Control Concepts for Parabolic Equations with an Application to the Control of Fluid Flow

MICHAEL HINZE[1] & ANDREAS KAUFFMANN[2]

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[1]Technische Universität Berlin, Fachbereich Mathematik, Straße des 17. Juni 136, D-10623 Berlin, email: hinze@math.tu-berlin.de

[2]Technische Universität Berlin, Fachbereich Mathematik, Straße des 17. Juni 136, D-10623 Berlin, email: kauffman@math.tu-berlin.de

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Abstract

Two approaches for obtaining control laws for time-dependent problems are presented. Applying instantaneous control an optimization problem is solved at each time step of the discretized problem. The reduced order modelling approach makes use of a suitable finite-dimensional approximation of the system under consideration and computes optimal controls for the reduced equations. Numerical examples demonstrate the scope of the approaches.

1 Introduction

Computing optimal controls numerically for realistic nonlinear time-dependent problems is very often unfeasible with most currently available computing facilities. This includes optimal control of fluid flow and optimal control of three-dimensional ignition models in chemical reactions, just to mention two examples. Hence, there is a need for suboptimal control strategies which incorporate the nonlinear character of the model and which are amenable to controlling the problem under consideration using presently available computing environments.

In this note we present two suboptimal control approaches for problems governed by time dependent partial differential equations.

- **Instantaneous control** [4, 6] The time-dependent equation is discretized w.r.t. time and on each time level an optimization problem is solved with an instantaneous cost function measuring the current cost of the control gain as well as the cost of the control. This approach may be interpreted as closed-loop control mechanism, as is worked out in [6].

- **Reduced order modelling** [9, 13] The nonlinear dynamics of the time-dependent equation is approximated using a reduced order model (ROM). An exact optimization for the reduced system gives suboptimal controls for the system under consideration. Various approaches differ in the construction of the basis functions that are used for the reduced models [7]. In this paper we use proper orthogonal decomposition (POD) to construct a basis for the reduced problem.

As numerical examples we apply instantaneous control to the flow around a cylinder and ROM to the control of the *small fuel loss model* [2, p.8], the latter describing the ignition process in solid fuel.

The paper is organized as follows. In Section 2 we describe the ROM approach and its applications in suboptimal control. In Section 3 we briefly present the instantaneous control approach with its interpretation as closed-loop control law. We close with numerical results presented in Section 4.

2 Reduced order modelling using POD

2.1 Idea of POD snapshot methods

The conjecture that the solutions of some dynamical systems reside on relatively low-dimensional manifolds or attractors [12, Part I] gives rise to the idea of approximating these solutions by Galerkin methods using small bases that in an appropriate sense approximate these attractors. One method to obtain such bases is proper orthogonal decomposition using snapshots [12], an approach that is sketched next.

To begin with let X be a Hilbert space and denote by $y : [0,T] \to X$ a solution of a dynamical system

$$\dot{y} - Ay = n(t,y), \quad y_{|t=0} = y_0. \tag{1}$$

Furthermore let $I := \{t_k : 1 \leq k \leq K\} \subset [0,T]$ be a set of $K \in \mathbb{N}$ time instances. To every time instance $t_k \in I$ we associate the *snapshot* $y^{(k)} := y(t_k)$ and form the correlation matrix

$$\mathcal{K} = (\kappa_{kl})_{k,l=1}^{K}, \quad \kappa_{kl} := \langle y^{(k)}, y^{(l)} \rangle_X. \tag{2}$$

Denoting by λ_k and $v_k \in \mathbb{R}^K$, $1 \leq k \leq K$ the eigenvalues and eigenvectors, respectively of the correlation matrix $\mathcal{K}$ we finally define the *POD modes* as

$$\phi^{(k)} = \sum_{l=1}^{K} v_{kl} y^{(l)}, \quad k = 1, \ldots, K. \tag{3}$$

It is not hard to show that these modes are pairwise orthogonal. For what follows we in addition suppose the *POD modes* to be normalized and define the Ansatz space for the Galerkin approximations by $X_K := \text{span}\,\{\phi^{(k)} : 1 \leq k \leq K\} \subset X$. Here we note that the snapshots $y^{(k)}$ in Eqs. (2) and (3) may be replaced by the differences $y^{(k)} - \bar{y}$, where $\bar{y}$ denotes the arithmetic mean of the snapshot set. This allows for a statistical interpretation of the approach.

2.2 Optimal control and POD snapshot methods

Let $z_d \in L^2(0,T;X)$ be given and define for $y, u \in L^2(0,T;X)$ the objective function by

$$J(y,u) := \frac{1}{2} \int_0^T \|y(t) - z_d(t)\|_X^2 \, dt + \frac{\nu}{2} \int_0^T \|u(t)\|_X^2 \, dt. \tag{4}$$

Here the first term measures the value in the L^2-norm of the quantity to be controlled and the second term in the same norm measures the control cost weighted with some parameter $\nu > 0$. In this paper we are considered with constructing suitable approximations to the solution(s) of the distributed tracking type optimal control problem

$$\min_{(y,u)} J(y,u) \quad \text{s.t.} \quad \dot{y} + Ay = n(t,y) + u, \; y_{|t=0} = y_0. \tag{5}$$

One straightforward approach in this direction is to minimize the functional J over the finite dimensional space X_K, which corresponds to a conformal Galerkin Method for problem (5). In order to apply this method we set $y_K(t) = \sum_{k=1}^K a_k(t)\phi^{(k)}$ and $u_K(t) = \sum_{k=1}^K b_k(t)\phi^{(k)}$ and denote by a_0 and $a_d(t)$ the projections of the initial value y_0 and the desired state $z_d(t)$ onto the space X_K. Plugging in these Ansatzes into the state equation in (5) one obtains the reduced nonlinear dynamical system

$$\dot{a} + A_K a = n_K(a) + b, \quad a_{|t=0} = a_0, \tag{6}$$

where the nonlinearity n_K is given by $(n_K(a))_k = \langle n(t, \sum_{l=1}^K a_l\phi^{(l)}), \phi^{(k)}\rangle_X$. The reduced optimization problem now becomes

$$\min_{(a,b)} J(y(a),u(b)) \quad \text{s.t.} \quad (6) \tag{7}$$

The optimal control $u = \sum b_k\phi^{(k)}$ of the finite dimensional optimization problem (7) is now taken as a suboptimal control in Eq. (5).

We now briefly sketch a method for constructing a feedback synthesis $u = u(y)$ in (5) using interpolation on a set of suboptimal solutions. To be more precise let $\{a_0^{(j)} : 1 \leq j \leq M\}$ be a grid of initial values for the reduced optimization problem (7) and denote the corresponding controls by $b^{(j)} := b|_{t=0} \in \mathbb{R}^K$. The feedback controller then works as follows: At some time t we observe the state $y(t)$ and compute the coefficients $a(t)$ of the projection of $y(t)$ onto the reduced space X_K. Then we seek for the initial value $a_0^{(j^*)}$ that is the closest (in a sense that has to be specified) to the vector $a(t)$ and use $u = \sum_{k=1}^K b_k^{(j^*)}\phi^{(k)}$ as control. Simple choices are based on piecewise constant interpolation, i.e. choose $j^* = \arg\min_j \|a_0^{(j)} - a(t)\|_{\mathbb{R}^K}$ or on linear interpolation. More sophisticated interpolation criteria may be based on curvature information of the field $\{a_0^{(j)} : 1 \leq j \leq M\}$ [8].

3 Instantaneous control

The approach that we take is based on a time discretization of equation (1). For this purpose let $0 = t_0 < t_1 < \ldots < t_m = T$ denote a equidistant grid on the time interval $[0,T]$ with step size $h = \frac{T}{m}$. The method then works as follows. At each discrete time level t_i a stationary control problem is solved for an approximate optimal control u_i^* and this control is used to steer the system from t_i to t_{i+1}, where a new approximate optimal control is determined.

As time discretization method we choose the semi-implicit Euler method (explicit w.r.t. the nonlinearity) and as cost function $J(y,u) = \frac{\nu}{2}\|u\|_X^2 + \frac{1}{2}\|y - z_d\|_X^2$. The optimization problem in every time step then has the form

$$\min J(y^{j+1}, u^{j+1}) \quad \text{s.t.} \quad (I + hA)y^{j+1} = y^j + hn^j + u^{j+1}, \; y^0 = y_0, \tag{8}$$

which, due to the quadratic character of the cost function, admits a unique solution (y^{j+1}, u^{j+1}) [5, Chapter II, Prop.1.2].

A closed-loop control law for the dynamical system (1) can be obtained by applying exactly one step of the gradient algorithm with step size $\rho > 0$ and initial guess $u_0^{j+1} = 0$ to the solution of problem (8)) at every time level t_i [6, Algorithm 3.1]. Proceeding in this manner we are led to the difference scheme

$$(I + hA)y^{j+1} = y^j + hn^j - \rho BB^*(y^j - z_d^j) - h\rho BB^*(n^j - f^j), \quad y^0 := y_0, \tag{9}$$

which in turn is semi-implicit time integration of the dynamical system

$$\dot{y} + Ay = n - \frac{\rho}{h} BB^*(y - z_d) - \rho BB^*(n - f), \quad y(0) = y_0, \tag{10}$$

see [6, Theorem 4.1], where we have abbreviated $B := (I + hA^*)^{-1}$ and $f := Az_d$. In [6, Theorem 5.4] the authors among other things prove that for certain constellations of the parameters involved in the schemes (10) and (9) the iterates $\{y^j\}_{j\in\mathbb{N}}$ in the finite dimensional case converge to the state z_d as $j \to \infty$ and h tends to zero.

4 Numerical examples

We close with giving a brief discussion of the numerical results obtained by the POD control approach applied to the *small fuel loss model* [2, p.8] and of the instantaneous method applied to control the flow around cylinder [6].

The *small fuel loss model*, also known as the *perturbed Gelfand problem* is a simplified model of combustion of solid fuel:

$$\dot{y} = \Delta y + \delta \exp\left(\frac{y}{1 + \epsilon y}\right) + u, \qquad y|_{t=0} = y_0, y|_{\partial\Omega} = 0. \tag{11}$$

In order to obtain the setting of Section 2 we set $X = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$ [10].

For the numerical tests we use $\Omega = (-1, +1) \subset \mathbb{R}^1$, $\delta = 1$ and $\epsilon = 0.175$. Snapshots $y^{(k)} = y(t_k)$ are taken at times $t_1 = 1$, $t_2 = 2.5$ and $t_3 = 4$ where y is the solution of (11) with $y_0 = 5\cos\left(\frac{\pi}{2}x\right)$ and $u \equiv 0$. The state to be reconstructed is chosen constant as $z_d = 100$ and the parameter ν is set to $\nu = 1$. The modes $\phi^{(k)}$ are presented in Fig. 1. As numerical tests show the dynamics of the ignition process are mainly determined by the mode ϕ_3. The numerical results of the optimal control problem (5) for different initial values y_0 are compared to those of the feedback synthesis with linear interpolation, sketched in Section 2 in Tab. 1. As one can see the quality of the controls is very good. As grid for the feedback synthesis the $a_0^{(j)} = (0,0,\alpha_j)^T\}$ are taken with α_j shown in Fig. 2.

Table 1: Solid fuel model: Feedback synthesis compared to optimal control

$\nu = 1$	$J(y_{tab})$	$J(y_{opt})$	$\lvert J(y_{tab}) - J(y_{opt})\rvert_{rel}$
$y_0 \equiv 0$	1.267396e+04	1.265863e+04	0.001211
$y_0 \equiv 100$	6.995310e+03	6.980263e+03	0.002156
$y_0 \equiv 1000$	1.336676e+05	1.324065e+05	0.009524
$y_0 = 1\chi_{(x\geq 0)}(x)$	1.258975e+04	1.257442e+04	0.001219
$y_0 = 1000\chi_{(x\geq 0)}(x)$	3.957604e+04	3.952468e+04	0.001299
$y_0 = 10000\chi_{(x\geq 0)}(x)$	4.652615e+06	4.526897e+06	0.027771

The instantaneous stabilization method (9) is applied to the control of laminar unsteady flow around a cylinder in a two-dimensional spatial domain Ω. The control gain is given by the Stokes flow $z_d\chi_{\Omega_s}$ in an observation volume Ω_s behind the cylinder. In order to put this problem into the framework developed in Section 3 we define A to be the Stokes solution operator in the space $V := \{H^1(\Omega,\mathbb{R}^2); \mathrm{div} v = 0\}$ and the nonlinearity as $n = n(y,\nabla y) := -(y\nabla)y$. In this setting the state equation (1) in variational formulation may be rewritten as Burgers equation in the space V, i.e. find $y(t) \in g(t) + V$ with $y(0) = y_0$ and

$$\dot{y} + Ay = n, \tag{12}$$

where g contains the Dirichlet boundary conditions. The suboptimal control scheme (10) applied to the control of flow around a cylinder is now given by the dynamical system

$$\dot{y} + Ay = n - \frac{\rho}{h}BB^*(y - z_d\chi_{\Omega_s}) - \rho BB^*(n - f), \quad u(0) = u_0, \tag{13}$$

where $f := Az_d\chi_{\Omega_s}$. We note that the operator B is the self-adjoint solution operator of the quasi-Stokes problem in V, i.e.

$$w \;=\; Bh \quad \Longleftrightarrow \quad (I + hA)w = h \quad \text{in } V.$$

As computational domain we choose the configuration taken for the two-dimensional benchmark tests in [11]. As numerical solver we use an extension of the Navier-Stokes solver developed by Bänsch in [1]. As parameters we choose Re $=$ 100, $\nu = 1.e-2$ and $h = 0.0075$. The observation volume is taken as $\Omega_s = [4,6] \times [-1,1]$. The time histories of control and of the descent parameter ρ used in scheme (10) are documented in Tab. 2. In this table ρ^* denotes an approximation of the optimal descent parameter in the gradient algorithm [4,

Table 2: Time history of control and the gradient descent parameters ρ

Time	$0 \le t \le 1$	$1 \le t \le 5$	$5 \le t \le 10$	$10 \le t \le 20$
ρ	no control	0.1	$\rho \approx \rho^*$	no control
Time	$20 \le t \le 40$	$40 \le t \le 100$	$100 \le t \le 120$	$120 \le t \le 200$
ρ	$\rho \approx \rho^*$	no control	$\rho \approx \rho^*$	no control

Section 4]. In Fig. 4 the values of costs (a) and control costs (b) are shown. As one can see the costs decrease immediately when control is switched on. If for ρ an approximation of the optimal gradient step-size is used the additional decrease of cost is remarkable. Note, that since the scale of the plot is logarithmic the control cost are set to their last value $\neq 0$.

Fig. 3 shows the uncontrolled Navier-Stokes flow (b) together with the controlled flow (c) at $t = 2$. As one can see the recirculation in a neighbourhood of the observation volume Ω_s vanishes completely. This effect is further amplified when control action is kept on.

Further numerical investigations on the control of flow around a cylinder can be found in the papers of Choi [3] and Tang & al. [13].

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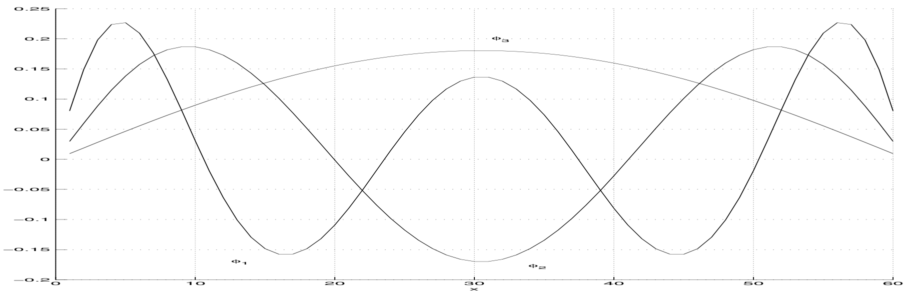

Figure 1: POD modes

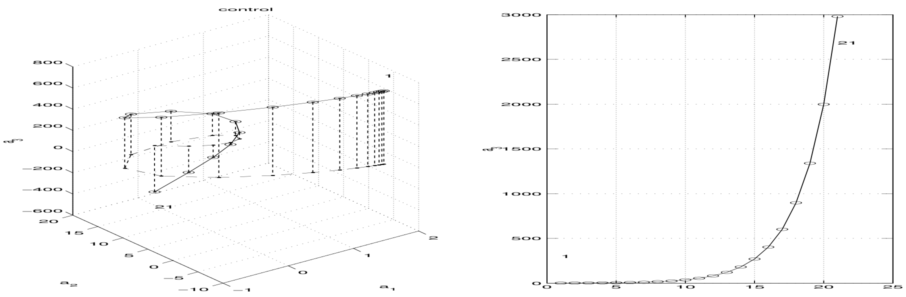

Figure 2: Stored controls $b_k \in \mathbb{R}^3$ (left) for $\alpha_k \in \mathbb{R}$ (right)

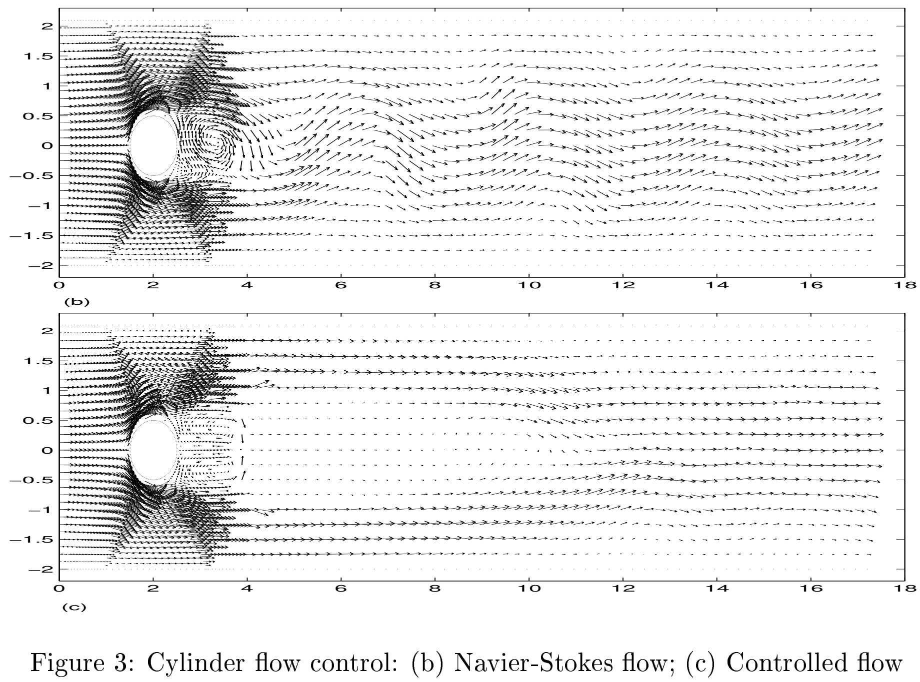

Figure 3: Cylinder flow control: (b) Navier-Stokes flow; (c) Controlled flow

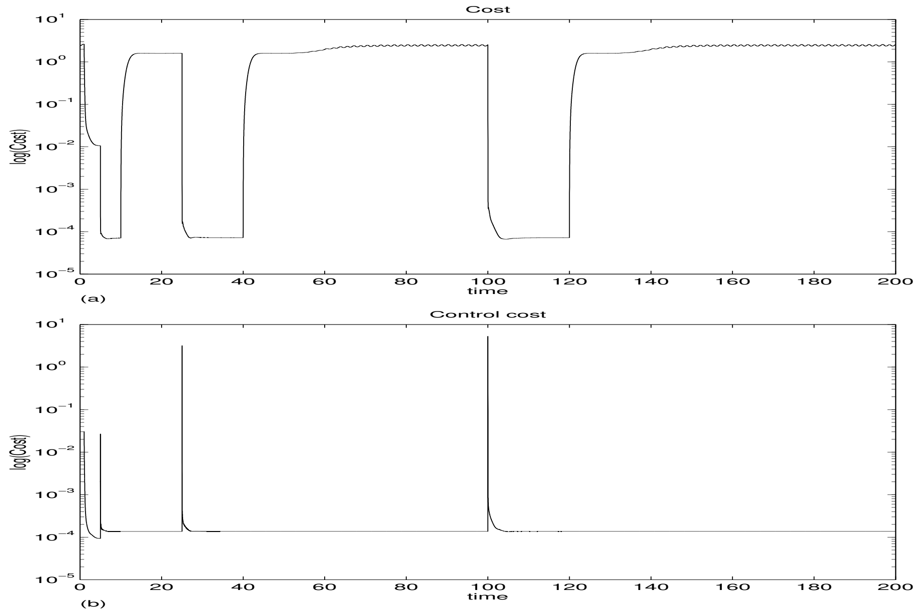

Figure 4: Cylinder control. Observation in $[4, 6] \times [-1, 1]$: (a) Cost; (b) Control Cost

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