L^2(V)} + \nu(\nabla y, \nabla v)_{L^2(L^2)} + \langle (y\cdot\nabla)y, v\rangle = 0, \text{ for all } v \in L^2(V) \\ \tau\, y = Bu \text{ in } W_\Sigma, u \in \overline{U} \\ y(0) = y_0 \text{ in } \mathbb{H}, \end{cases}$$

where $J_1\colon L^2(\mathbb{V}) \to \mathbb{R}$, $J_2\colon \mathbb{H} \to \mathbb{R}$, $J_3\colon U \to \mathbb{R}$ are bounded from below, weakly lower semi–continuous, twice Fréchet differentiable with locally Lipschitzian second derivative, and J_3 is radially unbounded, i.e.

$$J_3(u) \to \infty \ \text{ for } \ |u|_U \to \infty.$$

Proposition 2.1. There exists a solution u^* to (P).

It will be convenient to postpone the proof until an appropriate equivalent formulation of (P) is derived.

From Theorem 1.4 and its proof we deduce that the solution to the dynamical systems constraint in (P) can be represented by

$$y = \hat{y} + y_L$$

where $\hat{y}$ is the solution to (7) and y_L the solution to (2) with $g = Bu$, $u \in \overline{U}$. Further y_L can be expressed as

$$y_L = \bar{y} + \widetilde{Bu}, \tag{15}$$

where $\bar{y}$ satisfies the Stokes problem (4) with $\tilde{g} = \widetilde{Bu}$. As a consequence

$$y = \tilde{y} + \widetilde{Bu}, \ \text{ with } \ \tilde{y} = \hat{y} + \bar{y}, \tag{16}$$

and $\tilde{y} \in W$ satisfies

$$\left\{\begin{array}{l} \langle \tilde{y}_t, v\rangle_{L^2(V^*),L^2(V)} + \nu(\nabla\tilde{y}, \nabla v)_{L^2(L^2)} + \int_0^T [c(\tilde{y},\tilde{y},v) + c(\tilde{y},\widetilde{Bu},v) \\ +c(\widetilde{Bu},\tilde{y},v) + c(\widetilde{Bu},\widetilde{Bu},v)]dt = \\ -\langle \widetilde{Bu}_t, v\rangle_{L^2(V^*),L^2(V)} - \nu(\nabla\widetilde{Bu},\nabla v)_{L^2(L^2)}, \text{for all } v \in L^2(V), \\ \tau\,\tilde{y} = 0 \quad \text{in} \quad W_\Sigma, \\ \tilde{y}(0) = y_0 - \widetilde{Bu}(0) \in H. \end{array}\right. \tag{17}$$

This suggests the following equivalent formulation of (P):

$$\left\{\begin{array}{l} \min J(\tilde{y} + \widetilde{Bu}, (\tilde{y} + \widetilde{Bu})(T), u) \\ e(\tilde{y}, u) = 0, \ u \in \overline{U}, \end{array}\right. \tag{$\tilde{P}$}$$

where $e\colon W \times \overline{U} \to L^2(V^*) \times H$, with

$$\begin{aligned} \langle e_1(\tilde{y},u), v\rangle_{L^2(V^*),L^2(V)} = &\\ &\langle \tilde{y}_t, v\rangle_{L^2(V^*),L^2(V)} + \nu(\nabla\tilde{y},\nabla v)_{L^2(L^2)} \\ &+ \int_0^T [c(\tilde{y},\tilde{y},v) + c(\tilde{y},\widetilde{Bu},v) + c(\widetilde{Bu},\tilde{y},v) + c(\widetilde{Bu},\widetilde{Bu},v)]dt \\ &+ \langle \widetilde{Bu}_t, v\rangle_{L^2(V^*),L^2(V)} + \nu(\nabla\widetilde{Bu},\nabla v)_{L^2(L^2)}, \end{aligned} \tag{18}$$

and

$$e_2(\tilde{y}) = \tilde{y}(0) - (y_0 - \widetilde{B}u(0)).$$

Proof of Proposition 2.1

Since J is bounded from below there exists a minimizing sequence $\{u_n\}_{n=1}^{\infty}$ in $\overline{U}$. By radial unboundedness of J_3 it follows that $\{u_n\}_{n=1}^{\infty}$ is bounded in U. Let $y_n = y(u_n)$ denote the solutions to the controlled Navier–Stokes equations with $u = u_n$. These can equivalently be expressed as

$$y_n = \tilde{y}_n + \widetilde{B}u_n,$$

where $\tilde{y}_n$ denotes the solution to $e(\tilde{y}_n, u_n) = 0$ with e defined in (18). Due to Theorem 1.4 with $g = Bu_n$ we have that

$$\{|(y_n)_t|_{L^2(V^*)} + |y_n|_{L^2(\mathbb{V})} + |y_n(T)|_{\mathbb{H}}\}_{n=1}^{\infty} \quad \text{is bounded.} \tag{19}$$

Since $B \in \mathcal{L}(U, W_\Sigma)$ it follows that

$$\{|(\tilde{y}_n)_t|_{L^2(V^*)} + |\tilde{y}_n|_{L^2(V)} + |\tilde{y}_n(T)|_H\}_{n=1}^{\infty} \quad \text{is bounded,} \tag{20}$$

as well. Consequently there exist $u^* \in U$ and $\tilde{y}^* \in W$, and subsequences of $\{u_n\}$ and $\{\tilde{y}_n\}$ such that, suppressing subindex notation

$$\begin{cases} u_n \rightharpoonup u^* \text{ in } U, \tilde{y}_n \rightharpoonup \tilde{y}^* \text{ in } W, \tilde{y}_n(T) \rightharpoonup \tilde{y}^*(T) \text{ in } H, \\ \text{and } \tilde{y}_n(0) \rightharpoonup \tilde{y}(0) \text{ in } H. \end{cases} \tag{21}$$

To argue weak convergence of $\tilde{y}_n(T)$ and $\tilde{y}_n(0)$ to $\tilde{y}^*(T)$ and $\tilde{y}(0)$ respectively, recall that W is continuously embedded into $C([0,T]; H)$. Hence $v \mapsto v(t)$ from W to H is a bounded linear mapping for every $t \in [0,T]$ and the claimed weak convergence follows. Note that (21) also implies that

$$\tilde{y}^*_{n,t} \rightharpoonup y^*_t \text{ in } L^2(V^*) \text{ and } \tilde{y}_n \to \tilde{y}^* \text{ in } L^2(H). \tag{22}$$

Here we use the fact the V is injected compactly in H, and that consequently W is compact in $L^2(H)$, ([CF], pg. 69). Due to the fact that $B \in \mathcal{L}(U, W_\Sigma)$ and by construction of the extension we have

$$\widetilde{B}u_n \rightharpoonup \widetilde{Bu}^* \text{ in } \mathbb{W}. \tag{23}$$

This implies that

$$(\widetilde{B}u_n)_t \rightharpoonup (\widetilde{Bu}^*)_t \text{ in } L^2(V^*) \text{ and } \widetilde{B}u_n \to \widetilde{Bu}^* \text{ in } L^2(\mathbb{H}), \tag{24}$$

where compactness of $\mathbb{V}$ in $\mathbb{H}$ was used. Moreover $u \to \widetilde{B}u(t)$ is a continuous linear mapping from U to $\mathbb{H}$ for every $t \in [0,T]$. Consequently (on a further subsequence)

$$\widetilde{B}u_n(0) \rightharpoonup \widetilde{Bu}^*(0) \text{ in } \mathbb{H} \text{ and } \widetilde{B}u_n(T) \rightharpoonup \widetilde{Bu}^*(T) \text{ in } \mathbb{H}. \tag{25}$$

To argue that $y^* = \tilde{y}^* + \widetilde{Bu}^*$ is feasible for (P) note first that $u^* \in \overline{U}$ since $\overline{U}$ is a closed affine subspace. Then consider

$$\langle e_1(\tilde{y}_n, u_n), v\rangle_{L^2(V^*),L^2(V)} = 0, \quad \text{for all } v \in L^2(V), \tag{26}$$

with e_1 defined in (18). Utilizing (21), (22), (24), (25) we can pass to the limit for $v \in W^{1,2}(\mathcal{D}_{\text{div}})$, where $\mathcal{D}_{\text{div}} = \{v \in \mathcal{D}(\Omega)\colon \text{div } v = 0\}$. It follows that $\tilde{y}^*$ satisfies $e_1(y^*, u^*) = 0$ in $L^2(V^*)$ and consequently $y^* = y(u^*)$ is the solution to the Navier–Stokes equations. To finally pass to the limit in the cost functional J observe that $\{\tilde{y}_n^*\}$ converges weakly to y^* in $L^2(\mathbb{V})$. Combined with (21), (23), (25) and weak lower semi–continuity of norms we have

$$J(y(u^*), y(T; u^*), u^*) \le \liminf_{n\to\infty} J(y(u_n), y(T; u_n), u_n).$$

This completes the proof. □
We close this section with three examples for the choices of U_0 and U which fit into the framework of this section.

Example 2.2. Here were argue that $U = L^2(H^1_n(\partial\Omega)^2) \cap W^{1,2}(L^2(\partial\Omega)^2)$, endowed with the canonical inner product, where $H^1_n(\partial\Omega)^2 = \{g \in H^1(\partial\Omega)^2\colon \int_{\partial\Omega} g\, n\; dx = 0\}$, and $U_0 = \{g \in U\colon g(0,\cdot)n = 0 \text{ on } \partial\Omega\}$ satisfy the above assumptions on U and U_0. For this purpose let $g \in U$ and define $y(g(0))$ as the solution in $\mathbb{V}$ to

$$\begin{cases} -\Delta y + \nabla p = 0 & \text{in} \quad \Omega \\ \text{div } y = 0 & \text{in} \quad \Omega \\ y = g(0) & \text{on} \quad \partial\Omega. \end{cases} \tag{27}$$

Further define $\hat{g}$ as the (variational) solution to

$$\begin{cases} y_t - \nu\Delta y + \nabla p = 0 & \text{in} \quad Q \\ \text{div } \; y = 0 & \text{in} \quad Q \\ \tau\, y = g & \text{on} \quad \Sigma \\ y(0) = y(g(0)) & \text{in} \quad \Omega, \end{cases} \tag{28}$$

which satisfies $y \in \mathbb{W}$ and

$$|\hat{g}|_{\mathbb{W}} \le C|g|_U,$$

for a constant C independent of $g \in U$. Consequently, $|g|_{W_\Sigma} \le C|g|_U$, and if we choose B as the injection operator from U into W_Σ, we have $B \in \mathcal{L}(U, W_\Sigma)$. Let us assume that $y_0 \in \mathbb{V}$. Define $g_0 \in U$ by $g_0(t) = y_{0|\partial\Omega}$ for all $t \in [0,T]$ and note that $\tilde{g}_0(0) - y_0 \in H$. Moreover $\tilde{g}(0) \in H$ for every $g \in U_0$. Hence (14) holds. This choice of U corresponds to the one in [GM]. □

Example 2.3. Let $U = \{g = g_1(t)\, g_2(x)\colon g_1 \in W^{1,2}(\mathbb{R}^2),\; g_2 \in H^{1/2}_n(\partial\Omega)\}$ endowed with the canonical inner product, and $U_0 = \{g \in U\colon g_1(0) = 0\}$. Let $y_0 \in \mathbb{V}$ and define $g_0(t) = y_{0|\partial\Omega} \in U$. Determine $\tilde{g}_0$ according to (27)–(28). Setting $u_0 = g_0$, and B = injection from U into W_Σ, then with this choice of u_0, U_0 and B the requirements (14) hold. Concerning the numerical realization of the $|g|_{H^{1/2}_n(\partial\Omega)}$– norm, note that $|\nabla y(g)|_{L^2(\Omega)^2} + |g|_{L^2(\Omega)^2}$ is an equivalent norm on $H^{1/2}_n(\partial\Omega)$, where y denotes the solution to (27) with $g(0) = g$. □

Example 2.4. Let τ denote the unit counterclockwise tangent to Ω and set $U = \{g = g_n n + g_\tau \tau \colon g_n \in L^2(H_n^{1/2}(\partial\Omega) \cap H^{3/4}(H^{-1}(\partial\Omega)),\ g_\tau \in L^2(H^{1/2}(\partial\Omega)) \cap H^{1/4}(L^2(\partial\Omega))\}$, endowed with the canonical inner product. The trace operator $g_n \to g_n|_{t=0}$ is continuous from $L^2(H^{1/2}(\partial\Omega)) \cap H^{3/4}(H^{-1/2}(\partial\Omega))$ to $H^{-1/2}(\partial\Omega)$. Hence the set $U_0 = \{g \in U \colon g_n(0) = 0\}$ is well defined. For every $g \in U$ there exists $\tilde{g} \in \mathbb{W}$ such that $|\tilde{g}|_{\mathbb{W}} \leq C|g|_U$ for a constant C independent of $g \in U$, see ([FGH], Theorem 4.2). Let $y_0 \in \mathbb{V}$ and define $g_0 \in U$ by $g_0(t) = y_{0_{|\partial\Omega}}$ for all $t \in [0,T]$. It follows that $\tilde{g}_0(0) - y_0 \in H$ and $\tilde{g}(0) \in H$, for all $g \in U_0$. Hence (14) holds with $u_0 = g_0$, and B the injection of U into W_Σ. - To numerically realize the norm $|g|_U$, for $g \in U_0$, note that the trace τ onto the lateral boundary Σ satisfies $\tau \in \mathcal{L}(\mathbb{W}, U)$, see ([FGH], §4.1). Hence the set

$$\{y(g) \colon g \in U_0, y(g) \text{ solution to (28) with } y(0) = 0\}$$

is a closed subset in $\mathbb{W}$. Consequently $|y(g)|_{\mathbb{W}}$ defines an equivalent norm on U_0, by the closed range theorem.

3 Derivatives

The analysis as well as the numerical realization of the optimal control solution u^* of (P) relies on derivative information of J and e. In this section we therefore focus on the characterization of the derivatives of these mappings. We shall, at times, denote $(\tilde{y}, u) \in W \times \overline{U}$ by x.

Proposition 3.1. The mapping $e = (e_1, e_2) \colon W \times \overline{U} \to L^2(V^*) \times H$ is infinitely often continuously Fréchet differentiable with Lipschitz continuous second derivative. The action of the first two derivatives of e^1 at $(y, u) \in W \times \overline{U}$ in direction $(w, s) \in W \times U_0$ and directions $((w,s),(v,r)) \in (W \times U_0)^2$, respectively, are given by

$$\begin{aligned}\langle e^1_x(\tilde{y}, u)(w,s), \phi\rangle_{L^2(V^*),L^2(V)} &= \langle w_t + \widetilde{B}s_t, \phi\rangle_{L^2(V^*),L^2(V)} + \nu(\nabla w + \nabla\widetilde{B}s, \nabla\phi)_{L^2(L^2)} \\ &+ \int_0^T [c(\tilde{y}, w, \phi) + c(w, \tilde{y}, \phi)]ds + \int_0^T [c(w, \widetilde{B}u, \phi) + c(\tilde{y}, \widetilde{B}s, \phi) \\ &\qquad + c(\widetilde{B}s, \tilde{y}, \phi) + c(\widetilde{B}u, w, \phi) + c(\widetilde{B}s, \widetilde{B}u, \phi) + c(\widetilde{B}u, \widetilde{B}s, \phi)]ds,\end{aligned}$$

$$\begin{aligned}\langle e^1_{xx}(\tilde{y}, u)((w,s),(v,r)), \phi\rangle_{L^2(V^*),L^2(V)} &= \int_0^T [c(v, w, \phi) + c(w, v, \phi)]ds \\ &+ \int_0^T [c(w, \widetilde{B}r, \phi) + c(v, \widetilde{B}s, \phi) + c(\widetilde{B}s, v, \phi) + c(\widetilde{B}r, w, \phi) \\ &\qquad + c(\widetilde{B}s, \widetilde{B}r, \phi) + c(\widetilde{B}r, \widetilde{B}s, \phi)]ds,\end{aligned}$$

for all $\phi \in W$. The action of the first derivative of e^2 at $(\tilde{y}, u) \in W \times \overline{U}$ in direction $(w,s) \in W \times U_0$ is given by

$$e^2_x(\tilde{y}, u)(w,s) = w(0) - \widetilde{B}s(0).$$

Moreover, $e^2_{xx} \equiv 0$ and all derivatives of e of order greater than or equal to three vanish.

Proof. Since e_2 is linear, we only consider e_1. To analyze the differentiability of e_1 we consider the eight additive terms on the right hand side of (18). The first three terms are identical to those considered in [HK], Proposition 3.1, and the last two are linear. Thus it suffices to consider terms four through six on the right hand side of (18). We shall argue that

$$G_1 \colon W \times U \to L^2(V^*) \text{ given by } G_1(\tilde{y}, u) = c(\tilde{y}, \widetilde{Bu}, \cdot)$$

is a continuous bilinear mapping. This implies that G_1 is twice continuously Fréchet differentiable and the second derivative is clearly Lipschitz continuous since it is independent of $(\tilde{y}, u)$. Bilinearity of G_1 is evident. Thus let $\tilde{y}_1, \tilde{y}_2 \in W$ and $u_1, u_2 \in U$. Then by Lemma 1.3 (ii)

$$|G(\tilde{y}, u_1) - G(\tilde{y}_2, u_2)|^2_{L^2(V^*)} =$$

$$= \sup_{|\phi|_{L^2(V)=1}} \int_0^T |c(\tilde{y}_1(t) - y_2(t), v(t), \widetilde{Bu}_1(t) + c(\tilde{y}_2(t), v(t), \widetilde{Bu}_1(t) - \widetilde{Bu}_2(t))| dt$$

$$\leq \sqrt{2} \sup_{|\phi|_{L^2(V)=1}} \int_0^T (|\tilde{y}_1(t) - \tilde{y}_2(t)|^{1/2} |\nabla(\tilde{y}_1(t) - \tilde{y}_2(t)|^{1/2} |\widetilde{Bu}_1(t)|^{1/2} |\nabla \widetilde{Bu}_1(t)|^{1/2} |\nabla \phi(t)|$$

$$+ |\tilde{y}_2(t)|^{1/2} \nabla \tilde{y}_2(t)|^{1/2} |\widetilde{Bu}_1(t) - \widetilde{Bu}_2(t)|^{1/2} |\nabla(\widetilde{Bu}_1(t) - \widetilde{Bu}_2(t))|^{1/2} |\nabla \phi(t)|) dt$$

$$\leq \sqrt{2} \int_0^T (|\tilde{y}_1(t) - \tilde{y}_2(t)| \, |\nabla \tilde{y}_1(t) - \nabla \tilde{y}_2(t)| \, |\widetilde{Bu}_1(t)| \, |\nabla \widetilde{Bu}_1(t)| +$$

$$|\tilde{y}_2(t)| \, |\nabla \tilde{y}_2(t)| \, |\widetilde{Bu}_1(t) - \widetilde{Bu}_2(t)| \, |\nabla \widetilde{Bu}_1(t) - \nabla \widetilde{Bu}_2(t)|) dt$$

$$\leq \sqrt{2} C (|\tilde{y}_1 - \tilde{y}_2|_W |u_1|_U + |u_1 - u_2|_U |\tilde{y}_2|_W)(|\tilde{y}_1 - \tilde{y}_2|_{L^2(V)} |u_1|_U + |\tilde{y}_2|_{L^2(V)} |u_1 - u_2|_U),$$

where C depends on the embedding constants of W into $C(H)$ and $\mathbb{W}$ into $C(\mathbb{H})$ and $\|B\|_{\mathcal{L}(U, W_\Sigma)}$. The desired continuity of G_1 follows from this estimate. Differentiability of the terms $\int c(\widetilde{Bu}, \tilde{y}, v) dt$ and $\int_0^T c(\widetilde{Bu}, \widetilde{Bu}, v) dt$ with respect to $\tilde{y}$ and u can be verified analogously. $\square$

Remark 3.2. Note that under the assumptions of Proposition 3.1, e^1_x can be expressed as

$$\langle e^1_x(\tilde y,u)(w,s),\phi\rangle_{L^2(V^*),L^2(V)} = \langle w_t + (\widetilde{Bs})_t, \phi\rangle_{L^2(V^*),L^2(V)}$$
$$+\nu(\nabla w+\nabla\widetilde{Bs},\nabla\phi)_{L^2(L^2)}+\int_0^T [c(y,w+\widetilde{Bs},\phi)+c(w+\widetilde{Bs},y,\phi)]dt, \quad \text{where} \quad y=\tilde y+\widetilde{Bu}.$$

Similarly, e^1_{xx} can be expressed as

$$\langle e^1_{xx}(\tilde y,u)(w,s)(v,r),\phi\rangle_{L^2(V^*),L^2(V)} =$$
$$\int_0^T [c(v+\widetilde{Br},w+\widetilde{Bs},\phi)+c(w+\widetilde{Bs},v+\widetilde{Br},\phi)]dt.$$

Subsequently we shall repeatedly utilize the linearized equations and their adjoints

$$e_{\tilde y}(\tilde y,u)v=(f,v_0) \quad \text{in} \quad L^2(V^*)\times H \quad \text{for} \quad v\in W \tag{29}$$

and

$$e^*_{\tilde y}(\tilde y,u)w=g \quad \text{in} \quad W^* \quad \text{for} \quad w\in L^2(V)\times H. \tag{30}$$

It will be convenient to express (29), (30) in variational form. For (29) we obtain

$$\begin{cases} \langle v_t+(v\cdot\nabla)y+(y\cdot\nabla)v-\nu\Delta v,\phi\rangle_{L^2(V^*),L^2(V)} = \\ \langle f,\phi\rangle_{L^2(V^*),L^2(V)},\ \phi\in L^2(V), \\ v(0)=v_0, \end{cases} \tag{31}$$

where $y=\tilde y+\widetilde{Bu}$. Moreover if $g\in W^*$ has the special form

$$\langle g,\phi\rangle_{W^*,W}=\langle g_1,\phi\rangle_{L^2(V^*),L^2(V)}+(g_0,\phi(T))_H \quad \text{for} \quad \phi\in W,$$

with $g_1\in L^2(V^*)$, $g_0\in H$, then (30) can be expressed as

$$\begin{cases} \langle -w_t+(\nabla y)^t w-(y\cdot\nabla)w-\nu\Delta w,\phi\rangle_{W^*,W}=\langle g_1,\phi\rangle_{W^*,W},\ \phi\in W, \\ w(T)=g_0 \quad \text{in} \quad H. \end{cases} \tag{32}$$

In the following proposition we use the notation

$$H^{2,1}(Q)=\{v\in L^2(V\cap H^2(\Omega)^2)\colon v_t\in L^2(H)\},$$

equipped with the norm

$$|v|^2_{H^{2,1}(Q)}\colon=|v|^2_{L^2(H^2(\Omega)^2)}+|v_t|^2_{L^2(H)}.$$

Proposition 3.3. Let $y \in \mathbb{W}$, $v_0 \in H$, $f \in L^2(V^*)$, $g_1 \in L^2(V^*)$ and $g_0 \in H$. Then (31) admits a unique variational solution $v \in W$, and (32) has a unique variational solution $w \in L^2(V)$ with $w_t \in L^{\frac{4}{3}}(V^*) \cap W^*$, $w \in C(H)$, and the first equation in (32) holding in $L^{4/3}(V^*) \cap W^*$. Moreover the following estimates hold:

$$\begin{aligned} (i) \quad & |v_t|_{L^2(V^*)} + |v|_{L^2(V)} + |v|_{L^\infty(H)} \le C(|y|_{L^2(V)}, |y|_{L^\infty(H)})\{|f|_{L^2(V^*)} + |v_0|_H\}, \\ (ii) \quad & |w|_{L^2(V)} + |w_t|_{L^{4/3}(V^*)} \le C(|y|_{L^2(V)}, |y|_{L^\infty(H)})\{|g_1|_{L^2(V^*)} + |g_0|_H\}. \end{aligned}$$

If in addition $y \in L^\infty(\mathbb{V})$, then $w \in W$and

(iii) $\quad |w|_{L^2(V)} + |w_t|_{L^2(V^*)} \le C(|y|_{L^\infty(\mathbb{V})})\{|g_1|_{L^2(V^*)} + |g_0|_H\}$.

For $\partial\Omega \subset C^2$, $y \in \mathbb{W} \cap L^\infty(\mathbb{V}) \cap L^2(H^2(\Omega)^2)$, $v_0 \in V$, $g_0 \in V$, and $f \in L^2(H)$, $g_1 \in L^2(H)$ the unique solutions v and w of (31), (32) are elements of $H^{2,1}(Q)$ and satisfy the a–priori estimates

(iv) $\quad |v_t|_{H^{2,1}(Q)} \le C(|y|_{L^\infty(V)}, |y|_{L^2(H^2(\Omega)^2)})\{|f|_{L^2(H)} + |v_0|_H\}$,

(v) $\quad |w|_{H^{2,1}(Q)} \le C(|y|_{L^\infty(V)}, |y|_{L^2(H^2(\Omega)^2)})\{|g_1|_{L^2(H)} + |g_0|_V\}$.

If, moreover, $f \in L^\infty(H)$, then $v \in L^\infty(H^2(\Omega)^2)$. Throughout the constants depend continuously on their arguments and are independent of (f, v_0), (g_1, g_0).

Proof. The proof is analogous to that of Proposition 3.3 in [HK]. It only differs in that y satisfies homogenous boundary conditions in [HK], and inhomogeneous boundary conditions in the present paper. As a result the estimates of the trilinear form $c(\cdot,\cdot,\cdot,)$ are slightly different. Based on Lemma 1.3 one can follow the proof in [HK] to obtain (i)–(v). Let us turn to verify $v \in L^\infty(V \cap H^2(\Omega)^2)$ if $f \in L^\infty(H)$. We utilize Δv as a test function in (31) and obtain

$$\frac{1}{2}\frac{d}{dt}|v|_V^2 + \nu|\Delta v|_{L^2}^2 \ \le |c(v, y, \Delta v)| + |c(y, v, \Delta v)| + |f|_{L^2}|\Delta v|_{L^2},$$

all quantities being evaluated at $t \in (0, T)$. By Lemma 1.3 (vi) and (iv) there exists a constant C_ν depending on ν (and Ω) such that

$$\frac{1}{2}\frac{d}{dt}|v|_V^2 + \nu|\Delta v|_{L^2}^2 \le C_\nu(|v|_H|v|_{H^2}|y|_{\mathbb{V}}^2 + |y|_{\mathbb{H}}|y|_{\mathbb{V}}|v|_V|v|_{H^2} + |f|_H + \frac{\nu}{4}|\Delta v|_{L^2}^2.$$

Since $|w|_{H^2} \le C|\Delta w|_{L^2}$ for a constant C independent of $w \in V \cap H^2(\Omega)^2$, it follows that

$$\frac{1}{2}\frac{d}{dt}|v|_V^2 + \nu|\Delta v|_{L^2}^2 \le \widetilde{C}_\nu(|v|_H^2|y|_{\mathbb{V}}^4 + |y|_{\mathbb{H}}^2|y|_{\mathbb{V}}^2|v|_V^2 + |f|_H + \frac{\nu}{2}|\Delta v|_{L^2}^2$$

for a constant $\widetilde{C}_\nu$ depending on ν. Consequently

$$\frac{d}{dt}|v(t)|_V^2 + \nu|\Delta v(t)|_{L^2}^2 \le 2\widetilde{C}_\nu(|v(t)|_H^2|y(t)|_{\mathbb{V}}^4 + |y(t)|_{\mathbb{H}}^2|y(t)|_{\mathbb{V}}^2|v(t)|_V^2 + |f(t)|_H)$$

for a.e. $t \in (0,T)$. From (iv) it is known that $v \in L^\infty(V)$. By assumption $y \in L^\infty(\mathbb{V})$ and $f \in L^\infty(H)$. Consequently
$\Delta v \in L^\infty(L^2(\Omega)^2)$ and $v \in L^\infty(H^2(\Omega)^2)$. $\Box$

We now turn to differentiability and representation of the derivative of the cost–functional in (P) with respect to u. For this purpose we introduce

$$\hat{J}(u) = J(\tilde{y} + \widetilde{Bu}, (\tilde{y} + \widetilde{Bu})(T), u), \tag{33}$$

where $e(\tilde{y}, u) = 0$. As a consequence of Proposition 3.3 the mapping $e_{\tilde{y}}(\tilde{y}, u) \colon W \to L^2(V^*) \times H$ is a homeomorphism. Hence by Proposition 3.1 and the implicit function theorem the first derivative $\tilde{y}'(u)\delta u$ at $u \in \overline{U}$ in direction $\delta u \in U_0$ of the mapping

$$u \to \tilde{y}(u)$$

from $\overline{U}$ to W, can be expressed as

$$\tilde{y}'(u)\delta u = -e_{\tilde{y}}^{-1}(\tilde{y}, u)\; e_u(\tilde{y}, u)\delta u \in W. \tag{34}$$

A a consequence of the implicit function theorem and Proposition 3.1 the mapping $u \mapsto \tilde{y}(u)$ is twice continuously differentiable. Applying the chain rule we obtain

$$\begin{aligned}\langle \hat{J}'(u), \delta u\rangle_U = {} & \langle J_1'(y), \tilde{y}'(u)\delta u\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_2'(y(T)), \tilde{y}'(u)\delta u(T)\rangle_{\mathbb{H}} \\ & + \langle J_3'(u), \delta u\rangle_U + \langle J_1'(y), \widetilde{B\delta u}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_2'(y(T)), \widetilde{B\delta u}(T)\rangle_{\mathbb{H}}.\end{aligned}$$

Note that H and $L^2(V)$ are closed subspaces of $\mathbb{H}$ and $L^2(\mathbb{V})$, respectively, and that $\tilde{y}'(u)\delta u \in L^2(V)$, $\tilde{y}'(u)\delta u(T) \in H$. Hence the restrictions of $J_1'(y)$ and $J_1'(y(T))$ to $L^2(V)$ and H define unique elements in $L^2(V^*)$ and H. We shall denote these restrictions by the same symbol. Note then that $\hat{J}(u)$ can equivalently be expressed as

$$\begin{aligned}\langle \hat{J}'(u), \delta u\rangle_U = {} & \langle J_1'(y), \tilde{y}'(u)\delta u\rangle_{L^2(V^*),L^2(V)} + \langle J_2'(y(T)), \tilde{y}'(u)\delta u(T)\rangle_H \\ & + \langle J_3'(u), \delta u\rangle_U + \langle J_1'(y), \widetilde{B\delta u}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_2'(y(T)), \widetilde{B\delta u}(T)\rangle_{\mathbb{H}}.\end{aligned}$$

For convenience we recall that the domains and ranges of e_u^* and $e_{\tilde{y}}^*$ are

$$e_u^*(x) \colon L^2(V) \times H \to U \quad \text{and} \quad e_{\tilde{y}}^{-*}(x) \colon W^* \to L^2(V) \times H.$$

From (34) we have

$$\begin{aligned}\langle \hat{J}'(u), \delta u\rangle_U = {} & \langle e_u^*(x) e_y^{-*}(x) J_1'(y), \delta u\rangle_U \\ & - \langle (e_u^*(x) e_y^{-*}(x)) J_2'(y(T)), \delta u\rangle_U + \langle J_3'(u), \delta u\rangle_U \\ & + \langle J_1'(y), \widetilde{B\delta u}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_2'(y(T)), \widetilde{B\delta u}(T)\rangle_{\mathbb{H}}.\end{aligned}$$

Note that $\Phi_1(y) = J_1'(y) + J_2'(y(T))$ defines an element in W^* via

$$\langle \Phi_1(y), \phi\rangle_{W^*,W} = \langle J_1'(y), \phi\rangle_{L^2(V^*),L^2(V)} + \langle J_2'(y(T)), \phi(T)\rangle_H, \quad \text{for} \;\; \phi \in W.$$

Here $J_1'(y)$ and $J_2'(y(T))$ denote the restrictions to $L^2(V)$ and H as introduced above. Introducing $\lambda = -e_{\tilde{y}}^{-*}(x)\Phi_1(y) \in L^2(V) \times H$ we have

$$
(35) \qquad \begin{cases} \langle \hat{J}'(u), \delta u\rangle_U & = \langle e_u^*(x)\lambda, \delta u\rangle_U + \langle \hat{J}_1'(y), \widetilde{B}\delta u\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} \\ & +\langle J_2'(y(T)), \widetilde{B}\delta u(T)\rangle_{\mathbb{H}} + \langle J_3'(u), \delta u\rangle_U. \end{cases}
$$

Now we turn to λ and note that $e_{\tilde{y}}^*(x)\lambda = -\Phi_1(y)$ in W^* implies that $\lambda = (\lambda_1, \lambda_0) \in L^2(V) \times H$ satisfies

$$
\begin{aligned} \langle \lambda_1, e_{1,\tilde{y}}(x)\phi\rangle_{L^2(V),L^2(V^*)} + \langle \lambda_0, e_{2,\tilde{y}}(x), \phi\rangle_H = & -\langle J_1'(y), \phi\rangle_{L^2(V^*),L^2(V)} \\ & - \langle J_2'(y(T)), \phi\rangle_H \ \text{ for all } \ \phi \in W, \end{aligned}
$$

or alternatively

$$
(36) \qquad \begin{cases} \langle \lambda_1, \phi_t\rangle_{L^2(V),L^2(V^*)} + \nu(\nabla\lambda_1, \nabla\phi)_{L^2(L^2)} + \int_0^T [c(y, \phi, \lambda_1) \\ +c(\phi, y, \lambda_1)]dt + \langle \lambda_0, \phi(0)\rangle_H = \\ -\langle J_1'(y), \phi\rangle_{L^2(V^*),L^2(V)} - \langle J_2'(y(T)), \phi(T)\rangle_H. \end{cases}
$$

Thus we can argue, utilizing Proposition 3.3, that $\lambda_1 \in W$, $\lambda_1(0) = \lambda_0$, and that λ_1 is the variational solution to

$$
(37) \qquad \begin{cases} \langle -\lambda_{1,t} - \nu\Delta\lambda_1 - (y\cdot\nabla)\lambda_1 + (\nabla y)^T\lambda_1, \phi\rangle_{L^2(V^*),L^2(V)} = \\ -\langle J_1'(y), \phi\rangle_{L^2(V^*),L^2(V)},\ \phi \in L^2(V), \\ \lambda_1(T) = -J_2'(y(T)), \end{cases}
$$

where the first equation is satisfied in $L^2(V^*)$. Here $J_1'(y)$ and $J_2'(y(T))$ are still considered as restrictions to $L^2(V)$ and H respectively.

Our next objective is to obtain a Riesz representation of $\hat{J}'(u)$ from (35). For this purpose we shall introduce the dual pressure μ to (37). We henceforth assume

$$
(H1) \qquad \begin{cases} \partial\Omega \in C^2, y \in \mathbb{W} \cap L^\infty(\mathbb{V}) \cap L^2(H^2(\Omega)^2), \\ J_1'(y) \in L^2(L^2(\Omega)^2) \ \text{ and } \ J_2'(y(T)) \in V. \end{cases}
$$

Now we also decompose the action of $J_1'(y)$ on $L^2(H)$ and its orthogonal complement. For this purpose let $P\colon L^2(\Omega)^2 \to H$ denote the orthogonal projection (Leray projection, c.f. [CF], pg 9). It induces in a canonical manner a projection $\mathcal{P}\colon L^2(L^2(\Omega)^2) \to L^2(H)$. We decompose $J_1'(y) \in L^2(L^2(\Omega)^2)$ as

$$
J_1'(y) = \mathcal{P}J_1'(y) + (I - \mathcal{P})J_1'(y).
$$

Observe that λ_1 is the solution to (37) with $J_1'(y)$ replaced by $\mathcal{P}J_1'(y)$. From Proposition 3.3 and (H 1) we have $\lambda_1 \in H^{2,1}(Q)$. Let

$$
\gamma(t) = -\lambda_{1,t} - \nu\Delta\lambda_1 - (y\cdot\nabla)\lambda_1 + (\nabla y)^T\lambda_1 + \mathcal{P}J_1'(y) + (I - \mathcal{P})J_1'(y),
$$

and note that $\gamma \in L^2(L^2(\Omega)^2)$, and $\langle \gamma(t), v\rangle_{L^2(\Omega)} = 0$, for every $v \in V$. Hence there exists μ such that $\nabla\mu(t) = \gamma(t)$ for every $t \in [0,T]$ and $\mu \in L^2(H^1(\Omega) \cap L^2_0(\Omega))$, ([T], Chapter I). Thus

$$
\left\{
\begin{array}{l}
-\lambda_{1,t} - \nu\Delta\lambda_1 - (y\cdot\nabla)\lambda_1 + (\nabla y)^T\lambda_1 + \nabla\mu(t) = -J_1'(y) \\
\mathrm{div}\ \ \lambda_1(t) = 0 \\
\lambda_1(T) = -J_2'(y(T)),
\end{array}
\right.
\tag{38}
$$

with the first equation now holding in $L^2(L^2(\Omega)^2)$. We thus obtain with Proposition 3.1 and (38)

$$
\begin{aligned}
&- \langle e_u^* e_{\tilde{y}}^* \Phi_1, \delta u\rangle_U = \langle e_u^*(x)\lambda, \delta u\rangle_U = \langle \lambda_1, e_{1,u}(x)\delta u\rangle_{L^2(V),L^2(V^*)} + \langle \lambda_1(0), e_{2,u}(x)\delta u\rangle_H \\
&= \langle \lambda_1, (\widetilde{B\delta u})_t\rangle_{L^2(V),L^2(V^*)} + \nu(\nabla\lambda_1, \nabla\widetilde{B\delta u})_{L^2(L^2)} \\
&+ \int_0^T \left[c(y, \widetilde{B\delta u}, \lambda_1) + c(\widetilde{B\delta u}, y, \lambda_1)\right] dt + \langle \lambda_1(0), \widetilde{B\delta u}(0)\rangle_H \\
&= -(\lambda_{1,t}, \widetilde{B\delta u})_{L^2(L^2)} - \nu(\Delta\lambda_1, \widetilde{B\delta u})_{L^2(L^2)} + \nu\left(\frac{\partial\lambda_1}{\partial n}, B\delta u\right)_{L^2(\Sigma)} \\
&+ \int_0^T \left[-(y\nabla\lambda_1)\widetilde{B\delta u}) + (\nabla y)^T\lambda_1 \cdot \widetilde{B\delta u}\right] dt + \langle \lambda_1(T), \widetilde{B\delta u}(T)\rangle_H \\
&+ \langle \nabla\mu, \widetilde{B\delta u}\rangle_{L^2(L^2)} \\
&= -(J_1'(y), \widetilde{B\delta u})_{L^2(L^2)} + \nu\left(\frac{\partial\lambda_1}{\partial n}, B\delta u\right)_{L^2(\Sigma)} - \langle J_2'(y(T)), \widetilde{B\delta u}(T)\rangle_{L^2} \\
&- \langle \mu\, n, B\delta u\rangle_{L^2(\Sigma)}.
\end{aligned}
$$

Combining this last equality with (35) implies that

$$
\langle \hat{J}'(u), \delta u\rangle_U = \langle B^*\left(\nu\frac{\partial\lambda_1}{\partial n} - \mu\, n\right) + J_3'(u), \delta u\rangle_U
$$

for all $\delta u \in U$, where B^* denotes the adjoint of B considered as operator from U to $L^2(\Sigma)$. We summarize the above computations in

Theorem 3.4. The mapping $u \to \hat{J}(u) = J(\tilde{y} + \widetilde{Bu}, (\tilde{y} + \widetilde{Bu})(T), u)$ is Fréchet differentiable. If $(H1)$ holds, then its Riesz–representation is given by $\hat{J}'(u) = B^*(\nu\frac{\partial\lambda_1}{\partial n} - \mu\, n) + J_3'(u)$, where (λ_1, μ) is the solution in $H^{2,1}(Q) \times L^2(H^1(\Omega) \cap L^2_0(\Omega))$ to (38).

We turn to the characterization of the second derivative of $\hat{J}(u) = J(y, y(T), u)$, where $y = \tilde{y}(u) + \widetilde{Bu}$. Let $T(x) \colon U \to \mathbb{W} \times U$ be defined by

$$
T(x) = \begin{pmatrix} T_1(x) \\ T_2(x) \end{pmatrix} = \begin{pmatrix} \tilde{y}'(u)\delta u + \widetilde{B} \\ Id_U \end{pmatrix} = \begin{pmatrix} -e_{\tilde{y}}^{-1}(x)e_u(x) + \widetilde{B} \\ Id_U \end{pmatrix}
$$

where, as before, $x = (\tilde{y}, u)$ and $\tilde{y} = \tilde{y}(u)$. For $u \in \overline{U}$ and $\delta u, \delta v \in U_0$ we find

$$\langle \hat{J}'(u), \delta u \rangle_U = \\ \langle J_1'(y), T_1(x)\delta u \rangle_{L^2(\mathbb{V}^*), L^2(\mathbb{V})} + \langle J_2'(y(T)), T_1(x)\delta u(T) \rangle_{\mathbb{H}} + \langle J_3'(u), \delta u \rangle_U$$

and

$$\begin{aligned} \langle \hat{J}''(u)\delta u, \delta v \rangle_U = & \langle J_1''(y)T_1(x)\delta u, T_1(x)\delta v \rangle_{L^2(\mathbb{V}^*), L^2(\mathbb{V})} \\ & + \langle J_2''(y(T))(T_1(x)\delta u)(T), (T_1(x)\delta v)(T) \rangle_{\mathbb{H}} + \langle J_3''(u)\delta u, \delta v \rangle_U \\ & + \langle J_1'(y), \tilde{y}''(u)(\delta u, \delta v) \rangle_{L^2(V^*), L^2(V)} + \langle J_2'(y(T)), \tilde{y}''(u)(\delta u, \delta v)(T) \rangle_H. \end{aligned}$$

Note that $e_{\tilde{y}}(x)\tilde{y}'(u)\delta u + e_u(x)\delta u = 0$ and

$$\begin{aligned} & e_{\tilde{y}\tilde{y}}(x)(\tilde{y}'(u)\delta u, \tilde{y}'(u)\delta v) + e_{\tilde{y}u}(x)(\tilde{y}'(u)\delta u, \delta v) + \\ & + e_{u\tilde{y}}(\delta u, y'(u)\delta v) + e_{uu}(x)(\delta u, \delta v) + e_{\tilde{y}}(x)\tilde{y}''(u)(\delta u, \delta v) = 0. \end{aligned} \tag{39}$$

From the properties of J and e it follows that $u \to \hat{J}''$ is twice Fréchet differentiable and that the second derivative is locally Lipschitz continuous. It is more involved to find a simple representation for $\hat{J}''(u)$. Note that $e_{\tilde{y}}(x)$ has an obvious extension to $\mathbb{W}$ which we denote by $\bar{e}_{\tilde{y}}(x)$ and that

$$e_u(x) = \bar{e}_{\tilde{y}}(x)\widetilde{B}, \tag{40}$$

as operators from U_0 to $L^2(V^*) \times H$. In an analogous manner we have

$$e_{\tilde{y}u}(x) = \bar{e}_{\tilde{y}\tilde{y}}(x)(\cdot, \widetilde{B}\cdot) \quad \text{and} \quad e_{uu}(x) = \bar{e}_{\tilde{y}\tilde{y}}(x)(\widetilde{B\cdot}, \widetilde{B\cdot}).$$

Consequently (39) can be expressed as

$$\bar{e}_{\tilde{y}\tilde{y}}(x)(T_1(x)\delta u, T_1(x)\delta v) + e_{\tilde{y}}(x)\, \tilde{y}''(u)(\delta u, \delta v) = 0, \tag{41}$$

for all $(\delta u, \delta v) \in U_0 \times U_0$, with equality in $L^2(V^*) \times H$.
Inserting (40) into the expression for $\hat{J}''(u)$ gives

$$\begin{aligned} \langle \hat{J}''(u)\delta u, \delta v \rangle_U = & \langle J_1''(y)T_1(x)\delta u, T_1(x)\delta v \rangle_{L^2(\mathbb{V}^*), L^2(\mathbb{V})} \\ & + \langle J_2''(y(T))(T_1(x)\delta u)(T), (T_1(x)\delta v)(T) \rangle_{\mathbb{H}} + \langle J_3''(u)\delta u, \delta v \rangle_U \\ & - \langle \Phi_1(y), e_{\tilde{y}}^{-1}(x)\, \bar{e}_{\tilde{y}\tilde{y}}(x)(T_1(x)\delta u, T_1(x)\delta v) \rangle_{W^*, W}. \end{aligned}$$

Recalling that $e_{\tilde{y}}^{-*}(x)\Phi_1(y) = -\lambda \in L^2(V) \times H$ we have

$$\begin{aligned} - \langle \Phi_1(y), e_{\tilde{y}}^{-1}(x)\bar{e}_{\tilde{y}\tilde{y}}(x)(T_1(x)\delta u, T_1(x)\delta v) \rangle_{W^*, W} = \\ = \langle \lambda, \bar{e}_{\tilde{y}\tilde{y}}(x)(T_1(x)\delta u, T_1(x)\delta v) \rangle_{L^2(V)\times H, L^2(V^*)\times H}. \end{aligned}$$

Therefore

$$
(42)\qquad \hat{J}''(u) = T^*(x)\begin{pmatrix} J_1''(y) + J_2''(y(T)) + \langle \lambda, \bar{e}_{\tilde{y}\tilde{y}}(x)(\cdot,\cdot)\rangle & \vdots & 0 \\ \cdots\cdots\cdots & \vdots & \cdots \\ 0 & \vdots & J_3''(u) \end{pmatrix} T(x),
$$

where $(J_1''(y) + J_2''(y(T)) + \langle \lambda, \bar{e}_{\tilde{y}\tilde{y}}(x)(\cdot,\cdot)\rangle)\, T_1(x) \in \mathbb{W}^*$ stands for

$$
\begin{aligned}
\langle J_1''(y)T_1(x)\delta u, \cdot\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_2''(y(T))(T_1(x)\delta u)(T), \cdot\rangle_{\mathbb{H}} + \\
+ \langle \lambda, \bar{e}_{\tilde{y}\tilde{y}}(x)(T_1(x)\delta u, \cdot)\rangle_{L^2(V)\times H, L^2(V^*)\times H}.
\end{aligned}
$$

Note that

$$
(43)\qquad \begin{aligned}
&\langle \lambda, \bar{e}_{\tilde{y}\tilde{y}}(x)(T_1(x)\delta u, \cdot)\rangle_{L^2(V)\times H, L^2(V^*)\times H} = \\
&\int_0^T \left[c(T_1(x)\delta u, \cdot, \lambda^1) + c(\cdot, T_1(x)\delta u, \lambda^1)\right] dt.
\end{aligned}
$$

We next compute the action of $\hat{J}''(u)$ considered as bounded linear operator from $U_0 \subset U$ into U. For $\delta u \in U_0$ let $T_1(x)\delta u = g \in \mathbb{W}$, and note that by definition of $T_1(x)$

$$
\tau g = B\delta u \quad \text{and} \quad e_u(x)\delta u = e_{\tilde{y}}(x)(\widetilde{B\delta u} - g).
$$

From (40) we have $\bar{e}_{\tilde{y}}(x)g = 0$ in $L^2(V^*) \times H$. It follows that g is the unique solution in $\mathbb{W}$ to

$$
(44)\qquad \begin{cases}
\langle g_t, v\rangle_{L^2(V^*),L^2(V)} + \nu(\nabla g, \nabla v)_{L^2(L^2)} \\
\qquad\qquad + \displaystyle\int_0^T (c(y,g,v) + c(g,y,v))ds = 0, v \in L^2(V), \\
\tau g = B\delta u \ \text{ in } \ W_\Sigma \\
g(0,\cdot) = 0.
\end{cases}
$$

We shall also make use of the following lemma.

Lemma 3.5. Assume that $(H1)$ holds and let $\lambda_1 \in H^{2,1}(Q)$ and $g \in \mathbb{W}$ be as defined above. Then

$$
\nabla\lambda_1\, g \in L^2(L^2(\Omega)^2).
$$

Proof. Recall from [LU, p. 47], that there exists a constant C such that

$$
|\phi|_{L^4(\Omega)} \le C(|\phi|_{L^2(\Omega)} + |\nabla\phi|^{1/2}_{L^2(\Omega)}|\phi|^{1/2}_{L^2(\Omega)}) \ \text{ for all } \ \phi \in H^1(\Omega).
$$

Since $g \in \mathbb{W} \cap C([0,T]; L^2(\Omega)^2)$ it follows that $g \in L^4(L^4(\Omega)^2)$. Similarly, since $\lambda_1 \in H^{2,1}(Q) \cap C([0,T]; H^1(\Omega))$, we have $\nabla\lambda_1 \in L^4(L^4(\Omega)^4)$. Consequently $\nabla\,\lambda_1\, g \in L^2(L^2(\Omega)^2)$

as desired. $\square$

Observe that $\Phi_2(x) = J_1''(y)g + J_2''(y(T))g(T) + \langle\lambda, \bar{e}_{\tilde{y}\tilde{y}}(x)(g,\cdot)\rangle$ defines an element in W^* via

$$\langle\Phi_2(x), \phi\rangle_{W^*,W} = \langle J_1''(y)g, \phi\rangle_{L^2(V^*),L^2(V)} + \langle J_2''(y(T))g(T), \phi(T)\rangle_H \\ + \langle\lambda, \bar{e}_{\tilde{y}\tilde{y}}(x)(g,\phi)\rangle_{L^2(V)\times H, L^2(V^*)\times H}.$$

Let $w\colon = e_{\tilde{y}}^{-*}(x)\Phi_2(x) \in L^2(V)\times H$. We henceforth require the following additional assumptions:

$$(H2) \quad J_1''(y) \in \mathcal{L}(\mathbb{W}, L^2(L^2(\Omega)^2),\ J_2''(y(T)) \in \mathcal{L}(V)$$

as well as

$$(H3) \qquad J_1'(y) \in L^\infty(L^2(\Omega)^2).$$

Throughout the remainder of this section we assume that $(H1)-(H3)$ are satisfied. From the last assertion in Proposition 3.3 it then follows that $\lambda_1 \in L^\infty(H^2(\Omega)^2)$ and consequently $\lambda_1 \in L^\infty(L^\infty(\Omega))$ and $\nabla\lambda_1\, g \in L^2(L^2(\Omega)^2)$. Combining this fact with Lemma 3.5 and $(H2)$ it follows that

$$J_1''(y)g + (\nabla g)^T\lambda_1 - \nabla\lambda_1\, g \in L^2(L^2(\Omega)^2).$$

Then with the argument leading to (38) we have $w = (w_1, w_0)$ with $w_1 \in W$ and $w_1(0) = w_0$. We henceforth write w for w_1. Due to $(H2), (H3)$ and Proposition 3.3 moreover $w \in H^{2,1}(Q)$ and there exists $\chi \in L^2(H^1(\Omega)\cap L^2_0(\Omega))$ such that

$$\text{(45)} \qquad \begin{cases} -w_t - \nu\Delta w - (y\cdot\nabla)w & +(\nabla y)^T w + \nabla\chi = \\ & J_1''(y)(g) + (\nabla g)^T\lambda_1 - \nabla\lambda_1\, g \\ \tau\, w = 0 & \\ w(T) = J_2''(y(T)). & \end{cases}$$

With a computation analogous to the one above the statement of Theorem 3.4 we have

$$\text{(46)} \qquad \begin{aligned} T_1^*(x)\Phi_2(x) &= -e_u^*(x)e_{\tilde{y}}^{-*}(x)\Phi_2(x) + (\widetilde{B})^*\Phi_2(x) \\ &= -e_u^*(x)w + (\widetilde{B})^*\Phi_2(x) = B^*(\nu\tfrac{\partial w}{\partial n} - \chi n). \end{aligned}$$

We thus obtain

Theorem 3.6. The mapping $u \to \hat{J}''(u)$ is twice continuously Fréchet differentiable with locally Lipschitzian derivative. If $(H2)$ holds, then the action of $J''(u)$ at $\delta u \in U_0$ is given by

$$\hat{J}''(u)\delta u = B^*(\nu\frac{\partial w}{\partial n} - \chi n) + J_3''(u)\delta u,$$

where (w,χ) is the solution of the backwards linearized equations (45) which, in turn depends on g, solution to the forwards linearized equations (44).

4 Second Order Sufficient Optimality and Newton's Method

In this short section we give a sufficient condition for u^* to be a minimum for (P) and we formulate Newton's method in a form that is amenable for implementation. Throughout we assume that the generic regularity assumptions on J_i, $i = 1, 2, 3$, stated just below the formulation of (P) are satisfied. Then

$$u \to \hat{J}(u), u \in \bar{U} \subset U,$$

with $\hat{J}$ defined in (33) is twice continuously differentiable from $\bar{U} = u_0 + U_0 \subset U$ to $\mathbb{R}$. The conditions on the smallness of J_1' and J_2' in the following theorem are motivated by tracking–type cost functionals.

Theorem 4.1. Let u^* denote a solution to (P) and let $y^* = y(u^*)$ be the associated velocity field. If $J_1''(y^*)$ and $J_2''(y^*(T))$ are positive semi–definite, $J_3''(u^*)$ is uniformly positive definite, and $|J_1'(y^*)|_{L^2(V^*)}$ and $|J_2'(y^*(T))|_{\mathbb{H}}$ are sufficiently small, then there exists a neighborhood $N(u^*)$ of u^* in $\bar{U} \subset U$ and $\kappa > 0$ such that

$$\hat{J}''(u)(\delta u, \delta u) \geq \kappa \ |\delta u|_U^2 \ \text{ for all } \ \delta u \in U_0 \tag{47}$$

and $u \in N(u^*)$.

Proof. Sine $\hat{J}(u)$ is twice continuously differentiable it suffices to verify (47) for $u = u^*$. By assumption there exists $\bar{\kappa} > 0$ such that

$$\langle J_3''(u^*)\delta u, \delta u\rangle_U \geq \bar{\kappa}|\delta u|_U^2 \ \text{ for all } \ \delta u \in U.$$

We consider the expression for $\hat{J}''(u)$ given just above (39) and find for $\delta u \in U_0$

$$\begin{aligned}
\langle \hat{J}''(u^*)\delta u, \delta u\rangle_U &\geq \langle J_3''(u^*)\delta u, \delta u\rangle_U + \langle J_1'(y^*), \tilde{y}''(u)(\delta u, \delta u)\rangle_{L^2(V^*), L^2(V)} \\
&+ \langle J_2'(y^*(T)), \tilde{y}''(u^*)(\delta u, \delta u)(T)\rangle_{\mathbb{H}} \\
&\geq \bar{\kappa}|\delta u|_U^2 - |J_1'(y^*)|_{L^2(V^*)}|\tilde{y}''(u^*)(\delta u, \delta u)|_{L^2(V)} \\
&- |J_2'(y^*(T))|_{\mathbb{H}}|\tilde{y}''(u^*)(\delta u, \delta u)(T)|_{\mathbb{H}}.
\end{aligned}$$

Since $u \to \tilde{y}(u)$ is twice continuously differentiable from $\bar{U}$ to W there exists $K > 0$ such that

$$\max(|\tilde{y}''(u^*)(\delta u, \delta u)|_{L^2(V)}, |(\tilde{y}''(u^*)(\delta u, \delta u))(T)|_{\mathbb{H}}) \leq K \ |\delta u|^2$$

for all $\delta u \in U_0$. Consequently

$$\langle \hat{J}''(u^*)\delta u, \delta u\rangle_U \geq (\bar{\kappa} - K(|J_1'(y^*)|_{L^2(V^*)} + |J_2'(y^*(T))|_{\mathbb{H}}))|\delta u|_{\mathbb{H}}^2,$$

and the claim follows if $|J_1'(y^*)|_{L^2(V^*)}$ and $|J_2'(y^*(T))|_{\mathbb{H}}$ are sufficiently small. $\Box$

Theorem 4.1 gives a sufficient condition for second order convergence of Newton's method provided that the initial condition u_0 is chosen sufficiently close to u^*.

Let us briefly consider Newton's method for solving (P). Henceforth we assume that $(H1) - (H3)$ are satisfied.

Algorithm 4.2. (Newton Method)

1. Choose $u_0 \in N(u^*)$, set $k = 0$
2. Do until convergence
   (i) solve $\hat{J}''(u^k)\delta u^k = -\hat{J}'(u^k)$
   (ii) update $u^{k+1} = u^k + \delta u^k$
   (iii) set $k = k + 1$.

Given u^k, the evaluation of $\hat{J}'(u^k)$ requires to solve the Navier–Stokes equations for $y^k = y(u^k)$ and the linearized adjoint equations (38) with $y = y^k$ for $(\lambda^k, \mu^k) = (\lambda(u^k), \mu(u^k))$. With these quantities available, $\hat{J}'(u)$ can be computed from the formula given in Theorem 3.4. Let us turn to the Hessian $\hat{J}''(u^k)$. As we have seen in the derivation leading to Theorem 3.6 each evaluation of $\hat{J}''(u^k)$ in a direction δu requires two solves. Thus, assuming that U is finite dimensional with dimension n and utilizing the symmetry of $\hat{J}''(u^k)$, evaluation of $\hat{J}''(u^k)$ requires $\frac{n(n+1)}{2}$ solves, which, due to computing and storage limitations, can be infeasible if n is large. In this case one resorts to iterative solutions of 2(i). We then refer to 2. as the outer and to 2(i) as the inner loop. For convenience we recall from Section 3 the steps necessary to carry out one evaluation of $\hat{J}''(u^k)$ in direction $\delta u \in U_0$ during the inner loop. At this stage (u^k, y^k, λ^k) are available.

(a) Solve for $g \in \mathbb{W}$:

$$\left\{\begin{array}{l} g_t - \nu\Delta g + (y^k \cdot \nabla)g + (g \cdot \nabla)y^k = 0, \quad \text{in } \ Q, \\ \tau\, g = B\,\delta u, \quad \text{on } \ \Sigma \\ g(0,\cdot) = 0. \end{array}\right. \tag{48}$$

(b) Solve for $(w, \chi) \in W \cap H^{2,1}(Q) \times L^2(H^1(\Omega) \cap L^2_0(\Omega))$:

$$\left\{\begin{array}{ll} -w_t - \nu\Delta w - (y^k \cdot \nabla)w & +(\nabla y^k)^T w - \nabla\chi = \\ & J_1''(y^k)(g) + (\nabla g)^T\lambda^k - \nabla\lambda^k\, g \\ \tau\, w = 0 & \\ w(T) = J_2''(y(T)). & \end{array}\right. \tag{49}$$

(c) Set $\hat{J}''(u^k)\delta u = B^*(\nu\,\frac{\partial w}{\partial n} - \chi n) + J_3''(u^k)\delta u$.

The first equation in (48) must be interpreted as variational equation with test functions in $L^2(V)$. A complete implementation of Newton's method also requires a globalization strategy. We shall not focus on this issue here.

We close with stating

Theorem 4.3. Let u^* denote a solution to problem (P) and let the assumptions of Theorem 4.1 be satisfied. Then there exists a neighborhood $\mathcal{U}(u^*)$ such that u^* is the only solution to (P) in $\mathcal{U}(u^*)$ and for every starting value $u^0 \in \mathcal{U}(u^*)$ the iterates $\{u^k\}_{k\in\mathbb{N}}$ of Algorithm 4.2 converge quadratically to u^*.

5 Numerical Approach

5.1 Realization of control norm and structure of implementation

The numerical treatment of the boundary control mechanism and the control costs involving fractional order Sobolev spaces require special attention. In the first part of this subsection we describe a framework for a numerical realization. The second part is devoted to an outline of the most important features of the implementation of the minimization algorithm.

Let us return to Example 2.4. In earlier implementations the fractional order Sobolev spaces were typically avoided by rounding up to the next higher integer Sobolev space. Here we try to avoid such a procedure. We shall further modify the set-up of Example 2.4 by allowing the control action only to take place on a subset Γ_c of $\partial\Omega$ and choose

$$B\,u = \begin{cases} u & \text{on} \quad [0,T] \quad \times \quad \Gamma_c \\ 0 & \text{on} \quad [0,T] \quad \times \quad \partial\Omega. \end{cases}$$

Concerning the function space of admissible controls u we recall that for $\varphi \in H^{1/2}(\Gamma_c)$ the extension of φ by zero on $\partial\Omega \setminus \Gamma_c$ is not necessarily an element of $H^{1/2}(\partial\Omega)$. For this purpose we define

$$H_c^{1/2}(\Gamma_c) = \{\varphi \in H^{1/2}(\Gamma_c)\colon \int_{\Gamma_c} \xi(s)^{-1}\ |\varphi(s)|^2 ds < \infty\},$$

where $\xi(s)$ is the distance of s to $\partial\Gamma_c$, where $\partial\Gamma_c$ denotes the boundary of Γ_c, [G, F][pg 43]. Elements of $H^{1/2}(\Gamma_c)$ have the property that their extension by zero are in $H^{1/2}(\partial\Omega)$. To avoid technical issues we assume that Γ_c is a connected component of $\partial\Omega$ which can be mapped by an affine mapping into one of the canonical coordinate axes. We also assume that the initial velocity satisfies $y_0 \in H$, so that u_o introduced at the beginning of Section 2 can be taken to be 0. Finally we choose U as

$$\begin{aligned} U =&\{u = u_n n + u_\tau \tau\colon u_n \in L^2(H_c^{1/2}(\Gamma_c)) \cap H^{3/4}(H^{-1}(\Gamma_c)), \int_{\Gamma_c} u_n = 0, \\ & u_\tau \in L^2(H_c^{1/2}(\Gamma_c)) \cap H^{1/4}(L^2(\Gamma_c))\}, \end{aligned}$$

endowed with the canonical norm. To obtain a practical numerical realization of the norm in U we introduce the following operators:

$$-\Delta_c\colon H^2(\Gamma_c) \cap H_0^1(\Gamma_c) \to L^2(\Gamma_c),$$

the Laplacian on Γ_c with homogeneous boundary conditions (replaced by the Laplace-Beltrami-operator, if Γ_c is curvilinear), and

$$-\Delta_t\colon H_{0,}^2(0,T) \to L^2(0,T),$$

the Laplacian in 1-D, where

$$H^2_{0,}(0,T) = \{\varphi\colon (0,T) \to \mathbb{R} : \varphi(0) = \varphi_t(T) = 0\}.$$

With these operators we express the norm on U as

$$|u|^2_U = \int_0^T |(-\Delta_c)^{1/4} u|^2_{L^2(\Gamma_c)^2} + \int_0^T |(-\Delta_t)^{3/8} u_n|^2_{H^{-1}(\Gamma_c)} + \int_0^T |(-\Delta_t)^{1/8} u_\tau|^2_{L^2(\Gamma_c)}.$$

This is justified by results on the characterization of domains of fractional powers of elliptic operators and their relation to fractional-order Sobolev spaces ([F], Theorem 1.2(b), Theorem 1.4 (a), Theorem 3.4).

In the control problem (P) we choose

$$J_1(y) := \frac{1}{2}|y - z|_{L^2(Q_o)^2},\ J_2(y(T)) := 0,$$

and

$$J_3(u) := \frac{\alpha}{2}\int_0^T |u|^2_{L^2(\Gamma_c)^2} + \frac{\beta}{2}\int_0^T |(-\Delta_c)^{1/4} u|^2_{L^2(\Gamma_c)^2} + \frac{\gamma}{2}\int_0^T |(-\Delta_t)^{3/8} u_n|^2_{H^{-1}(\Gamma_c)} + \frac{\delta}{2}\int_0^T |(-\Delta_t)^{1/8} u_\tau|^2_{L^2(\Gamma_c)}.$$

Note that for $\alpha \geq 0$ and $\beta, \gamma, \delta > 0$ the functional $J_3(u)$ induces a norm on U which is equivalent to $|\cdot|_U$.

Let us now turn to the representation of the gradient of $\hat{J}(u)$ under the assumptions of Theorem 3.4. Splitting the gradient into its normal and tangential components we find

$$\hat{J}'(u) = \begin{pmatrix} \nu\partial_n\lambda_1 \cdot n - \mu + \alpha u_n + \beta(-\Delta_c^{1/2} u)\cdot n - \gamma\Delta_c^{-1}(-\Delta_t)^{3/4}\, u_n \\ \nu\partial_n\lambda_1 \cdot \tau + \alpha u_\tau + \beta(-\Delta_c^{1/2} u)\cdot \tau + \delta(-\Delta_t)^{1/4}\, u_\tau \end{pmatrix}.$$

In Newton's Algorithm (4.2) we use

$$J_{1_{uu}}(u) = \begin{pmatrix} J^1_{1_{uu}}(u) & 0 \\ 0 & J^2_{1_{uu}}(u) \end{pmatrix} = \begin{pmatrix} \alpha Id + \beta(-\Delta_c)^{1/2} + \gamma(-\Delta_c)^{-1}(-\Delta_t)^{3/4} & 0 \\ 0 & \alpha Id + \beta(-\Delta_c)^{1/2} + \delta(-\Delta_t)^{1/4} \end{pmatrix}$$

as preconditioning operator for the Newton system 2.(i).

We recall that, given a control u the evaluation of the gradient of $\hat{J}$ at a point u amounts to solving (5) with $g := Bu$ for the state y and (38) for the adjoint variable λ^1. The Newton system 2.(i) itself is solved iteratively by the preconditioned conjugate gradient method. As preconditioner the operator $J_{1_{uu}}(u)$ is used. For given r the resulting subsystems $J_{1_{uu}}(u)h = r$ in each iteration of the conjugate gradient method are also solved iteratively with a conjugate gradient method. All together this amounts to an inexact Newton method with three levels of nested iterations. We note that this approach is justified in a neighborhood of a local solution u^* of the optimal control problem by the positive definiteness of $\hat{J}''(u^*)$, provided the assumptions of Theorem 4.1 are satisfied.

5.2 Numerical Experiment

Here we present a numerical example that first of all demonstrates the feasibility of utilizing Newton's method for optimal boundary control of the two-dimensional instationary Navier-Stokes equations in a workstation environment despite the formidable size of the optimization problem. The total number of unknowns (primal-, adjoint-, and control variables) in the example below will be of order $1.1{*}10^6$. The time horizon could still be increased or the mesh size decreased by utilizing reduced storage techniques at the expense of additional cpu-time, but we shall not pursue this aspect here.

We consider boundary control of the two-dimensional backward facing step flow at Re=100 and Re=200. The flow configuration together with the uncontrolled flow is shown in Fig. 1. The inflow is parabolic and at the outflow natural boundary conditions are imposed. We therefore omit the zero mass flux condition imposed on the normal components of the controls in the control space U. We compute optimal controls on the time horizon $[0, T]$ for $T = 2.5$ with time step size $\delta t = 0.025$. This results in 100 gridpoints for an equidistant time grid. As time integration scheme for the instationary Navier-Stokes system, its linearizations and the corresponding adjoint equations we use semi-implicit Euler schemes (implicit discretization of diffusive terms, explicit discretization of transport terms). In the numerical realization of the operator $\hat{J}''(u)$ the schemes for the adjoint equations are chosen in such a way that the symmetry properties of this operator are conserved on the discrete level, compare [HA]. For the spatial discretization we use Taylor Hood finite elements on a grid containing 1152 triangles with 2417 velocity and 633 pressure nodes. In our numerical experiments $\Gamma_c := \{2\} \times (-0.75, 0)$, i.e. the spatial control boundary is a subset of the back wall. To discretize the control domain Γ_c we use the equidistant grid induced by the velocity discretization. Its gridsize in our example then is 0.0625, which corresponds to 11 control points on Γ_c.

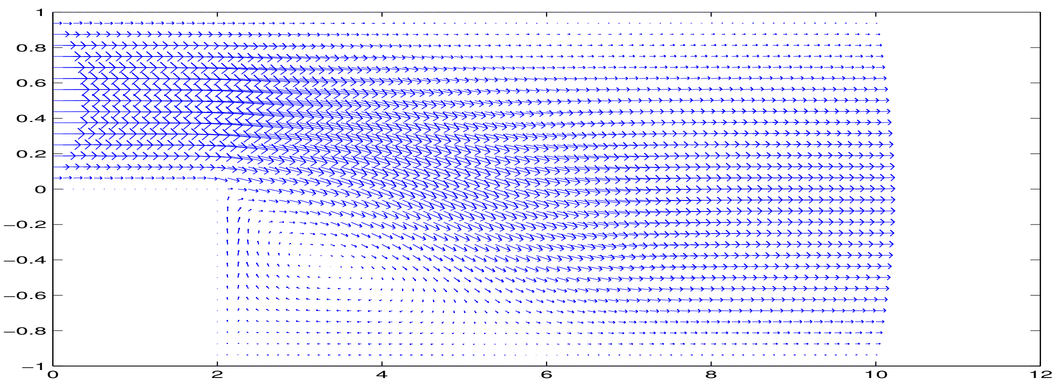


Figure 1: Flow configuration with uncontrolled flow

For the numerical realization of the fractional operators in $J'(u)$ and $J_{1_{uu}}$ we first observe that for appropriate discretisations, the matrix representations A_c of $-\Delta_c$ and A_t

of $-\Delta_t$, respectively, are positive definite. Denoting by U_c the matrix whose columns are composed of orthonormal eigenvectors of A_c and by $\lambda_1^c, \ldots, \lambda_n^c$ the eigenvalues of A_c, the powers of A_c can be expressed as

$$A_c^{m/n} = U_c^t \mathrm{diag}(\lambda_1^{c\,m/n}, \ldots, \lambda_n^{c\,m/n}) U_c.$$

A similar expression holds for A_t. In our numerical calculations $A_c \in \mathbb{R}^{9\times 9}$ and $A_t \in \mathbb{R}^{99\times 99}$ are obtained from the usual finite difference approximations of the operators $-\Delta_c$ and $-\Delta_t$, respectively. To compute the eigenvalues and eigenvectors of this matrices we use the EISPACK package.

The control objective is to suppress recirculation of the flow in the backwall region. For this purpose we track the Stokes flow z in the time-space observation domain $Q_o := (0,T) \times (4,6) \times (-1,0)$. Thus, the functional J_1 is given by

$$J_1(y) := \frac{1}{2} \int\limits_{Q_o} |y - z|^2 \, dx dt.$$

Fig. 2 shows the evolution of the cost functional and of the control costs for different sets of parameters $\alpha, \beta, \gamma, \delta$. It turns out that smaller parameter values allow stronger control actions and result in better reduction of the overall costs as well as the tracking costs, which are not depicted in the figures.

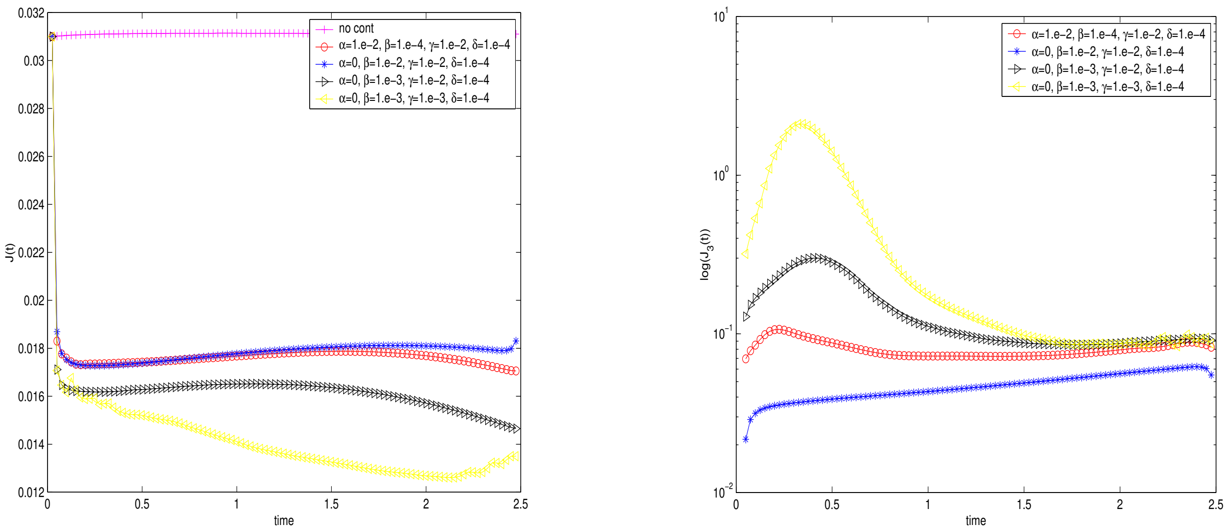


Figure 2: Evolution of cost (left) and control cost (right, logarithmic plot) at Re=100 for $\alpha = 1.e-2, \beta = 1.e-4, \gamma = 1.e-2, \delta = 1.e-4$(red), $\alpha = 0, \beta = 1.e-2, \gamma = 1.e-2, \delta = 1.e-4$(blue), $\alpha = 0, \beta = 1.e-3, \gamma = 1.e-2, \delta = 1.e-4$(black), $\alpha = 0, \beta = 1.e-3, \gamma = 1.e-3, \delta = 1.e-4$(yellow)

In Fig. 3 the results for

$$\tilde{J}_3(u) := \frac{\alpha}{2}\int_0^T |u|^2_{L^2(\Gamma_c)^2} + \frac{\beta}{2}\int_0^T |(-\Delta_c)^{1/4}u|^2_{L^2(\Gamma_c)^2} + \frac{\gamma}{2}\int_0^T |(-\Delta_t)^{3/8}u_n|^2_{L^2(\Gamma_c)} + \frac{\delta}{2}\int_0^T |(-\Delta_t)^{1/8}u_\tau|^2_{L^2(\Gamma_c)}.$$

are presented, i.e. the H^{-1}-norm on Γ_c is replaced by the L^2-norm. For fixed weights in the cost functional this results in reduced control actions and slightly increased tracking errors.

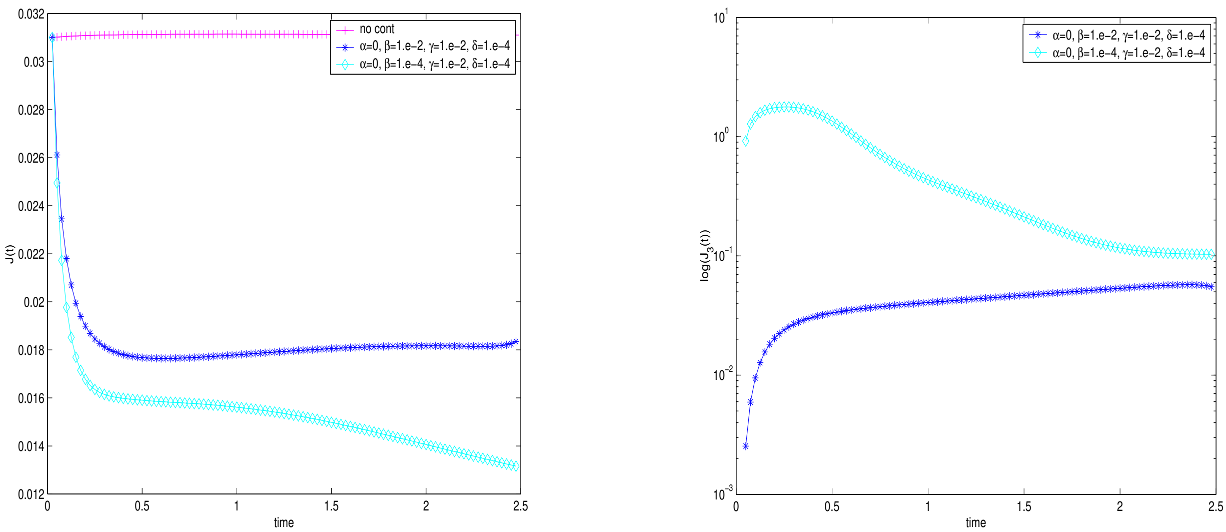


Figure 3: Evolution of cost (left) and control cost (right, logarithmic plot) with $\tilde{J}_3(u)$ for $\alpha = 0, \beta = 1.e-2, \gamma = 1.e-2, \delta = 1.e-4$(blue), and $\alpha = 0, \beta = 1.e-4, \gamma = 1.e-2, \delta = 1.e-4$(cyan), Re=100

Now we take as control part in the cost functional

$$\tilde{\tilde{J}}_3(u) := \frac{\alpha}{2}\int_0^T |u|^2_{L^2(\Gamma_c)^2} + \frac{\beta}{2}\int_0^T |(-\Delta_c)^{1/2}u|^2_{L^2(\Gamma_c)^2} + \frac{\gamma}{2}\int_0^T |(-\Delta_t)^{1/2}u_n|^2_{L^2(\Gamma_c)} + \frac{\delta}{2}\int_0^T |(-\Delta_t)^{1/2}u_\tau|^2_{L^2(\Gamma_c)},$$

which relates to Example 2.2. It allows optimality conditions without fractional order differential operators at the expense of using a stronger norm than necessary. This, in turn, results in more moderate control actions than in the previously considered cases. We refer to Fig. 5 in this respect. In Fig. 4 a snapshot of the flow together with the boundary control action obtained with J_3 at $t = .25$ is shown. For $\tilde{\tilde{J}}_3$ the control action is weaker. One observes that the control possesses a significant component parallel to the back wall, which

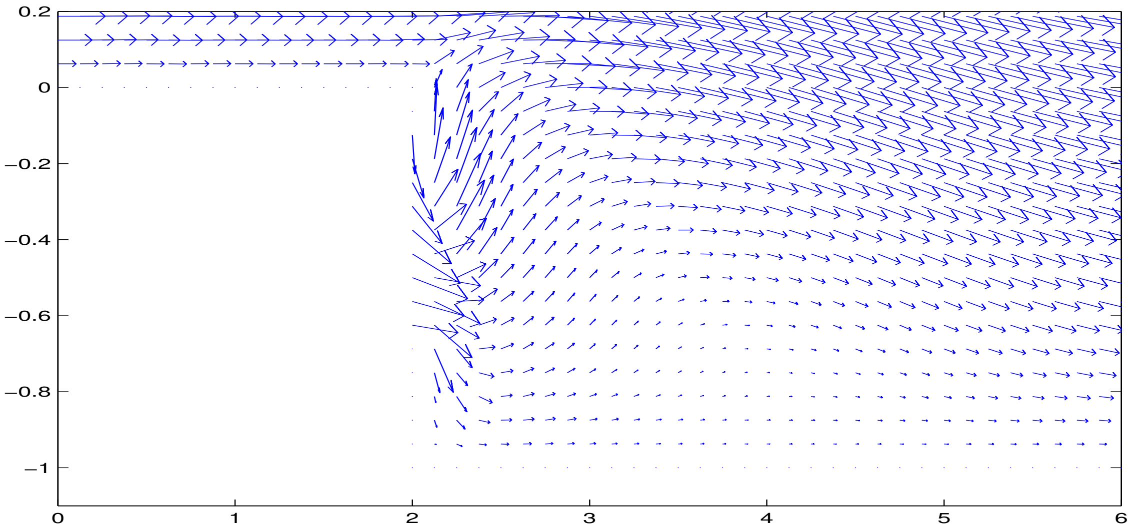


Figure 4: Zoom on flow together with boundary control actions at $t = 0.25$ for $J_3(u)$, $\alpha = 1.e-2, \beta = 1.e-4, \gamma = 1.e-3, \delta = 1.e-5$, Re=100

is more pronounced in the first case. As we can anticipate this has the effect that the control drags the fluid into the region near the backwall and thus, avoids recirculation of the fluid. The behavior obtained with $\tilde{J}_3(u)$ is similar to that obtained for $J_3(u)$. The qualitative behavior of the overall and control costs for $\tilde{\tilde{J}}_3(u)$ is very similar to that presented in Figs. 2,3. In Fig. 5 we compare the evolution of control costs and of $\frac{1}{2}|y(t) - z(t)|^2_{L^2(\Omega_o)}$ for the three considered cost functionals with $\alpha = 0, \beta = 1.e-2, \gamma = 1.e-2, \delta = 1.e-4$. As the figure shows usage of $J_3(u)$ in the cost functional which is based on fractional order Sobolev spaces with the weakest regularity requirements gives the best results followed by $\tilde{J}_3(u)$ and $\tilde{\tilde{J}}_3(u)$.

Our numerical tests indicate that the influence of the parameter δ on the behavior of the controls is weaker than that of the remaining parameters α, β and γ, which seem to be equally important.

In all numerical computations the termination criterion for Newton's method is given by

$$\max\left\{\frac{|\hat{J}'(u^k)|}{|\hat{J}'(u^0)|}, \frac{\hat{J}(u^k) - \hat{J}(u^{k-1})}{\hat{J}(u^0)}\right\} \leq 1.e-3.$$

The termination criterion for the j-th iterate δu_j^k in the conjugate gradient method is chosen as

$$\frac{|\hat{J}''(u^k)\delta u_j^k + \hat{J}'(u^k)|}{|\hat{J}'(u^0)|} \leq \epsilon \frac{|\hat{J}'(u^k)|}{|\hat{J}'(u^0)|} \text{ or } j \geq 50, \tag{50}$$

where $\epsilon \in (0,1)$. This results in linear convergence behavior of the inexact Newton method

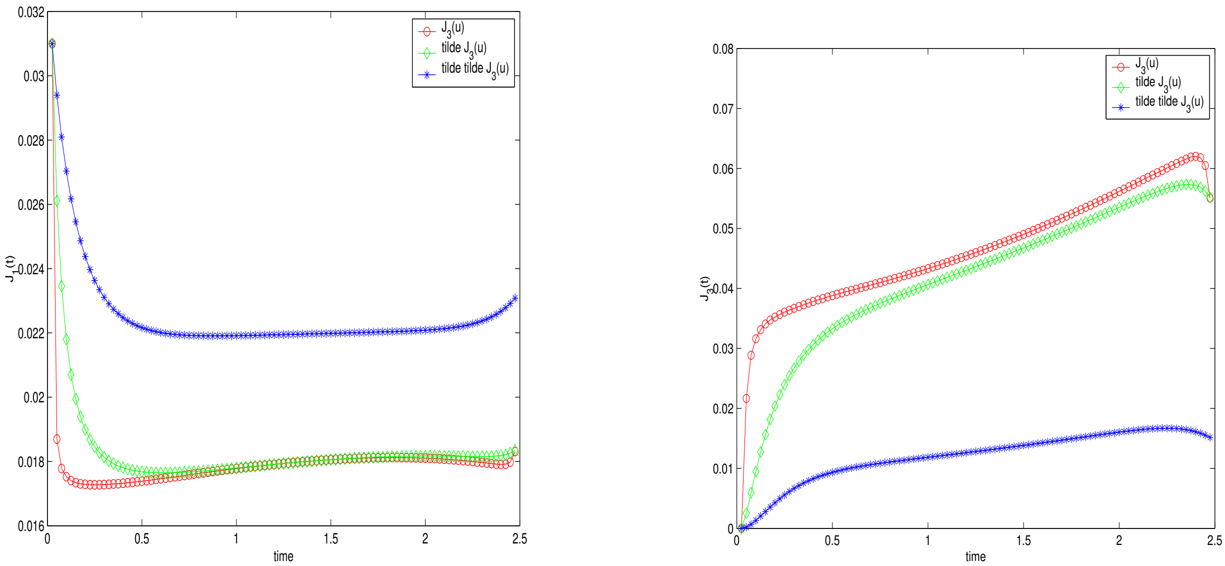


Figure 5: Evolution of $\frac{1}{2}|y(t)-z(t)|^2_{L^2(\Omega_o)}$ (left) and of control costs for the three considered cost functionals with $\alpha = 0, \beta = 1.e-2, \gamma = 1.e-2, \delta = 1.e-4$, Re=100

which is also observed numerically, see Tab. 1, right columns, where the results for $\epsilon = 0.5$ are depicted. If the termination criterion of the inner iteration is altered to

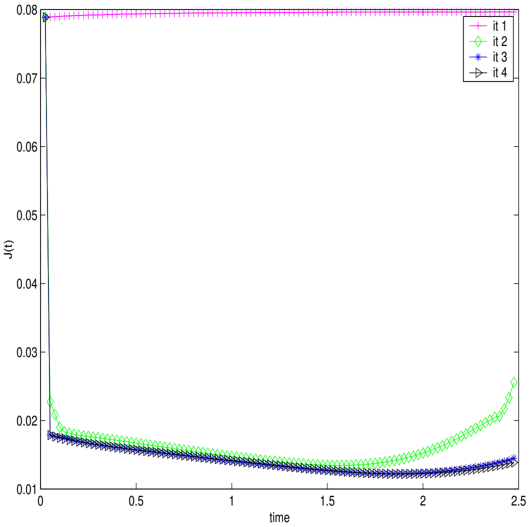
$$
\frac{|\hat{J}''(u^k)\delta u_j^k + \hat{J}'(u^k)|}{|\hat{J}'(u^0)|} \le \min\left\{ \left(\frac{|\hat{J}'(u^k)|}{|\hat{J}'(u^0)|} \right)^{\frac{3}{2}}, \epsilon \frac{|\hat{J}'(u^k)|}{|\hat{J}'(u^0)|} \right\} \quad \text{or } j \ge 50 \tag{51}
$$

with $\epsilon = 0.1$ this results in super linear convergence (left columns). However, the number of outer iterations is too small to claim generic super linear convergence.

It	CG-steps	$\frac{\lvert\delta u^k\rvert_U}{\lvert\delta u^{k-1}\rvert_U}$	CG-steps	$\frac{\lvert\delta u^k\rvert_U}{\lvert\delta u^{k-1}\rvert_U}$
1	-	-	-	-
2	7	1.	3	1.
3	15	0.172	7	0.382
4	14	0.036	10	0.311

Table 1: Performance of Newton's method for $\tilde{J}_3(u)$ and $\alpha = 1.e-2, \beta = 1-e-4, \gamma = 1.e-3, \delta = 1.e-5$: termination criterion (51) with $\epsilon = 0.1$ left columns, termination criterion (50) with $\epsilon = 0.5$ right columns

The numerical results presented in Figs. 2, 3, 5, 6 are obtained with $\epsilon = 0.1$. For most sets of parameter constellations 3-6 iterations of Newton's method are sufficient to satisfy the outer termination criterion. However, similar to the case of distributed controls reported in [HK], the graphs of the cost functionals are visually nearly indistinguishable after two Newton iterations, compare Fig. 6 where the results for $J_3(u)$ with $\alpha = 1.e{-}2, \beta = 1.e-4, \gamma = 1.e-3, \delta = 1.e-5$ and Re=200 are shown.



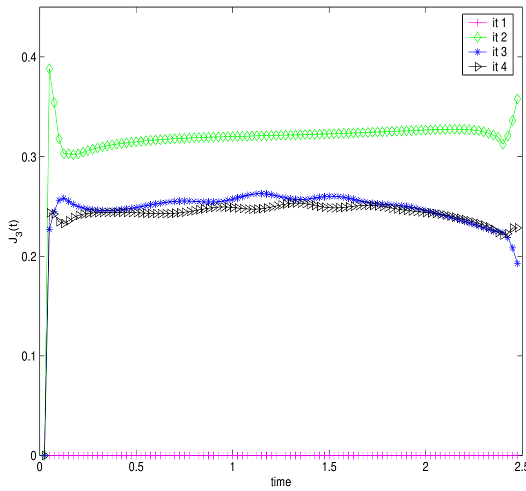


Figure 6: Evolution of cost (left) and control cost (right) for $\alpha = 1.e-2, \beta = 1.e-4, \gamma = 1.e-3, \delta = 1.e-5$ at different Newton iterations, Re=200

References

[B] M. Berggren: *Numerical solution of a flow–control problem: vorticity reduction by dynamic boundary action*, SIAM J. Control and Optimization **19** (1998), 829–860.

[Be] T. Bewley: *Flow control: new challenges for a new Renaissance*, preprint.

[CF] P. Constantin and C. Foias: *Navier–Stokes Equations* , The University of Chicago Press, 1988.

[DL] R. Dautray and J.–L. Lions: *Mathematical Analysis and Numerical Methods for Science and Technology*, **Vol V**, Springer–Verlag, New York, 1992.

[F] D. Fujiwara: *Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order*, Proc. Japan Acad. **43** (1967), 82–86.

[FGH] A. V. Fursikov, M. D. Gunzburger and L. S. Hou: *Boundary value problems and optimal boundary control for the Navier–Stokes systems: The two–dimensional case*, SIAM J. Control and Optimization **36** (1998), 852–894.

[G] P. Grisvard: *Elliptic Operators in Nonsmooth Domains*, Pitman, London, 1985.

[GB] O. Ghattas and J.–H. Bark: *Optimal Control of two–and three–dimensional incompressible Navier–Stokes flows*, J. Comp. Physics, **136** (1997), 231–244.

[GM] M. D. Gunzburger and S. Manservisi: *The velocity tracking problem for Navier–Stokes flows with boundary control*, SIAM J. Control and Optimization **39** (2000), 594–634.

[GR] V. Girault and P.–A. Raviart: *Finite Element Methods for Navier–Stokes Equations*, Springer–Verlag, Berlin, 1986.

[HA] W.–W. Hager: *Runge-Kutta methods in optimal control and the transformed adjoint system*, Numer. Math. **87** (2000), 247-282.

[GHS] M. D. Gunzburger, L. S. Hou and Th. P. Svobodny: *Analysis and finite approximation of optimal control problems for the stationary Navier–Stokes equations with Dirichlet control*, M^{2}AN, Mathematical Modelling and Numerical Analysis, **25** (1991), 711–748.

[H] M. Heinkenschloss: *Formulation and analysis of a sequential quadratic programming method for the optimal Dirichlet boundary control of Navier–Stokes flow*, Optimal Control: Theory, Algorithms, and Applications, Kluwer Academic Publishers B. V. 1998, 178–203.

[HK] M. Hinze and K. Kunisch: *Second order methods for optimal control of time–dependent fluid flow*, SIAM J. Control and Optimization **40** (2001), 925-946.

[HR] L. S. Hou and S. S. Ravindran: *A penalized Neumann control approach for solving an optimal Dirichlet control problem for the Navier–Stokes equations*, SIAM J. Control and Optimization, **36** (1998), 1795–1814.

[L] M. Laumen: *Newton's mesh independence principle for a class of optimal shape design problems*, SIAM J. Control and Optimization **37** (1999), 1088.

[LU] O. Ladyzhenskaya and N. Uraltseva: *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[T] R. Temam: *Navier–Stokes equation*, North Holland Publ. Comp., Amsterdam, 1979.

[T1] R. Temam: *Infinite–Dimensional Dynamical Systems in Mechanics and Physics*, Springer–Verlag, New York, 1988.