On the numerical treatment of quasiminimal surfaces

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Abstract

Let u denote a quasiminimal surface (QMS) bounded by a polygon $\Gamma \in \mathbb{R}^q (q \geq 2)$ with N+3 distinct vertices in the sense of Shiffman. A linear finite element method is presented for the approximation of u. Furthermore, an error estimation in terms of the angles at the vertices of Γ and some examples of computed quasiminimal surfaces are given.

Typographically identified list of symbols

α - alpha
β - beta
Γ - uppercase gamma
γ - lowercase gamma
δ - delta
ϵ - epsilon
η - eta
θ - theta
κ - kappa
λ - lambda
μ - mu
ν - nu
τ - tau
∂ - partial
Ω - uppercase omega
ϕ - uppercase phi
φ - phi
π - pi

1 Introduction

Let Γ be a Jordan curve in $\mathbb{R}^q$. The classical problem of Plateau asks for a disc-type surface u of least area spanning Γ. If we introduce isothermal coordinates on u, we may parameterize u by a vector valued function $u(w) = (u_1(w), \ldots, u_q(w))$ over the unit disc B such, that $u_{|\partial B} \ : \ \partial B \ \to \ \Gamma$ is a parameterization of Γ and $|u_{x_1}|^2 - |u_{x_2}|^2 \ = \ (u_{x_1}, u_{x_2}) \ = \ 0 \ in \ B \ (w = x_1 + ix_2)$. Since u is assumed to be of last area among all surfaces spanning Γ, necessarily every parameterization $u(w)$ of u is a stationary value of the area functional

$$A(v) \ := \ \int_B |v_{x_1} \times v_{x_2}| \ dx \tag{1}$$

in the class of all admissible parameterizations $v(w)$ over the unit disc B of u. This variational argument leads to the following system of nonlinear partial differential equations for $u(w)$.

$$\begin{array}{rcl} \Delta u & = & 0 \quad \text{in } B \\ |u_{x_1}|^2 - |u_{x_2}|^2 \ = \ (u_{x_1}, u_{x_2}) & = & 0 \quad \text{in } B \\ u_{|\partial B} \text{ is a parameterization of } \Gamma & & \end{array} \tag{2}$$

On the other hand, every solution of (2) will parameterize a surface u of vanishing mean curvature (away from branch points where $\nabla u(w) = 0$) spanning the curve Γ. Since, necessarily, a surface of least area spanning Γ must have mean curvature 0, the system (2) may be considered as the Euler-Lagrange equations of Plateau's minimization problem. Every solution of (2) is called *minimal surface spanning* Γ. But (2) no longer requires the surface u to be absolutely area- minimizing. In general, solutions of (2) may have branch points, self-intersections, and be physically unstable - properties that would not be expected for area-minimizing surfaces spanning Γ.

Disc-type minimal surfaces spanning a closed polygon Γ correspond in a one to one manner to the critical points of Courant's function $d(\kappa)$. Therefore, Plateau's problem for the polygon Γ is connected with the critical points of a function with finitely many variables. The function $d(\kappa)$ is defined as follows. Let Γ be a polygon with N+3 distinct vertices $a_1, \ldots, a_{N+3}$ $(1 \le j \le N+3)$. Consider mappings $v : \overline{B} \to \mathbb{R}^q$ of class $C^0(\overline{B}, \mathbb{R}^q) \cap C^2(B, \mathbb{R}^q)$ such that $v(-1) = a_{N+1}, \ v(-i) = a_{N+2}, \ v(1) = a_{N+3}$ and $v(e^{i\kappa_j}) = a_j \ (1 \le j \le N+3)$, where $\kappa = (\kappa_1, \ldots, \kappa_N)$ is an N-tuple of parameter values κ_j satisfying $0 < \kappa_1 < \ldots < \kappa_N < \pi$. Let $F'(\kappa)$ be the class of such mappings which map the arc $\overline{\gamma}_j := \{e^{i\phi}; \kappa_j \le \phi \le \kappa_{j+1}\}$ weakly monotonically onto the interval $[a_j, a_{j+1}]$ on the straight line $a_j + \Gamma_j$ through the points a_j, a_{j+1} $(\kappa_j \equiv \kappa_l \ if \ j \equiv l \ mod \ N+3)$. Let

$$D(v) := \int_B |\nabla v|^2 dx \tag{3}$$

denote Dirichlet's integral. We set

$$d(\kappa) := \inf_{v \in F'(\kappa)} D(v). \tag{4}$$

Then, problem (4) has a unique solution $v(\kappa)$, called *minimal vector* (see [5, ch. 6]). The function $d(\kappa)$ is known to be of class C^1. Unfortunately it is unknown whether d is of class C^2, therefore Courant's function is not suited to develop a Morse theory for polygonal minimal surfaces. In the paper of Marx [15] based on a joint work of Marx and Shiffman concerning polygonally bounded minimal surfaces the Marx-Shiffman function $\Theta(\kappa)$ is introduced. Marx and Shiffman have enlarged the class $F'(\kappa)$ to the class $F(\kappa)$ of mappings which map the arc γ_j into the straight lines $a_j + \Gamma_j$. We set

$$\Theta(\kappa) := \inf_{v \in F(\kappa)} D(v). \tag{5}$$

The unique solution $u(\kappa)$ of problem (5) is called *quasiminimal surface*. Then, we clearly have $d(\kappa) \geq \Theta(\kappa)$ and we can have $d(\kappa) > \Theta(\kappa)$ for certain values of κ (see [13]). In [15] it is stated that $\Theta(\kappa)$is of class C^∞. In [8], Heinz proved that $\Theta(\kappa)$ is real analytic. Moreover, in [9] Heinz proved that the critical points of $\Theta(\kappa)$ correspond to solutions of a generalized Plateau problem for Γ (*generalized* means ; the solution $u(\kappa)$ can overshoot the vertices and we only know that $u(\gamma_j) \subset a_j + \Gamma_j$). The Morse index of such generalized solutions was computed by Sauvigny [17] by studying the second derivative of the function $\Theta(\kappa)$. Note, that the two functions $d(\kappa)$ and $\Theta(\kappa)$ are closely connected as they coincide in the critical points of $d(\kappa)$ (see [16]). In this context it would be desirable to determine classes of polygons for which the function $\Theta(\kappa)$ has only finitely many critical points. For arbitrary polygons this question is still open. For a special class of extreme polygons Sauvigny had fully investigated these question ([16]). To formulate his theorem we have to introduce some definitions. A polygon $\Gamma \subset \mathbb{R}^q$ is called extremal iff all vertices a_j lie on the boundary of a bounded convex set in $\mathbb{R}^q$. Let $\alpha_j\pi \in (0, \pi)$ denote the outer angle of Γ at the vertex a_j. Then the quantity

$$k(\Gamma) := \sum_{j=1}^{N+3} \alpha_j \pi \tag{6}$$

is called the total curvature of the polygon Γ. $\omega \in \overline{B}$ is called a branch point of the mapping $u : \overline{B} \to \mathbb{R}^q$ iff $\nabla u(\omega) = 0$. With these definitions we have

Theorem 1.1 *Let $\Gamma \subset \mathbb{R}^q$ $(q \geq 3)$ be an extremal polygon of total curvature*

$$k(\Gamma) < 4\pi \ (q = 3) \text{ or } k(\Gamma) < \frac{10}{3}\pi \ (q \geq 4). \tag{7}$$

Then Γ bounds exactly one minimal surface u which is free of branch points in $\overline{B}$.

In the present paper we are concerned with the first step in tackling the generalized Plateau problem numerically, namely the numerical evaluation of $\Theta(\kappa)$. Another paper of the author [12] deals with the numerical evaluation of $\nabla\Theta$ and $D^2\Theta$. With the help of Newton's method we are then able to compute polygonally bounded minimal surfaces via zeroes of $\nabla\Theta$. The computation of the eigenvalues of $D^2\Theta$ yields the Morse index of the minimal surface u and we are able to decide whether the minimal surface is stable or not.

The paper is organized as follows. In chapter 2 we introduce the basic notations. Moreover we summarize important results on quasiminimal surfaces. In view of the implementation of a linear finite element method for the approximation of the variational problem (5) we will introduce a slightly modified form of the variational problem (5) in chapter 3. Furthermore we give existence and uniqueness results for the modified problem and it turns out, that the unique solutions of problem (5) and of the modified problem coincide *a.e.* on $\overline{B}$. Chapter 4 deals with the discrete version of the modified variational problem. To obtain a conforming finite element method we construct a suitable finite element space. Furthermore, we proof existence and uniqueness for the solution of the modified problem. With the help of a suitable interpolation operator we are able to proof an error estimation for the unique discrete solution in terms of the geometry of the problem in chapter 5. In chapter 6 the finite element implementation of the discrete variational problem is described and in chapter 7 some numerical examples are presented.

2 Notation and preparations

Let $B := \{w = x_1 + ix_2; |w| < 1\} \equiv \{x = (x_1, x_2); |x| < 1\}$ denote the unit disc and let $\Gamma \subset \mathbb{R}^q$ $(q \geq 2)$ be a closed polygon with $N+3$ distinct vertices $a_1, \ldots, a_{N+3}$ arranged in cyclical order. With

$$\varphi_k := \frac{a_{k+1} - a_k}{|a_{k+1} - a_k|} \quad (1 \leq k \leq N+3; a_{N+4} := a_1) \tag{8}$$

we assume the pairs φ_k, φ_{k+1} to be linearly independent for $1 \leq k \leq N+2$. With S_k, T_k we denote the reflections at the lines $\Gamma_k : t\varphi_k$ $(t \in \mathbb{R})$, $a_k + \Gamma_k$ $(1 \leq k \leq N+3)$, respectively. Let $\kappa = (\kappa_1, \ldots, \kappa_N) \in \mathbb{R}^N$ denote an $N-tuple$ of parameter values κ_j satisfying

$$0 < \kappa_1 < \ldots < \kappa_N < \pi. \tag{9}$$

It comes from the equivariance of problem (5) under the action of the three dimensional conformal group of the unit disc that we have to fix

$$\kappa_{N+1} := \pi, \quad \kappa_{N+2} := \frac{3\pi}{2}, \quad \kappa_{N+3} := 2\pi. \tag{10}$$

We set

$$\gamma_k := \{w = e^{i\phi}; \kappa_k < \phi < \kappa_{k+1}\} \quad (1 \leq k \leq N+3;\ \kappa_{N+4} \equiv \kappa_1 + 2\pi) \tag{11}$$

and abbreviate $\beta_k := e^{i\kappa_k}$ $(1 \leq k \leq N+3)$. For $m \in \mathbb{N} \cup \{0\}$ and $1 \leq p \leq \infty$, $H^{m,p}(B)$ denotes the Sobolev space of all Lebesgue-measurable functions having

p-integrable derivatives up to order m equipped with the usual norm $\|\cdot\|_{m,p,B}$. With $|\cdot|_{m,p,B}$ we denote the corresponding seminorms. $H^{m,p}(B,\mathbb{R}^q)$ denotes the space of vector-valued functions $f=(f_1,\ldots,f_q)$ such that $f_i\in H^{m,p}(B)$ $(1\le i\le q)$. For $p=2$ we we use the notation $H^m\equiv H^{m,2}$ and $\|\cdot\|_{m,B}\equiv\|\cdot\|_{m,2,B}$. For $v\in H^1(B,\mathbb{R}^q)$

$$D(v):=\frac{1}{2}\int_B|\nabla v|^2dx$$

denotes Dirichlet's integral. Introducing the function space

$$F(\kappa)=\{v\in C^2(B,\mathbb{R}^q)\cap C^0(\overline{B},\mathbb{R}^q);v(\gamma_k)\subset a_k+\Gamma_k\} \tag{12}$$

the quasiminimal surface problem (5) can be written as

$$\Theta(\kappa)=\inf_{v\in F(\kappa)}D(v). \tag{13}$$

For fixed κ Heinz proved the existence of a unique solution $u(\kappa)$ of the variational problem (13). Moreover, Heinz proved that $u(\kappa)$ is analytical on $\overline{B}\setminus\{\beta_1,\ldots,\beta_{N+3}\}$. Furthermore, Heinz showed that the solution of (13) solves the following elliptic system of partial differential equations ([8])

$$\begin{array}{rcll}\Delta u(\kappa) & = & 0 & in\ B\\ u(\kappa)(\gamma_k) & \subset & a_k+\Gamma_k & 1\le k\le N+3\\ (\partial_r u(\kappa),\varphi_k) & = & 0 & on\ \gamma_k,\quad 1\le k\le N+3.\end{array} \tag{14}$$

Here ∂_r denotes the derivative in radial direction. Since for our purposes κ is fixed this argument will be dropped in the sequel. Next, we state some properties of the solution u of (14). Since u is assumed to be continuous on $\overline{B}$ we necessarily have

$$u(\beta_k)=a_k\qquad(1\le k\le N+3).$$

At the vertices of the polygon Γ the normal to the contour has a jump discontinuity. Therefore, the boundary conditions in (14) are nonlinear and singularities have to be expected at the points β_k $(1\le k\le N+3)$. Let $\alpha_k\pi\in(0,\pi)$ denote the outer angle of Γ at the vertex a_k and let $q=3$. Then, u at β_k locally maps either into the angle $\alpha_k\pi$ or into the angle $(1-\alpha_k)\pi$. In any case u locally may wind around the vertex a_k m_k times ($m_k\in\mathbb{N}\cup\{0\}$). Heinz has fully investigated this situation ([8]) and he proved the following singular expansions at the vertices of Γ, which are also valid for the case $q>3$. With $\partial_w:=\frac{1}{2}(\partial_{x_1}-i\partial_{x_2})$ we have for the solution u of (14) (see also [11])

Lemma 2.1 *For every $1\le k\le N+3$ there exists an integer $1\le p_k\le q$, p_k exponents $-1<p_1^k<\ldots<p_{p_k}^k\le 0$ and an $\epsilon_0>0$ such that for every ϵ with $0<\epsilon\le\epsilon_0$*

$$u_w(w)=\sum_{j=1}^{p_k}f_j^k(w)\ (w-\beta_k)^{p_j^k}\quad\forall w\in B_\epsilon(\beta_k)\cap\overline{B}. \tag{15}$$

The complex-valued coefficient functions f_j^k are holomorphic in $B_{\epsilon_0}(\beta_k)$.

Integration of (15) yields for $1 \leq k \leq N+3$

$$u(w) = a_k + Re(\sum_{j=1}^{p_k} g_j^k(w)\ (w - \beta_k)^{p_j^k+1}) \quad \forall w \in B_\epsilon(\beta_k) \cap \overline{B}, \tag{16}$$

with suitable coefficient functions $g_j^k(w)$. The exponents p_j^k are directly related to the eigenvalues of $S_{k-1}S_k^{-1}$ and therefore to the angles of the polygon Γ at a_k ($1 \leq k \leq N+3$) (see [10]). For what follows we define

$$p^k := min\{p_j^k\ ;\ 1 \leq j \leq p_k\}, \tag{17}$$

$$p_{min} := min\{p^k\ ;\ 1 \leq k \leq N+3\}. \tag{18}$$

With the help of the expansions (15) and (16) we infer the the following asymptotic behavior of u at β_j ($1 \leq j \leq N+3$).

$$u_w(w) \ = \ a(w-\beta_j)^{p^j+m^j} \ + \ O(|w-\beta_j|^{p^j+m_j+s}\ (w \to \beta_j) \tag{19}$$

with a complex constant $a \neq 0$ and $s > 0$. In the case $m_j > 0$ there is a branch point at β_j. In the case $q = 3$ we have from [10] and [18]

$$p_1^k = 0, \quad p_2^k = -\alpha_k, \quad p_3^k = \alpha_k - 1 \qquad (1 \leq k \leq N+3).$$

In this context we also refer to Dziuk's investigations concerning the boundary regularity of polygonally bounded minimal surfaces [6]. For $0 < \epsilon \leq \epsilon_0$ with a suitable ϵ_0 (compare Lemma 2.1) we set

$$\Omega_\epsilon := \cap_{k=1}^{N+3}\ (\overline{B} \setminus B_\epsilon(\beta_k)). \tag{20}$$

Since with φ_k from (8), the pairs $\varphi_k,\ \varphi_{k+1}$ ($1 \leq k \leq N+2$) are assumed to be linearly independent we have $p_{min} \in (-1, 0)$. With (15) and (16) we obtain

$$u \in H^{1,p}(B, \mathbb{R}^q) \qquad \forall\ 2 \leq p < -\frac{2}{p_{min}}. \tag{21}$$

Since u is analytical in $\overline{B} \setminus \{\beta_1, \ldots, \beta_{N+3}\}$ we have

$$u \in H^2(\Omega_\epsilon, \mathbb{R}^q) \qquad \forall\ 0 < \epsilon \leq \epsilon_0. \tag{22}$$

3 A modified variational problem

To obtain a conforming finite element method for the numerical approximation of the solution u of (13) we have to modify problem (13). For this purpose we replace the function space $F(\kappa)$ in (13) by

$$X := \{v \in H^1(B, \mathbb{R}^q)\ ;\ v(\gamma_k) \subset a_k + \Gamma_k \quad (1 \leq k \leq N+3)\}. \tag{23}$$

The corresponding space of test functions is

$$X_0 := \{v \in H^1(B, \mathbb{R}^q)\ ;\ v(\gamma_k) \subset \Gamma_k \quad (1 \le k \le N+3)\}. \tag{24}$$

Now we introduce the modified variational problem

$$\Theta = \inf_{v \in X} D(v). \tag{25}$$

For the variational problem (25) we have

Theorem 3.1 *Problem (25) has a unique solution. The solution u coincides with the solution u of (13) a.e. on* $\overline{B}$

Proof : First, we show that the solution u of (13) is also a solution of (25). Therefore we have to verify

$$D(u) - D(v) \ \le\ 0 \qquad \forall\ v \in X, \tag{26}$$

that means u solves (25) if

$$\int_B \nabla u\ \nabla v\ dx \ =\ 0 \qquad \forall\ v \in X_0. \tag{27}$$

To proof (27) we choose a smooth, scalar cut-off function η_ϵ with

$$\eta_{\epsilon_{|_{\overline{B}\setminus\Omega_{\frac{\epsilon}{2}}}}} \ =\ 0,\ \eta_{\epsilon_{|_{\overline{\Omega}_\epsilon}}} \ =\ 1 \text{ and } |\nabla\eta_\epsilon| \ \le\ \frac{c}{\epsilon}\ (c>0).$$

Since u is analytical in $\overline{\Omega}_\epsilon$, we have $\eta_\epsilon \partial_r u \ \in\ C^1(\overline{B}, \mathbb{R}^q)$. Therefore, the boundary conditions in (14) yield

$$0 \ =\ \int_{\partial B} \eta_\epsilon\ (\partial_r u, v)\ dS(x) \qquad \forall\ v \in X_0. \tag{28}$$

Integration by parts together with (14) yields

$$0 \ =\ \int_B \nabla u\ \nabla v\ \eta_\epsilon\ dx \quad +\quad \int_B \nabla u\ \nabla\eta_\epsilon\ v\ dx \ =:(a)\ +\ (b). \tag{29}$$

Since $\nabla u\ \nabla v\ \eta_\epsilon \ \to\ \nabla u\ \nabla v\ (\epsilon \to 0)$ a.e. in B, (a) tends to $\int_B \nabla u\ \nabla v\ dx\ (\epsilon \to 0)$. To obtain $(b) \to 0\ (\epsilon \to 0)$ choose s and r with $1 < s < \frac{2}{1-p_{min}}$ and $s^{-1}+r^{-1} = 1$, p_{min} from (18). Using (19), the embedding $H^1(B) \hookrightarrow L^r(B) \quad \forall\, 1 \le r < \infty$ and Hölder's inequality, we obtain for a suitable $1 \le k \le N+3$

$$\begin{aligned}
|\int_B \nabla u \nabla\eta_\epsilon\ v\ dx| \quad &\le \quad \sum_{j=1}^{N+3} \int_{(B_\epsilon \setminus B_{\frac{\epsilon}{2}})(\beta_j)\cap B} |\nabla u\ \nabla\eta_\epsilon\ v|\ dx \\
&\le \quad \frac{c}{\epsilon}\ \|v\|_{1,B} \left(\int_{(B_\epsilon \setminus B_{\frac{\epsilon}{2}})(\beta_k)\cap B} |x - \beta_k|^{p_{min} s}\ dx \right)^{\frac{1}{s}} \\
&\le \quad c\ \|v\|_{1,B}\ \epsilon^{(p_{min}s+2-s)\frac{1}{s}} \ \to 0\ (\epsilon \to 0),
\end{aligned}$$

since $p_{min}s - s > -2$. The uniqueness of a solution of (25) is obvious. $\diamond$

Remark 3.2 *We do not have* $F(\kappa) \subseteq X$.

Without proof we state a Poincaré-like inequality for functions in X_0.

Lemma 3.3

$$\exists\ c > 0\ \forall\ v \in X_0\ :\ \|v\|_B\ \leq\ c\ |v|_{1,B}\ \diamond \tag{30}$$

4 The discrete variational problem

To obtain a conforming finite element method approximating problem (25) we first have to construct the finite dimensional affine space X_h in such a way that $X_h \subseteq X$. For this purpose let $h > 0$ denote the gridsize of a given regular, conforming triangulation τ of the unit-disc. We set

$$\begin{aligned} B_h &:= \cup\{T \in \tau\}, & (31)\\ Y_h &:= \{v \in C^0(\overline{B}_h, \mathbb{R}^q)\ ; v_{|_T} \text{ linear } \forall\ T \in \tau\}, & (32)\\ X_h &:= \{v \in Y_h\ ; v(\gamma_k \cap \partial B_h) \subset a_k + \Gamma_k\ (1 \leq k \leq N+3)\}. & (33)\end{aligned}$$

To obtain the inclusion $X_h \subseteq X$ we have to define a suitable prolongation operator for functions of Y_h. Let η denote the outer unit normal vector to ∂B. For $x \in \mathbb{R}^2$, let $a(x) := dist(x, \partial B_h)$ measure the distance between x and ∂B_h. The prolongation operator is defined as follows

$$\begin{aligned} Pro: \qquad Y_h &\ \to\ H^1(B, \mathbb{R}^q) \\ Pro(v)(x) &\ := v(x),\ x \in \overline{B}_h \\ Pro(v)(x) &\ := v(x - a(x)\eta),\ x \in \overline{B} \setminus \overline{B}_h \end{aligned} \tag{34}$$

The definition of Pro implies that $Pro(v) \in X$, if $v \in X_h$. Because of the trace-theorem for functions of class H^1, Pro is a continuous operator. From now on we do not longer distinguish between v and $Pro(v)$. The discrete analogon to the variational problem (25) can now be written as

$$\Theta_h\ =\ \inf_{v \in X_h}\ D(v). \tag{35}$$

For the discrete variational problem (35) we have

Lemma 4.1 *The discrete variational problem (35) has a unique solution* u_h. *For* u_h *the following variational equality holds true*

$$\int_{B_h} \nabla u_h \nabla v\ dx\ =\ 0 \qquad \forall\ v \in X_{h0}, \tag{36}$$

where X_{h0} *denotes the discrete analogon to* X_0 *with respect to (24).*

With the help of the Lax-Milgram lemma [1, ch. 4] and Lemma 3.3 the proof is straight forward. $\diamond$

We finish this chapter with giving a description of how to get a triangulation τ of the unit disc in practice. Therefore we triangulate the unit-disc as shown in Figure 1 and denote the so obtained macro-triangulation with τ_0 . For $j \geq 1$, we recall two possibilities for obtaining a refinement τ_j of a given triangulation τ_{j-1}. In the first case some triangles $T \in \tau_{j-1}$ are partitioned into four congruent triangles (Figure 2). To obtain a conformal triangulation τ_j, maybe some neighboring-triangles, having a common edge with T, have to be halved. A second strategy is based on a bisecting algorithm (Figure 3). For a more detailed discussion on this topic we refer to [2]. The latter strategy yields a sequence $\{\tau_j\}_{j\in\mathbb{N}}$ of regular triangulations, whereas the first strategy may yield a sequence of triangulations with degenerating triangles. If an edge of a triangle, which is to be divided, contains two boundary nodes, proceed like in Figure 4. In this context regular means, that each τ_j $(j \in \mathbb{N})$ is conforming and the sequence $\{\tau_j\}_{j\in\mathbb{N}}$ is stable in terms of the smallest occurring angle of the macro-triangulation τ_0.

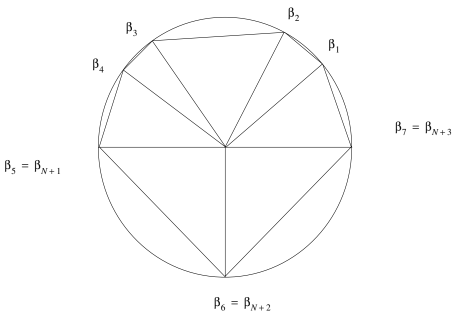

Figure 1:

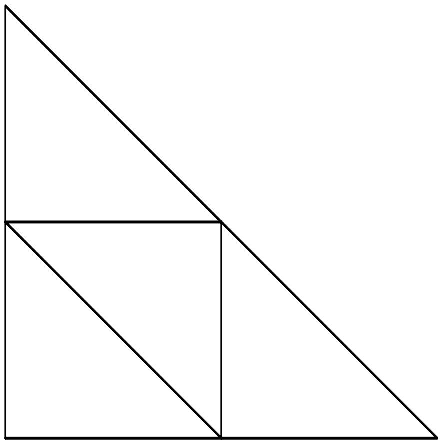

Figure 2:

5 The error estimation

In the present chapter we derive an error estimation for the difference between the solutions u of (13) and u_h of (35) in the H^1-norm. For this purpose we first define a suitable interpolation operator which maps vector valued functions of class $H^{m,p}$ into Y_h (see 32) for $m - \frac{2}{p} > 0$.

Definition 5.1 *Let τ be a triangulation with gridsize $h > 0$. Let $m \in \mathbb{N}, p \in \mathbb{R}$ with $2 < p \leq \infty$ and $m - \frac{2}{p} > 0$.*

$$I_h : \quad H^{m,p}(B, \mathbb{R}^q) \quad \to Y_h \tag{37}$$

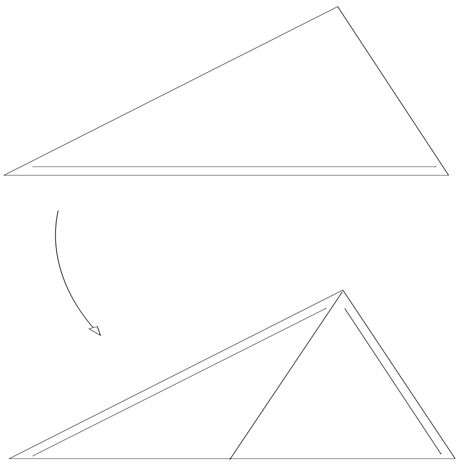

Figure 3:

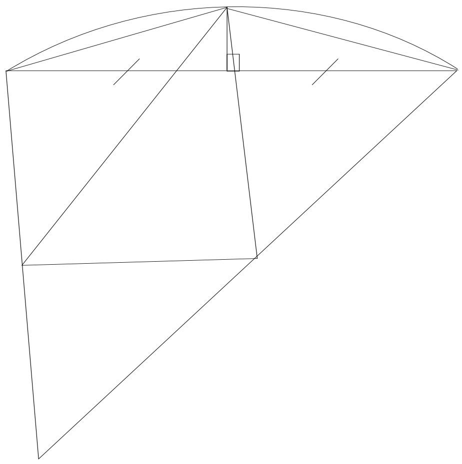

Figure 4:

$$I_h v(x) \qquad := x \quad \textit{on the nodes } x \textit{ of } \tau . \diamond$$

Definition 5.1 implies

$$I_h v \ \in \ X_h \ (X_{h0}) \quad iff \quad v \ \in \ X \ (X_0). \tag{38}$$

To estimate the interpolation error $\|u - I_h u\|_{1,B_h}$, we split B_h into two disjoint parts B_h^+, B_h^- , where

$$B_h^+ \ := \ \cup\{T \in \tau \text{ ; there is a } 1 \leq j \leq N+3 \text{ such that } \beta_j \text{ is a vertex of T}\} \tag{39}$$

and

$$B_h^- \ := \ B_h \setminus B_h^+.$$

In Lemma 5.2 we give an elementwise estimation of the interpolation error for the solution u of (13).

Lemma 5.2 *Let u be the solution of (13). For $T \subseteq \overline{B}_h^+$, the following estimation is valid*

$$\|u - I_h u\|_T \ + \ h|u - I_h u|_{1,T} \ \leq \ c \ meas(T)^{\frac{1}{2}+\frac{p-2}{2p}} \ |u|_{1,p,T} \qquad \forall \ 2 \leq p < -\frac{2}{p_{min}}, \tag{40}$$

and for $T \subseteq \overline{B}_h^-$, we obtain

$$\|u - I_h u\|_T \ + \ h|u - I_h u|_{1,T} \ \leq \ c \ h(T)^2 \ |u|_{2,T} \tag{41}$$

The constant c is positive and independent of u and $h(T)$.

Here, $h(T) := diam(T)$. The proof of lemma 5.2 follows from Theorem 3.1.5 in [4] under consideration of (21) and (22). $\diamond$
We are now able to state the major tool for estimating $\|u-u_h\|_{1,B_h}$, where u_h denotes the solution of the discrete variational problem (35).

Theorem 5.3 *With I_h from Definition 5.1 we have for the solution u of (13)*

$$\|u - I_h u\|_{B_h} \; + \; h|u - I_h u|_{1,B_h} \;\leq\; c\; h^{1+\frac{p-2}{p}}\; |u|_{1,p,B_h} \qquad \forall\; 2 \leq p < -\frac{2}{p_{min}} \tag{42}$$

with a constant $c > 0$ depending on ϵ_0 (compare Lemma 2.1) and linearly on N, where $N+3$ is the number of the vertices of the polygon Γ.

Proof : In B_h^+ we have with the help of (40) and Hölder's inequality for series under consideration of $\#\{T;\; T \subseteq \overline{B}_h^+\} = O(N)$

$$\begin{aligned} \|u - I_h u\|^2_{1,B_h^+} &\leq c \sum_{T \subseteq \overline{B}_h^+} h(T)^{\frac{2(p-2)}{p}}\; |u|_{1,p,B_h} \\ &\leq c(N)\; h^{\frac{2(p-2)}{p}}\; |u|^2_{1,p,B_h}. \end{aligned}$$

In B_h^- one has

$$\|u - I_h u\|^2_{1,B_h^-} \;\leq\; c\; h^2\; \left(|u|^2_{2,B_h^- \cap \Omega_{\epsilon_0}} + |u|_{2,B_h^- \setminus \Omega_{\epsilon_0}} \right).$$

The first term in the brackets is bounded by a positive constant depending on ϵ_0 from Lemma 2.1. To proof that the second term is of order $O(h^{2p_{min}})$ $(h \to 0)$ with p_{min} from (18) we first note that, since the triangulation τ is regular, there exists a constant $\delta > 0$ independent of the gridsize $h > 0$ with $dist(B_h^-, \beta_j) \geq \delta h$ $(1 \leq j \leq N+3)$. Differentiation of expansion (15) with respect to w then yields for $0 < h < 1$

$$\begin{aligned} |u|^2_{2,B_h^-} &\leq c \sum_{j=1}^{N+3} \int_{B_h^- \cap B_{\epsilon_0}(\beta_j)} |x - \beta_j|^{2(p_{min}-1} \; dx \\ &\leq c(N) \int_{\delta h}^{\epsilon_0} r^{2(p_{min}-1)+1}\; dr \\ &\leq c(N)\; h^{2p_{min}} \;+\; c(\epsilon_0). \diamond \end{aligned}$$

Therefore, the term in the brackets is of order $O(h^{2p_{min}})$ and the theorem is proved. We are now able to formulate the main theorem.

Theorem 5.4 *Let $h > 0$ denote the gridsize of a regular triangulation τ. Let u, u_h be the unique solutions of (13) and (35), respectively. Then we have*

$$\|u - u_h\|_{1,B_h} \;\leq\; h^{\frac{p-2}{p}}\; |u|_{1,p,B} \qquad \forall\; 2 \leq p < -\frac{2}{p_{min}} \tag{43}$$

with a constant $c > 0$ depending linearly on N.

Proof : With the help of Lemma 3.3 and Theorem 5.3 we obtain

$$\begin{aligned} \|u - u_h\|_{B_h} &\leq \|u - I_h u\|_{B_h} \;+\; \|I_h u - u_h\|_{B_h} \\ &\leq c\; \left(h^{1+\frac{p-2}{p}} |u|_{1,p,B} \;+\; |u - u_h|_{1,B_h} \right). \end{aligned}$$

Obviously it remains to estimate $|u-u_h|_{1,B_h}$. For $v \in X_h$ we have under consideration of the variational equalities (27) and (36)

$$|u-u_h|^2_{1,B_h} \;=\; \int_{B_H} \nabla(u-u_h)\nabla(u-v)\;dx \;-\; \int_{B\setminus B_h} \nabla u \nabla(v-u_h)\;dx.$$

Using Young's inequality, a straightforward estimation yields

$$|u-u_h|_{1,Bh} \;\leq\; c \inf_{v\in X_h} (|u-v|_{1,B_h} \;+\; |u|_{1,B\setminus B_h}). \tag{44}$$

Since $\text{meas}(B \setminus B_h) \leq ch^2$, it follows from Hölder's inequality that

$$\begin{aligned} |u|_{1,B\setminus B_h} \;&\leq\; \text{meas}(B\setminus B_h)^{\frac{p-2}{2p}}\; |u|_{1,p,B\setminus B_h} \\ &\leq\; c\; h^{\frac{p-2}{p}}\; |u|_{1,p,B} \qquad \forall\; 2 \leq p < -\frac{2}{p_{min}}. \end{aligned}$$

Setting $v := I_h u$ in (44), the assumption follows from Theorem 5.3. $\diamond$
We finish this chapter with some remarks.

Remark 5.5 *In the case $q \leq 3$ we have $p_{min} = \min_{1\leq k\leq N+3}\{-\alpha_k, \alpha_k - 1\}$, where $\alpha_k\pi$ denotes the outer angle at the vertex a_k of the polygon Γ $(1 \leq k \leq N+3)$. In this case we have a direct relation between the smallest angle of the polygon Γ and the order of convergence.* $\diamond$

Remark 5.6 *There are two possibilities for improving the order of convergence in (43). The first possibility is to refine locally at the singular points β_k $(1 \leq k \leq N+3)$. The second possibility is to consider instead of X_h the function space*

$$X_h \;\cup\; \langle h_1, \ldots, h_{N+3}\rangle,$$

where $h_k(x) \;:=\; a_k + |x-\beta_k|^{1+p^k}\varphi_k$ $(1 \leq k \leq N+3)$,p^k from (17) and φ_k from (8). For a more detailed analysis see for example [19, p. 257 ff]. $\diamond$

6 The Finite element implementation

In the present chapter we outline the implementation of a linear finite element method for the variational problem (13). It will turn out that the discrete variational problem (35) can be formulated as a minimization problem for a quadratic functional subjected to linear constraints.
Let NV, NR denote the number of nodes and boundary nodes, respectively of a triangulation τ with gridsize $h > 0$. Let $\{V_i;\; 1 \leq i \leq NV\}$ denote the set of nodes of τ. Let $\{b_i;\; 1 \leq i \leq NV\}$ denote the set of scalar, piecewise linear, continuous functions with the properties

$$\text{supp } b_i \;=\; \{T \in \tau;\; V_i \text{ is node of } \tau \text{ and vertex of } T\}$$

and

$$b_i(V_j) \; = \; \delta_i^j \qquad (1 \leq i, j \leq NV).$$

Therefore we have for Y_h from (32) $Y_h = span\{b_i; 1 \leq i \leq NV\}^q$ and it turns out that Dirchlet's integral for the functions

$$v(x) \; := \; \sum_{j=1}^{NV} v_j b_j(x) \quad (v_j \in \mathbb{R}^q)$$

is equal to a quadratic functional on $\mathbb{R}^{NVq}$, namely

$$D(v) \; = \; \frac{1}{2} \sum_{1 \leq i,j \leq NV} v_i^t l_{ij} v_j, \tag{45}$$

where $l_{ij} \; := \; \int_{B_h} \nabla b_i \nabla b_j \; dx \; (1 \leq i, j \leq NV)$ and $v_i = (v_i^1, \ldots, v_i^q) \in \mathbb{R}^q$.
For what follows we fix the following enumeration for the nodes $\{V_i; 1 \leq i \leq NV\}$ of the triangulation τ :

$$\begin{array}{lcll} V_i & := & \beta_i & (1 \leq i \leq N+3) \\ V_i & \in & \partial B & (N+4 \leq i \leq NR) \\ V_i & \in & B & (NR \leq i \leq NV). \end{array} \tag{46}$$

With this notation problem (35) can be formulated as

$$\begin{array}{ll} \min & {}_{v \in Y_h} \; D(v) \\ s.t. & i.) \; v_i \; = \; a_i \qquad (1 \leq i \leq N+3) \\ & ii.) \; v_i \; \in \; a_k + \Gamma_k \qquad (N+4 \leq i \leq NR \text{ and } V_i \in \gamma_k), \end{array} \tag{47}$$

Inserting (47) i.) into D we obtain a modified functional $h(v) = \frac{1}{2} v^t A v - b^t v + c$ of $(NV - N - 3)q$ variables. Here, $A \in M((NV - N - 3)q, \mathbb{R})$ denotes the stiffness matrix. We then can write (47) as an optimization problem with linear constraints,

$$\begin{array}{ll} \min & {}_{v \in \mathbb{R}^{(NV-N-3)q}} \; h(v) \\ s.t. & Ev = f. \end{array} \tag{48}$$

Here, $Ev = f$ assembles that every nodal vector v_i which corresponds to a boundary node $V_i \in \gamma_k$ of the triangulation τ is in $a_k + \Gamma_k$. Introducing the Lagrange multiplier $\lambda \in \mathbb{R}^{(NR-N-3)(q-1)}$, (48) may be transformed into an unconstraint problem

$$\min_{((v,\lambda) \in \mathbb{R}^{(NV-N-3)q} \times \mathbb{R}^{(NR-N-3)(q-1)})} (h(v) + (\lambda, Ev - f)) \tag{49}$$

and it is a well known fact (see [7]) that the solution u of (49) is characterized by the existence of a $\lambda^* \in \mathbb{R}^{(NR-N-3)(q-1)}$ with

$$\begin{pmatrix} A & E^t \\ E & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda^* \end{pmatrix} = \begin{pmatrix} b \\ f \end{pmatrix}. \tag{50}$$

Since E is surjective, (50) has a unique solution (u, λ^*) which may be calculated from

$$u \;=\; A^{-1}(b - E^t\lambda^*) \tag{51}$$

and

$$EA^{-1}E^t\lambda^* \;=\; -(f - EA^{-1}b). \tag{52}$$

Since A is positive definite and E is surjective, $EA^{-1}E^t$ is also positive definite. Thus, (52) can approximately be solved by using a usual conjugate-gradient-type algorithm. The solution λ^* of (52) has then to be inserted into (51) to obtain u.
A second approach for solving problem (48) is due to Luenberger (see [14]), who proposed to look for a solution of

$$\begin{pmatrix} A & E^t \\ E & 0 \end{pmatrix}^2 \begin{pmatrix} u \\ \lambda^* \end{pmatrix} = \begin{pmatrix} A & E^t \\ E & 0 \end{pmatrix} \begin{pmatrix} b \\ f \end{pmatrix}, \tag{53}$$

investing only one vector-matrix multiplication per iteration step of the conjugate gradient algorithm for system (53).
We finish this chapter with some remarks.

Remark 6.1 *Let h denote the gridsize of the triangulation τ. The condition of the systems (51) and (52) is of order $O(h^{-2})$, whereas the condition of problem (53) is of order $O(h^{-4})$. In (51), (52) we have to solve linear systems with the coefficient matrix A, so that a lot of preconditioning techniques are available to improve the order of convergence (for details see [3]).* $\diamond$

Remark 6.2 *We may introduce the augmented Lagrangian*

$$L_r(v, \lambda) \;:=\; h(v) + (\lambda, Ev - f) + \frac{r}{2}|Ev - f|^2 \qquad (0 \le r < \infty)$$

for the problem (48), which may be interpreted as a preconditioned access to system (50) (see [7] for an extensive analysis of this approach). $\diamond$

Remark 6.3 *Because of the geometrical nature of the constraints in (47) we may choose a basis of the affine space X_h in such a way that the problem (47) transforms into an unconstraint minimization problem for Dirichlet's integral over the space X_h. For this purpose, the affine basis of the space X_h has to chosen as follows. Every shape function corresponding to a boundary node $V_i \in \gamma_k$ is of the form $a_k + b_i\varphi_k, b_i$ a scalar shape function $(1 \le k \le N+3; N+3 \le i \le NR)$, φ_k from (8) and γ_k from (11). Shape functions corresponding to interior nodes are defined as usual. The resulting linear system of equations is positive definite and is of dimension $q(N+3)(NV - NR) + (N+3)(NR - N - 3)$.* $\diamond$

7 Examples

In this chapter we give some examples of computed QMS in $\mathbb{R}^3$. All surfaces were computed according to (51) and (52), (53), Remark 6.2 and Remark 6.3, respectively with coinciding results. To solve the linear systems we have used a usual conjugate gradient method. The surfaces which we obtained via a selfadaptive strategy have been computed due to Remark 6.3. To obtain triangulations as regular as possible we used a more suitable normalization as in (10).

In figures 5 and 6 a QMS spanning a polygonal contour with five vertices and the corresponding triangulation are shown. The contour is similar to that one used by Lewerenz to show that Courant's function $d(\kappa)$ and Shiffman's function $\Theta(\kappa)$ do not coincide for certain values of κ (see [13]). In figures 7 and 8 we show a self-intersecting QMS with eight vertices and the corresponding triangulation. In figure 9 and figure 10 a QMS spanning a contour with twenty vertices and the corresponding triangulation is shown. Figure 11 and figure 12 show a QMS in a polygon with six vertices together with a triangulation which has been obtained via a selfadaptive strategy. The refinement strategy is as follows. The normal vectors of the images of two neighboring triangles are compared. The triangles are assigned to be halved if the angle of the two normal vectors is larger than a prescribed tolerance. The figures 13 and 14 show a twisted knot together with a triangulation obtained via another selfadaptive strategy. This strategy works as follows. The gradient of the parameterization of the surface in the nodal points of the triangulation has to be computed. A triangle is assigned to be halved if the gradient in a node which corresponds to the triangle is lower than a prescribed value. The area of refinement locates a branch point.

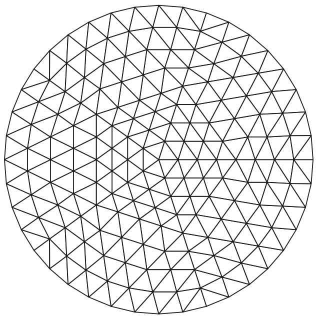

Figure 5:

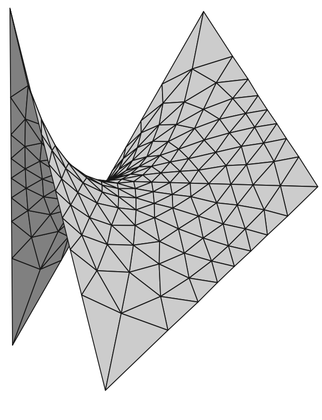

Figure 6:

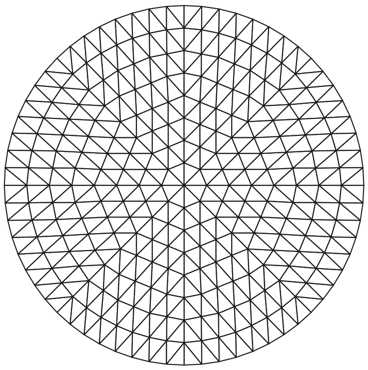

Figure 7:

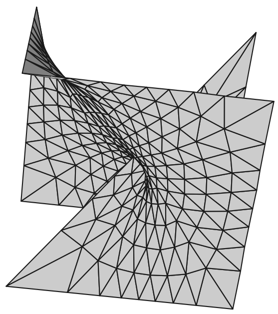

Figure 8:

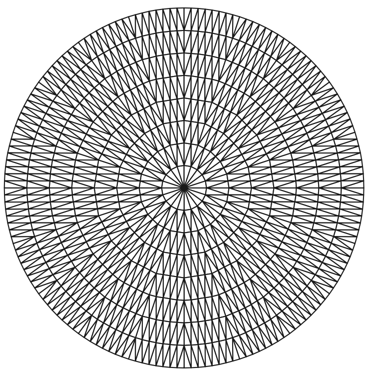

Figure 9:

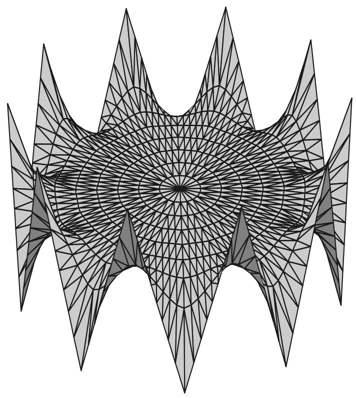

Figure 10:

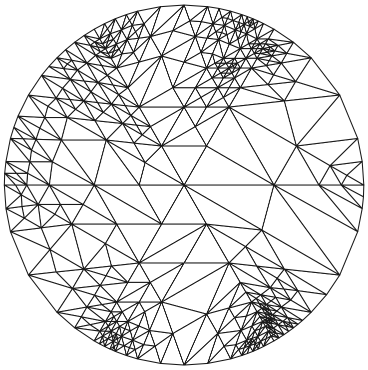

Figure 11:

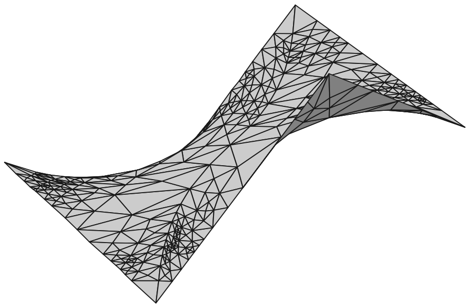

Figure 12:

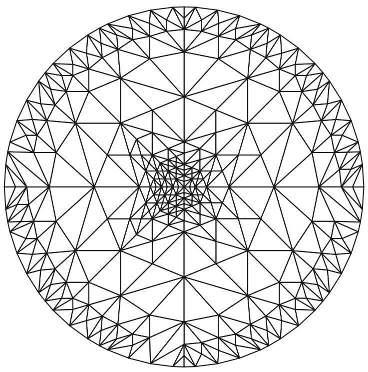

Figure 13:

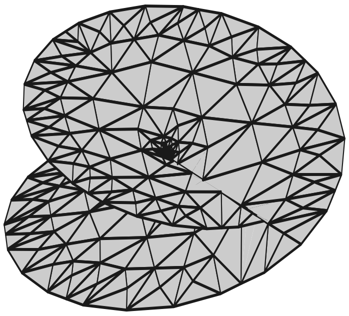

Figure 14:

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