Suboptimal Control of Fluid Flows[1]

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Preprint No. 567/1997

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Abstract

In the present paper suboptimal boundary control strategies for the time-dependent, incompressible flow over the backward-facing step are considered. The objective consists in the reduction of the recirculation bubble of the flow behind the step. A cost functional adapted to the optimization target is discussed. Numerical examples are given.

1 Problem formulation

The main objective of this note is the development of numerically reliable control techniques for fluid flows. As a model configuration serves the backward-facing-step. The aim of the control mechanisms consists in reducing the recirculation bubble behind the step, and thus reducing the re-attachment length of a backward-facing-step flow by controlling the flow at the boundary near the edge of the step. For the following presentation it will be convenient to refer to Fig. 1 which depicts the spatial domain Ω that constitutes the flow region and the subsets of the boundary that shall be referred to.

Let (u_1, u_2, u_3) denote the velocity of the fluid in the directions (x_1, x_2, x_3) and let p denote its pressure. The time-dependent Navier-Stokes equations on the space-time cylinder $\Omega^T := \Omega \times (0,T)$, $T > 0$ are given by

$$\begin{array}{rcl} u_t - \frac{1}{\mathrm{Re}}\Delta u + (u\nabla)u + \nabla p &=& f \text{ in } \Omega^T \\ -div\, u &=& 0 \text{ in } \Omega^T \\ u &=& \left\{ \begin{array}{cl} u_{in} & \text{on } \Gamma_{in}^T \\ \Phi & \text{on } \Gamma_c^T \\ 0 & \text{on } \Gamma_0^T \end{array} \right. \\ \frac{1}{Re}\partial_\eta u &=& p\eta \text{ on } \Gamma_{out}^T \\ u(0) &=& u_0, \end{array} \tag{1}$$

where, as usual Δ denotes the (3×3) matrix with Δ in its diagonal and zero otherwise, $(u \cdot \nabla)u$ stands for the vector with components $\sum_{j=1}^3 u_j \frac{\partial}{\partial x_j} u_i$ and $Re = \frac{UH}{\nu}$ denotes the Reynolds number with U denoting the bulk velocity of the inlet profile, H is the step hight and ν the kinematic viscosity. The inlet velocity profile is denoted by u_{in} and will be taken as a parabolic inflow profile in the numerical examples. On the outflow part of the boundary a natural (" do-nothing") boundary condition is imposed, where η denotes the unit normal to $\partial\Omega$ in outward direction. Finally, in order to influence the flow in the recirculation zone behind the step a control Φ will be applied along the boundary Γ_c. Here, the control function Φ represents blowing and sucking which will be chosen in such a manner that an appropriately defined performance index is minimized. Further dimension three is chosen for presenting the theoretical part of our approach.

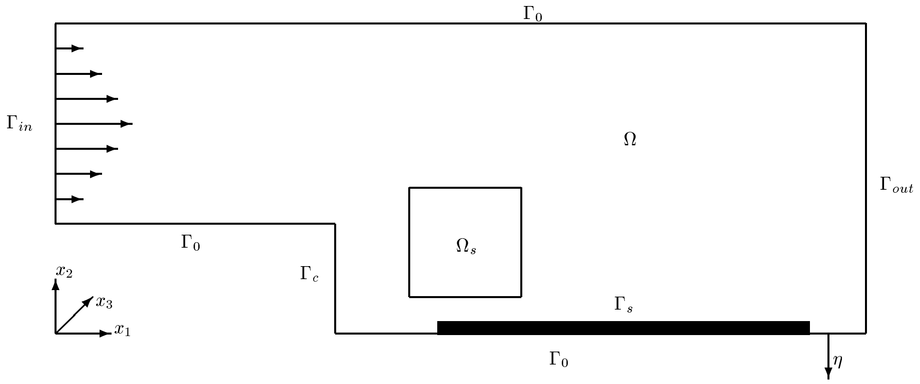

Figure 1: Flow region

The choice of the cost functional appropriate to capture the goal of influencing and reducing the recirculation bubble is certainly a delicate one. Here primarily functionals of the form

$$J(\phi) = \int_0^T \left[\int_{\Gamma_s} F(\partial_\eta u)\, d\Gamma_s \; + \; \frac{\gamma}{2} \int_{\Gamma_c} \phi^2 \, d\Gamma_c \right] dt \tag{2}$$

are considered, where $\gamma > 0$ and $F : \mathbb{R}^3 \to \mathbb{R}$ is a given, sufficiently smooth function. Referring to Fig. 1 one has $\eta = (0, -1, 0)^t$ on Γ_s and hence a special case of (2), is given by

$$J(\phi) = \int_0^T \left[\int_{\Gamma_s} \frac{\partial u_1}{\partial x_2} \Big(\frac{\partial u_1}{\partial x_2} - \Big| \frac{\partial u_1}{\partial x_2} \Big| \Big) \, d\Gamma_s \; + \; \frac{\gamma}{2} \int_{\Gamma_c} \phi^2 \, d\Gamma_c \right] dt. \tag{3}$$

The first part of the cost penalizes a negative wall velocity gradient in the sensor area Γ_s which is supposed to contain the re-attachment point. The 2nd summand weights the costs for the control acting along Γ_c. Other functionals involving shear stresses or drag along Γ_s can be expressed as special cases of (2). The optimal control problem which is the focus of this paper can be stated as follows.

$$(P)\left\{\begin{array}{rcll}
\multicolumn{4}{l}{\min J(\phi) = \int_0^T \left[\int_{\Gamma_s} F(\partial_\eta u)\, d\Gamma_s \;+\; \frac{\gamma}{2}\int_{\Gamma_c} \phi^2\, d\Gamma_c\right] dt} \\
\multicolumn{4}{l}{\text{s.t.}} \\
u_t - \frac{1}{Re}\Delta u + (u\cdot\nabla)u + \nabla p & = & f & \text{in } \Omega^T \\
-div\, u & = & 0 & \text{in } \Omega^T \\
u & = & \left\{\begin{array}{ll} u_{in} & \text{on } \Gamma_{in}^T \\ \phi & \text{on } \Gamma_c^T \\ 0 & \text{on } \Gamma_0^T \end{array}\right. & \\
\frac{1}{Re}\partial_\eta u & = & p\eta & \text{on } \Gamma_{out}^T \\
u(0) & = & u_0 &
\end{array}\right. \qquad (4)$$

The contents of the following sections is as follows. In Section 2 the optimality system for problem (P) is discussed briefly. It will be seen that its numerical realization is beyond todays workstation capabilities. Hence, following Choi, Kim, Moin and Temam [1] suboptimal control strategies amenable for workstation implementation are developed. Numerical results will be given in Section 3.

2 The instantaneous control strategy

Let ϕ^* denote a solution to (4) and let $(u^*, p^*) = (u(\phi^*), p(\phi^*))$ denote the corresponding optimal velocity field and pressure, respectively satisfying the Navier-Stokes system. Then (u^*, p^*, ϕ^*) satisfies a coupled system of nonlinear differential equations involving the Navier-Stokes equations with boundary values Φ^* on Γ_c^T forward in time, a linear parabolic equation involving the adjoint velocity and pressure (ξ^*, π^*) backward in time and an optimality condition that relates Φ^* to the adjoint variables (ξ^*, π^*). Solving this optimality system is almost unworkable. This suggests the use of suboptimal control strategies. A second motivation for such strategies is given by closed-loop control considerations as will briefly be explained.

The approach that is taken here is based on a time discretization of (4). At each discrete time level t_i a stationary control problem is solved for an optimal control ϕ^* and this control is used to steer the system from t_i to t_{i+1}, where a new optimal control is determined. It cannot be claimed that the controls obtained in this manner approximate the optimal control for (4) as the discretization parameter tends to zero. However, this procedure will be justified by effectiveness that it exhibits for numerical examples and the interpretation that it allows for suboptimal feedback controls.

To commence, let $m > 1$ be fixed and set $\delta t = \frac{T}{m}$, $t_i = i\delta t, i = 0, \ldots, m$. The discretization is chosen implicit in the viscous term and explicit in the nonlinear convective term. In this case only measurements of the shear wall stress will be necessary to compute the adjoint variables of the corresponding adjoint system. More precisely, consider at the i-th time level of the time integration scheme the minimization problem

$$
(P_i)\left\{\begin{array}{l}
\min J(\phi) = \int_{\Gamma_s} F(\partial_\eta u)\, d\Gamma_s + \frac{\gamma}{2}\int_{\Gamma_c} \phi^2\, d\Gamma_c \\
\text{s.t.} \\
\begin{array}{rcll}
u - c\Delta u + \delta t \nabla p & = & R_i & \text{in } \Omega \\
-div\, u & = & 0 & \text{in } \Omega \\
u & = & \left\{\begin{array}{ll} u_{in} & \text{on } \Gamma_{in} \\ \phi & \text{on } \Gamma_c \\ 0 & \text{on } \Gamma_0 \end{array}\right. & \\
\frac{1}{Re}\partial_\eta u & = & p\eta & \text{on } \Gamma_{out}
\end{array}
\end{array}\right. \tag{5}
$$

where $c = \frac{Re}{\delta t}$ and $R_i = u_{i-1} + \delta t(f_i - (u_{i-1}\cdot\nabla)u_{i-1})$. Note that the information from time level t_{i-1} to t_i is passed solely through the inhomogeneity R_i. Let ϕ_i denote a solution to (P_i) and set $(u_i, p_i) = (u(\phi_i), p(\phi_i))$. A short calculation shows that (u_i, p_i, ϕ_i) must satisfy an optimality system consisting of the primal equations which are the equality constraints in (5) and an adjoint equation, coupled with an optimality condition which are given next:

$$
\begin{array}{rcll}
\xi_i - c\Delta\xi_i + \nabla\pi_i & = & 0 & \text{in } \Omega \\
-\delta t div\, \xi_i & = & 0 & \text{in } \Omega \\
\xi_i & = & \left\{\begin{array}{ll} 0 & \text{on } \partial\Omega\setminus(\Gamma_{out}\cup\Gamma_s) \\ Re\nabla F(\partial_\eta u_i) & \text{on } \Gamma_s \end{array}\right. & \\
\frac{1}{Re}\partial_\eta \xi_i & = & \pi_i\eta & \text{on } \Gamma_{out}
\end{array} \tag{6}
$$

$$
\gamma\phi_i + \frac{1}{Re}\partial_\eta\xi_i = \pi_i\eta \quad \text{on } \Gamma_c. \tag{7}
$$

In the optimality system for (P_i) the coupling between primal and adjoint equations for the problems (P_i) occurs only along the observation boundary Γ_s. The gradient of J at Φ_i in direction χ is given by

$$
\nabla J(\Phi_i)\chi = \int_{\Gamma_c} (\gamma\phi_i + \frac{1}{Re}\partial_\eta\xi_i - \pi_i\eta)\,\chi\, d\Gamma_c. \tag{8}
$$

The optimal control problem (4) has thus been replaced by a sequence of stationary problems (P_i) which can be realized on workstations. Moreover, if one interprets (7) as a feedback control law, then only observations of the state along the boundary Γ_s are required. To solve (5) numerically several different approaches are possible. Putting speed above accuracy a gradient algorithm with (almost) optimal step size is proposed to solve (P_i):

Gradient Algorithm

1. set $k = 0$ and choose $\phi^0 = \Phi_i^0$,
2. Compute
$$\rho^* = \arg\min_{\rho>0} H(\rho) := J(\phi^k - \rho\nabla J(\phi^k))$$
3. Set
$$\phi^{k+1} = \phi^k - \rho^*\nabla J(\phi^k)$$

4. Set $k = k + 1$ and goto 2.

In this algorithm the expensive part is step 2. A near to optimal step size rule for ρ^* based on a suitable linearization of H at the expense of two additional quasi-Stokes solves is developed in [2].

3 Numerical results

Instantaneous control is applied to problem (4) using the cost function given by Eq. (3). The optimization target is the reduction of the recirculation bubble behind the step and thus of the re-attachment length of the two-dimensional Navier-Stokes flow at $Re = 300$ (Fig. 2, top). The sensor area Γ_s is chosen as the interval $[3, 9]$ on the x-axis located at the bottom of the flow domain, control is applied at the top part of the back wall and the parameter γ which penalizes the control costs is set to $\gamma = 10^{-2}$. At each time instance one iteration of the gradient algorithm is performed using a near to optimal step size ρ^* [2]. For the discretization of the quasi-Stokes problems the Taylor-Hood finite element is used.

Fig. 2, top shows the uncontrolled Navier-Stokes flow with re-attachment point at 8.75. The application of the feedback control mechanism up to $T = 100$ yields the flow presented in Fig. 2, bottom with re-attachment point located at 3.5 on the x-axis. As one can see the recirculation bubble vanishes nearly completely.

Fig 3 shows the evolution of the costs (top) together with two zooms on the time interval. The costs of uncontrolled flow are 0.046. Control is invoked at $t = 0.2$ (middle), at $t = 80$ the costs become stationary at the value 0.00019 (bottom). Hence, the suboptimal control strategy proposed here achieves a remarkable reduction of the costs. The oscillations of the costs are due to the instantaneous character of the feedback-control mechanism.

Further numerical examples involving other cost functionals are presented in [2]. Different approaches to the control of fluid flows can be found in [3], [4], [5], [6] and the references cited therein.

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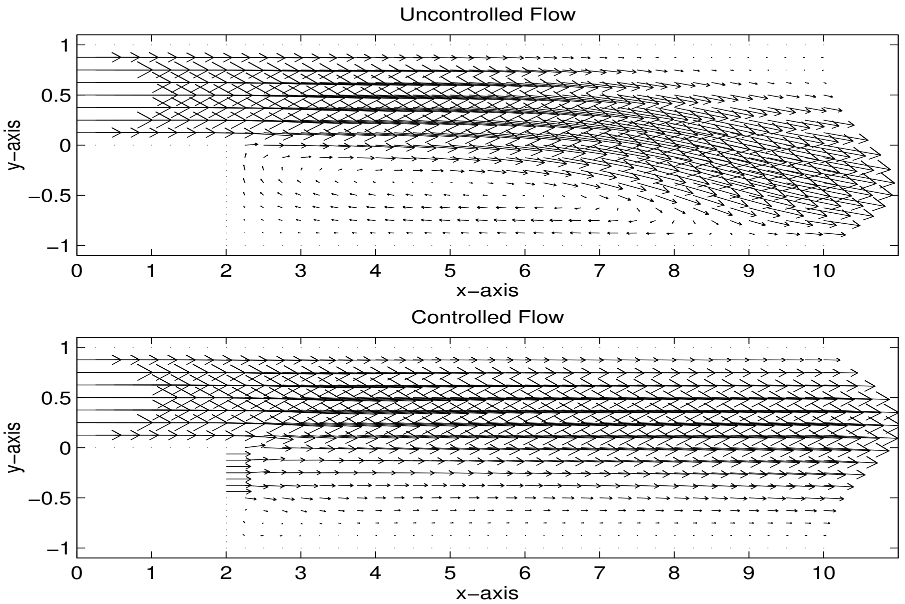

Figure 2: Top: Navier-Stokes flow at $Re = 300$, Bottom: Controlled flow at $T = 100$

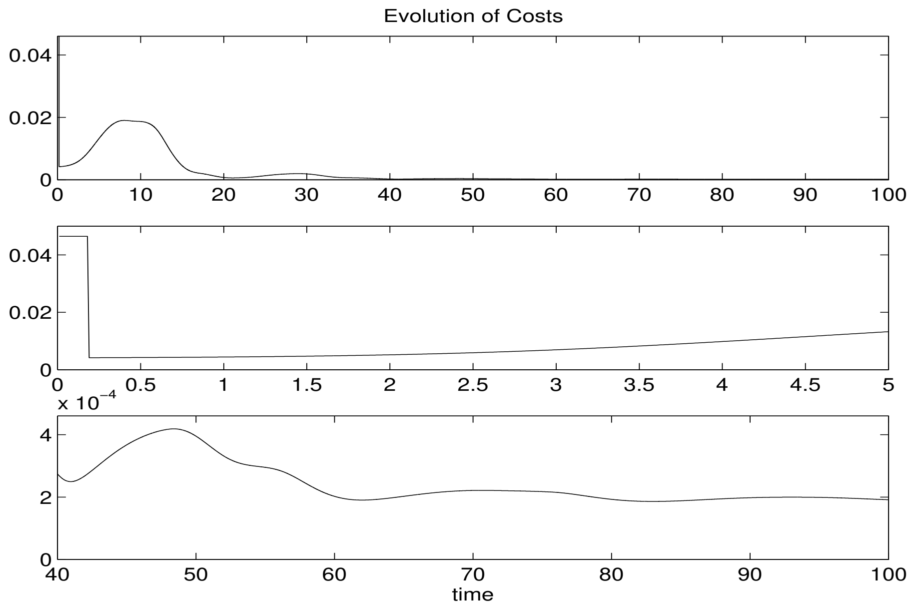

Figure 3: Top: Evolution of costs in $[0, 100]$, Middle: in $[0, 5]$, Bottom: in $[40, 100]$ zoomed

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