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Convergence of a finite element approximation to a state constrained elliptic control problem

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Contents

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Abstract: We consider an elliptic optimal control problem with pointwise state constraints. The cost functional is approximated by a sequence of functionals which are obtained by discretizing the state equation with the help of linear finite elements and enforcing the state constraints in the nodes of the triangulation. The corresponding minima are shown to converge in L^2 to the exact control as the discretization parameter tends to zero. Furthermore, error bounds for control and state are obtained both in two and three space dimensions. Finally, we present numerical examples which confirm our analytical findings.

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1 Introduction

The aim of this paper is to analyze a finite element discretization of a control problem with pointwise state constraints. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded, convex domain with a smooth boundary. For a given function $u \in L^2(\Omega)$ we denote by $y = \mathcal{G}(u)$ the solution of the Neumann problem

$$\begin{aligned} -\Delta y + y &= u \quad \text{in } \Omega, \\ \partial_\nu y &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Here ν denotes the outward pointing unit normal to $\partial\Omega$. It is well known that $y \in H^2(\Omega)$ and

$$\|y\|_{H^2} \leq C\|u\|_{L^2}. \tag{1.1}$$

We now consider the following control problem

$$\min_{u \in L^2(\Omega)} J(u) = \frac{1}{2}\int_\Omega |y - y_0|^2 + \frac{\alpha}{2}\int_\Omega |u - u_0|^2 \tag{1.2}$$
$$\text{subject to } y = \mathcal{G}(u) \text{ and } y(x) \leq b(x) \text{ in } \Omega.$$

Here, $\alpha > 0$ and $y_0, u_0 \in H^1(\Omega)$ as well as $b \in W^{2,\infty}(\Omega)$ are given functions. We denote by $\mathcal{M}(\bar{\Omega})$ the space of Radon measures which is defined as the dual space of $C^0(\bar{\Omega})$ and endowed with the norm

$$\|\mu\|_{\mathcal{M}(\bar{\Omega})} = \sup_{f \in C^0(\bar{\Omega}), |f| \leq 1} \int_{\bar{\Omega}} f d\mu.$$

The analysis of (1.2) is well understood and sketched in [14, Section 6.2.1] for the problem under consideration. Since the state constraints form a convex set and the cost functional is quadratic it is not difficult to establish the existence of a unique solution $u \in L^2(\Omega)$ to this problem. Moreover, from [3, Theorem 5.2] we infer (compare also [2, Theorem 2])

Theorem 1.1. *A function $u \in L^2(\Omega)$ is a solution of (1.2) if and only if there exist $\mu \in \mathcal{M}(\bar{\Omega})$ and $p \in L^2(\Omega)$ such that with $y = \mathcal{G}(u)$ there holds*

$$\int_\Omega p(-\Delta v + v) = \int_\Omega (y - y_0)v + \int_{\bar{\Omega}} v d\mu \qquad \forall v \in H^2(\Omega) \text{ with } \partial_\nu v = 0 \text{ on } \partial\Omega \tag{1.3}$$

$$p + \alpha(u - u_0) = 0 \qquad a.e. \text{ in } \Omega \tag{1.4}$$

$$\mu \geq 0,\ y(x) \leq b(x) \ a.e. \text{ in } \Omega \quad \text{and} \quad \int_{\bar{\Omega}} (b - y) d\mu = 0. \tag{1.5}$$

The study of (1.2) is complicated by the presence of the measure μ on the right hand side of (1.3). As a consequence, the solution p of this problem is no longer in $H^1(\Omega)$ but only in $W^{1,s}(\Omega)$ for all $1 \leq s < \frac{d}{d-1}$. This fact also accounts for the form of the weak formulation (1.3).

The aim of the present paper is to develop a finite element approximation of problem (1.2). The underlying idea consists in approximating the cost functional J by a sequence of functionals J_h where h is a mesh parameter related to a sequence of triangulations. The definition of J_h involves the approximation of the state equation by linear finite elements and enforces constraints on the state in the nodes of the triangulation. We shall prove that the minima of J_h converge in L^2 to the minimum of J as $h \to 0$ and that the states convergence strongly in H^1 as well as uniformly and derive corresponding error bounds.

To the authors knowledge only few attempts have been made to develop a finite element analysis for state constrained elliptic control problems. In [4] Casas proves convergence of finite element approximations to optimal control problems for semi-linear elliptic equations with finitely many state constraints. Casas and Mateos extend these results in [5] to a less regular setting for the states and prove convergence of finite element approximations to semi-linear distributed and boundary control problems.

Let us comment on further approaches that tackle optimization problems for pdes with state constraints. A *Lavrentiev-type regularization* of problem (1.2) is investigated in [11]. In this approach the state constraint $y \leq b$ in (1.2) is replaced by the mixed constraint $\epsilon u + y \leq b$, with $\epsilon > 0$ denoting a regularization parameter. It turns out that the associated Lagrange multiplier μ_ϵ belongs to $L^2(\Omega)$. The resulting optimization problems are solved either by interior-point methods or primal-dual active set strategies, compare [10]. The development of numerical approaches to tackle (1.2) is ongoing. An excellent overview can be found in [8, 9], where also further references are given.

The paper is organized as follows: in §2 we describe our discretization and establish convergence of controls and states to their continuous counterparts for two- and three–dimensional domains. An error analysis is carried out in §3. We obtain

$$\|u - u_h\|_{L^2},\ \|y - y_h\|_{H^1} = \begin{cases} O(h^{1-\epsilon}), & \text{if } d = 2 \\ O(h^{\frac{1}{2}-\epsilon}), & \text{if } d = 3 \end{cases}$$

($\epsilon > 0$ arbitrary) where u_h and y_h are the discrete control and state respectively. Roughly speaking, the idea is to insert the discrete solution into the continuous functional and vice versa. An important tool in the analysis is the use of L^∞–error estimates for finite element approximations of the Neumann problem developed in [13]. The need for uniform estimates is due to the presence of the measure μ in (1.3).

2 Finite element discretization

Let $\mathcal{T}_h$ be a triangulation of Ω with maximum mesh size $h := \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$ and vertices $x_1, \ldots, x_m$. We suppose that $\bar\Omega$ is the union of the elements of $\mathcal{T}_h$ so that element edges lying on the boundary are curved. In addition, we assume that the triangulation is quasi-uniform in the sense that there exists a constant $\kappa > 0$ (independent of h) such that each $T \in \mathcal{T}_h$ is contained in a ball of radius $\kappa^{-1}h$ and contains a ball of radius κh. Let us define the space of linear finite elements,

$$X_h := \{v_h \in C^0(\bar\Omega) \,|\, v_h \text{ is a linear polynomial on each } T \in \mathcal{T}_h\}.$$

In what follows it is convenient to introduce a discrete approximation of the solution operator $\mathcal{G}$. For a given function $v \in L^2(\Omega)$ we denote by $z_h = \mathcal{G}_h(v) \in X_h$ the solution of the discrete Neumann problem

$$\int_\Omega (\nabla z_h \cdot \nabla v_h + z_h v_h) \quad = \quad \int_\Omega v v_h \quad \text{for all } v_h \in X_h.$$

It is well–known that for all $v \in L^2(\Omega)$

$$\|\mathcal{G}(v) - \mathcal{G}_h(v)\| \;\leq\; Ch^2\|v\|, \tag{2.1}$$

$$\|\mathcal{G}(v) - \mathcal{G}_h(v)\|_{L^\infty} \;\leq\; Ch^{2-\frac{d}{2}}\|v\|. \tag{2.2}$$

Here, $\|\cdot\|$ denotes the L^2–norm. We propose the following approximation of the control problem (1.2):

$$\begin{aligned}&\min_{u\in L^2(\Omega)} J_h(u) := \frac{1}{2}\int_\Omega |y_h - P_h y_0|^2 + \frac{\alpha}{2}\int_\Omega |u - P_h u_0|^2 \\ &\text{subject to } \; y_h = \mathcal{G}_h(u) \text{ and } y_h(x_j) \leq b(x_j) \text{ for } j = 1, \ldots, m.\end{aligned} \tag{2.3}$$

Here, P_h denotes the L^2–projection, i.e.

$$\int_\Omega P_h z\, v_h = \int_\Omega z\, v_h \qquad \forall v_h \in X_h. \tag{2.4}$$

It is well–known that

$$\|z - P_h z\| \leq Ch\|z\|_{H^1} \qquad \forall z \in H^1(\Omega). \tag{2.5}$$

Problem (2.3) represents a convex infinite-dimensional optimization problem of similar structure as problem (1.2), but with only finitely many equality and inequality constraints which form a convex admissible set. Again we can apply [3, Theorem 5.2] which together with [2, Corollary 1] yields (compare also the analysis of problem (P) in [4])

Lemma 2.1. *Problem (2.3) has a unique solution $u_h \in L^2(\Omega)$. There exist $\mu_1, \dots, \mu_m \in \mathbb{R}$ and $p_h \in X_h$ such that with $y_h = \mathcal{G}_h(u_h)$ and $\mu_h = \sum_{j=1}^m \mu_j \delta_{x_j}$ we have*

$$\int_\Omega \big(\nabla p_h \cdot \nabla v_h + p_h v_h\big) = \int_\Omega (y_h - P_h y_0) v_h + \int_{\bar\Omega} v_h d\mu_h \qquad \text{for all } v_h \in X_h, \tag{2.6}$$

$$p_h + \alpha(u_h - P_h u_0) = 0 \text{ in } \Omega, \tag{2.7}$$

$$\mu_j \geq 0,\ y_h(x_j) \leq b(x_j), j = 1, \dots, m \text{ and } \int_{\bar\Omega} \big(I_h b - y_h\big) d\mu_h = 0. \tag{2.8}$$

Here, δ_x denotes the Dirac measure concentrated at x and I_h is the usual Lagrange interpolation operator.

Remark 2.2. From (2.7) we deduce that in problem (2.3) it is sufficient to minimize over controls $u \in X_h$ instead of $u \in L^2(\Omega)$ in order to obtain the same unique solution u_h. For the resulting finite dimensional optimization problem the result of Lemma 2.1 then follows from e.g. [12, Theorem 12.1].

We have the following convergence result.

Theorem 2.3. *Let $u_h \in L^2(\Omega)$ be the optimal solution of (2.3) with corresponding state $y_h \in X_h$ and adjoint variables $p_h \in X_h$ and $\mu_h \in \mathcal{M}(\bar\Omega)$. Then, as $h \to 0$ we have*

$$u_h \to u \text{ in } L^2(\Omega), \quad y_h \to y \text{ in } H^1(\Omega) \text{ and in } C^0(\bar\Omega),$$

where u is the solution of (1.2) with corresponding state y.

Proof. Let $\underline{b} := \min_{x \in \bar\Omega} b(x)$. Since $\underline{b} = \mathcal{G}_h(\underline{b})$ and $\underline{b} \leq b(x_j)$ for $j = 1, \dots, m$ we have

$$\frac{1}{2} \int_\Omega |y_h - P_h y_0|^2 + \frac{\alpha}{2} \int_\Omega |u_h - P_h u_0|^2 = J_h(u_h) \leq J_h(\underline{b}) \leq C(y_0, u_0, \underline{b}).$$

This implies that there exists a constant C which is independent of h such that

$$\|y_h\|,\ \|u_h\|,\ \|p_h\| \leq C \qquad \text{for all } 0 < h \leq 1. \tag{2.9}$$

Note that the bound on p_h follows from (2.7). In order to estimate μ_h we use $v_h \equiv 1$ in (2.6) and obtain for every $f \in C^0(\bar\Omega), |f| \leq 1$

$$\int_{\bar\Omega} f d\mu_h \leq \sum_{j=1}^m \mu_j |f(x_j)| \leq \sum_{j=1}^m \mu_j = \int_{\bar\Omega} 1 d\mu_h = \int_\Omega \big(p_h + P_h y_0 - y_h\big) \leq C$$

by (2.9). This yields

$$\|\mu_h\|_{\mathcal{M}(\bar\Omega)} \leq C \qquad \text{for all } 0 < h \leq 1. \tag{2.10}$$

In view of (2.9), (2.10) there exists a sequence $h \to 0$ and $\hat u, \hat p \in L^2(\Omega)$ as well as $\hat\mu \in \mathcal{M}(\bar\Omega)$ such that

$$u_h \rightharpoonup \hat u, \quad p_h \rightharpoonup \hat p \text{ in } L^2(\Omega), \qquad \text{and } \mu_h \rightharpoonup \hat\mu \text{ in } \mathcal{M}(\bar\Omega). \tag{2.11}$$

Since $\mathcal{G}$ is compact as an operator from $L^2(\Omega)$ into $C^0(\bar{\Omega})$ we have, after passing to a further subsequence if necessary,

$$\mathcal{G}(u_h) \to \mathcal{G}(\hat{u}) \qquad \text{in } C^0(\bar{\Omega}) \tag{2.12}$$

and hence

$$\|y_h - \mathcal{G}(\hat{u})\|_{L^\infty} \le \|\mathcal{G}_h(u_h) - \mathcal{G}(u_h)\|_{L^\infty} + \|\mathcal{G}(u_h) - \mathcal{G}(\hat{u})\|_{L^\infty} \le Ch^{2-\frac{d}{2}}\|u_h\| + \|\mathcal{G}(u_h) - \mathcal{G}(\hat{u})\|_{L^\infty}$$

so that $y_h \to \mathcal{G}(\hat{u}) =: \hat{y}$ in $C^0(\bar{\Omega})$ as $h \to 0$ by (2.9) and (2.12). A similar argument shows that $y_h \to \hat{y}$ in $H^1(\Omega)$.

Let us now pass to the limit in (2.6)–(2.8). To begin, let $v \in H^2(\Omega)$ with $\partial_\nu v = 0$ on $\partial\Omega$ and denote by $R_h v$ the Ritz projection of v. Recalling (2.11), (2.6) and the fact that $R_h v \to v$ in $C^0(\bar{\Omega})$ we obtain

$$\begin{aligned} \int_\Omega \hat{p}\big(-\Delta v + v\big) \quad &\leftarrow \quad \int_\Omega p_h\big(-\Delta v + v\big) = \int_\Omega \big(\nabla p_h \cdot \nabla v + p_h v\big) \\ &= \quad \int_\Omega \big(\nabla p_h \cdot \nabla R_h v + p_h R_h v\big) = \int_\Omega (y_h - P_h y_0) R_h v + \int_{\bar{\Omega}} R_h v \, d\mu_h \\ &\to \quad \int_\Omega (\hat{y} - y_0) v + \int_{\bar{\Omega}} v \, d\hat{\mu}. \end{aligned}$$

Using (2.11) we may pass to the limit in (2.7) and deduce $\hat{p} + \alpha(\hat{u} - u_0) = 0$ a.e. in Ω. Clearly, $\hat{\mu} \ge 0$; since $y_h \le I_h b$ in $\bar{\Omega}$ and $y_h \to \hat{y}$ in $C^0(\bar{\Omega})$ we have $\hat{y} \le b$ in $\bar{\Omega}$. Furthermore, recalling that $\int_{\bar{\Omega}} (I_h b - y_h) d\mu_h = 0$ we obtain in the limit

$$\int_{\bar{\Omega}} (b - \hat{y}) d\hat{\mu} = 0.$$

Lemma 1.1 now implies that $\hat{u}$ is a solution of (1.2); as the solution of this problem is unique we must have $u = \hat{u}$ and hence $y = \hat{y}$ and the whole sequence is convergent.

Let us finally prove that $u_h \to u$ in $L^2(\Omega)$. To begin, note that by (2.2)

$$\mathcal{G}_h\big(u - \gamma h^{2-\frac{d}{2}}\big) = \mathcal{G}_h(u) - \mathcal{G}(u) + \mathcal{G}(u) - \gamma h^{2-\frac{d}{2}} \le Ch^{2-\frac{d}{2}}\|u\| + b - \gamma h^{2-\frac{d}{2}} \le b$$

in $\bar{\Omega}$, provided that γ is large enough. Evaluating the above inequality at the nodes $x_1, \dots, x_m$ we see that $\mathcal{G}_h(u - \gamma h^{2-\frac{d}{2}})$ is admissible for the discrete problem and hence $J_h(u_h) \le J_h(u - \gamma h^{2-\frac{d}{2}})$ or

$$\frac{\alpha}{2}\|u_h - P_h u_0\|^2 \le \frac{\alpha}{2}\|u - \gamma h^{2-\frac{d}{2}} - P_h u_0\|^2 + \frac{1}{2}\|\mathcal{G}_h(u) - \gamma h^{2-\frac{d}{2}} - P_h y_0\|^2 - \frac{1}{2}\|y_h - P_h y_0\|^2.$$

Since $y_h \to y$, $\mathcal{G}_h(u) \to \mathcal{G}(u) = y$ in $L^2(\Omega)$ we infer that

$$\limsup_{h\to 0} \|u_h - P_h u_0\|^2 \le \|u - u_0\|^2 \le \liminf_{h \to 0} \|u_h - P_h u_0\|^2,$$

where the second inequality is a consequence of the weak convergence $u_h - P_h u_0 \rightharpoonup u - u_0$. Thus, $\|u_h - P_h u_0\|^2 \to \|u - u_0\|^2$ which implies $u_h - P_h u_0 \to u - u_0$ in L^2 and hence $u_h \to u_0$ in L^2. ∎

3 Error analysis

Let us now turn to the error analysis and start with a couple of auxiliary results.

Lemma 3.1. *Suppose that $u, u_h \in L^2(\Omega)$ are the optimal solutions of (1.2) and (2.3) respectively with corresponding states $y \in H^2(\Omega), y_h \in X_h$. Let $v \in L^2(\Omega)$ and $z = \mathcal{G}(v), z_h = \mathcal{G}_h(v)$. Then*

$$J(u) + \frac{1}{2}\int_\Omega |z-y|^2 + \frac{\alpha}{2}\int_\Omega |v-u|^2 + \int_{\bar\Omega}(b-z)d\mu \;=\; J(v) \tag{3.13}$$

$$J_h(u_h) + \frac{1}{2}\int_\Omega |z_h-y_h|^2 + \frac{\alpha}{2}\int_\Omega |v-u_h|^2 + \int_{\bar\Omega}\big(I_hb-z_h\big)d\mu_h \;=\; J_h(v) \tag{3.14}$$

Proof. An elementary calculation using (1.3) shows

$$\begin{aligned} J(v)-J(u) &= \frac{1}{2}\int_\Omega |z-y|^2 + \frac{\alpha}{2}\int_\Omega |v-u|^2 + \int_\Omega (z-y)(y-y_0) + \alpha\int_\Omega (u-u_0)(v-u) \\ &= \frac{1}{2}\int_\Omega |z-y|^2 + \frac{\alpha}{2}\int_\Omega |v-u|^2 + \int_\Omega p\big(-\Delta(z-y)+(z-y)\big) \\ &\quad -\int_{\bar\Omega}(z-y)d\mu + \alpha\int_\Omega (u-u_0)(v-u). \end{aligned}$$

Since $z = \mathcal{G}(v), y = \mathcal{G}(u)$ we have

$$\int_\Omega p\big(-\Delta(z-y)+(z-y)\big) = \int_\Omega p(v-u),$$

so that (1.4) and (1.5) finally imply

$$J(v)-J(u) = \frac{1}{2}\int_\Omega |z-y|^2 + \frac{\alpha}{2}\int_\Omega |v-u|^2 + \int_{\bar\Omega}(b-z)d\mu.$$

The second claim follows in a similar way. ∎

Remark 3.2. Note that in the above $z = \mathcal{G}(v), z_h = \mathcal{G}_h(v)$ do not necessarily have to be admissible for the minimization problems.

The next lemma examines in more detail the approximation of J by J_h.

Lemma 3.3. *Suppose that $v \in W^{1,s}(\Omega)$ for some $\frac{2d}{d+2} \le s \le 2$. Then*

$$|J(v)-J_h(v)| \le Ch^{2+\frac{d}{2}-\frac{d}{s}}\big(\|u_0\|_{H^1}\|v\|_{W^{1,s}} + \|v\|^2 + \|y_0\|^2_{H^1} + \|u_0\|^2_{H^1}\big).$$

Proof. Let $z = \mathcal{G}(v), z_h = \mathcal{G}_h(v)$. Then

$$J(v)-J_h(v) = \frac{1}{2}\int_\Omega \big(|z-y_0|^2 - |z_h-P_hy_0|^2\big) + \frac{\alpha}{2}\int_\Omega \big(|v-u_0|^2 - |v-P_hu_0|^2\big).$$

Using (2.4), (2.5), (2.1) and (1.1) we obtain

$$
\begin{aligned}
&|\int_\Omega \big(|z-y_0|^2 - |z_h - P_h y_0|^2\big)| = |\int_\Omega (z - y_0 - z_h + P_h y_0)(z - y_0 + z_h - P_h y_0)| \\
&= \ |\int_\Omega \big((z-z_h)(z - y_0 + z_h - P_h y_0) - (y_0 - P_h y_0)(z - y_0 - P_h(z-y_0))\big)| \\
&\leq \ C\|z - z_h\|\big(\|z\| + \|z_h\| + \|y_0\|\big) + Ch^2\|y_0\|_{H^1}\big(\|z\|_{H^1} + \|y_0\|_{H^1}\big) \\
&\leq \ Ch^2\big(\|v\|^2 + \|y_0\|_{H^1}^2\big).
\end{aligned}
$$

For the second term we obtain in a similar way

$$
\int_\Omega \big(|v-u_0|^2 - |v - P_h u_0|^2\big) = \int_\Omega (u_0 - P_h u_0) w = \int_\Omega (u_0 - P_h u_0)(w - P_h w),
$$

where $w = u_0 + P_h u_0 - 2v$ and where we have used (2.4). Applying Lemma 5.1 from the Appendix we infer

$$
\begin{aligned}
|\int_\Omega \big(|v-u_0|^2 - |v - P_h u_0|^2\big)| \ &\leq \ Ch^{2+\frac{d}{2}-\frac{d}{s}}\|u_0\|_{H^1}\|w\|_{W^{1,s}} \\
&\leq \ Ch^{2+\frac{d}{2}-\frac{d}{s}}\|u_0\|_{H^1}\big(\|u_0\|_{H^1} + \|v\|_{W^{1,s}}\big).
\end{aligned}
$$

This proves the lemma. ∎

Lemma 3.4. *Suppose that $v \in W^{1,s}(\Omega)$ for some $1 < s < \frac{d}{d-1}$. Then*

$$
\|\mathcal{G}(v) - \mathcal{G}_h(v)\|_{L^\infty} \leq Ch^{3-\frac{d}{s}}|\log h| \ \|v\|_{W^{1,s}}.
$$

Proof. Let $z = \mathcal{G}(v)$, $z_h = \mathcal{G}_h(v)$. Elliptic regularity theory implies that $z \in W^{3,s}(\Omega)$ from which we infer that $z \in W^{2,q}(\Omega)$ with $q = \frac{ds}{d-s}$ using a well–known embedding theorem. Furthermore, we have

$$
\|z\|_{W^{2,q}} \leq c\|z\|_{W^{3,s}} \leq c\|v\|_{W^{1,s}}. \tag{3.15}
$$

Using Theorem 2.2 and the following Remark in [13] we have

$$
\|z - z_h\|_{L^\infty} \leq c|\log h| \inf_{\chi \in X_h} \|z - \chi\|_{L^\infty}, \tag{3.16}
$$

which, combined with a well–known interpolation estimate, yields

$$
\|z - z_h\|_{L^\infty} \leq ch^{2-\frac{d}{q}}|\log h|\|z\|_{W^{2,q}} \leq ch^{3-\frac{d}{s}}|\log h|\|v\|_{W^{1,s}}
$$

in view (3.15) and the relation between s and q. ∎

Our next aim is to derive a uniform bound on $\|u_h\|_{W^{1,s}}$ for $s < \frac{d}{d-1}$.

Lemma 3.5. *Let $1 < s < \frac{d}{d-1}$. Then there exists a constant c, which is independent of h, such that*

$$
\|u_h\|_{W^{1,s}} \leq c \qquad \textit{for all } 0 < h \leq 1.
$$

Proof. In view of (2.7) we have

$$\|u_h\|_{W^{1,s}} \leq \frac{1}{\alpha}\|p_h\|_{W^{1,s}} + \|P_h u_0\|_{H^1} \leq \frac{1}{\alpha}\|p_h\|_{W^{1,s}} + c,$$

so that it is sufficient to bound $\|p_h\|_{W^{1,s}}$.
Let s' be such that $\frac{1}{s}+\frac{1}{s'}=1$ and suppose that $\phi \in L^{s'}(\Omega)$. Let us denote by $\psi \in W^{2,s'}(\Omega)$ the unique solution of the Neumann problem

$$\begin{aligned} -\Delta\psi + \psi &= \phi \quad \text{in } \Omega \\ \partial_\nu \psi &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Integration by parts and (2.6) yield

$$\begin{aligned} \int_\Omega p_h\,\phi &= \int_\Omega \big(\nabla p_h \cdot \nabla\psi + p_h\psi\big) = \int_\Omega \big(\nabla p_h \cdot \nabla R_h\psi + p_h R_h\psi\big) \\ &= \int_\Omega (y_h - P_h y_0)R_h\psi + \int_{\bar\Omega} R_h\psi d\mu_h, \end{aligned} \tag{3.17}$$

where $R_h\psi$ is the Ritz projection of ψ. Arguing similarly as in Theorem 1 of [1] one shows that there exists a unique solution $p^h \in W^{1,s}(\Omega)$ of the problem

$$\int_\Omega p^h\big(-\Delta v + v\big) = \int_\Omega (y_h - P_h y_0)v + \int_{\bar\Omega} v d\mu_h \quad \forall v \in H^2(\Omega) \text{ with } \partial_\nu v = 0 \text{ on } \partial\Omega. \tag{3.18}$$

Furthermore, there exists a constant $c = c(s) > 0$ such that

$$\|p^h\|_{W^{1,s}} \leq c\big(\|y_h - P_h y_0\| + \|\mu_h\|_{\mathcal{M}(\bar\Omega)}\big) \leq c \tag{3.19}$$

uniformly in h in view of (2.9) and (2.10). If we use $v=\psi$ in (3.18) and combine it with (3.17) we obtain

$$\begin{aligned} \int_\Omega (p^h - p_h)\phi &= \int_\Omega (y_h - P_h y_0)(\psi - R_h\psi) + \int_{\bar\Omega} (\psi - R_h\psi)d\mu_h \\ &\leq ch^2\|\psi\|_{H^2}\big(\|y_h\| + \|P_h y_0\|\big) + \|\psi - R_h\psi\|_{L^\infty}\|\mu_h\|_{\mathcal{M}(\bar\Omega)} \\ &\leq ch^2\|\psi\|_{H^2} + ch^{2-\frac{d}{s'}}|\log h|\|\psi\|_{W^{2,s'}} \\ &\leq ch^{2-\frac{d}{s'}}|\log h|\|\phi\|_{L^{s'}}. \end{aligned}$$

Note that we have again applied (3.16) in order to control $\|\psi - R_h\psi\|_{L^\infty}$. Since $\phi \in L^{s'}(\Omega)$ is arbitrary we infer

$$\|p^h - p_h\|_{L^s} \leq ch^{2-\frac{d}{s'}}|\log h|.$$

Interpolation and inverse estimates then give

$$\|\nabla p_h\|_{L^s} \leq c\|\nabla p^h\|_{L^s} + ch^{1-\frac{d}{s'}}|\log h| \leq c$$

by (3.19) and since $1-\frac{d}{s'} = \frac{d-1}{s}\big(\frac{d}{d-1} - s\big) > 0$. ∎

Let us finally turn to an error estimate for the optimal controls and the optimal states.

Theorem 3.6. *Let u and u_h be the solutions of (1.2) and (2.3) respectively. For every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$\|u - u_h\| + \|y - y_h\|_{H^1} \le C_\epsilon h^{2-\frac{d}{2}-\epsilon}.$$

Proof. Let us define $\tilde{y}^h := \mathcal{G}(u_h) \in H^2(\Omega)$ and $\tilde{y}_h := \mathcal{G}_h(u) \in X_h$. Then Lemma 3.1 implies

$$\begin{aligned} J(u) + \frac{1}{2}\int_\Omega |\tilde{y}^h - y|^2 + \frac{\alpha}{2}\int_\Omega |u_h - u|^2 + \int_{\bar\Omega} (b - \tilde{y}^h) d\mu &= J(u_h) \\ J_h(u_h) + \frac{1}{2}\int_\Omega |\tilde{y}_h - y_h|^2 + \frac{\alpha}{2}\int_\Omega |u - u_h|^2 + \int_{\bar\Omega} \big(I_h b - \tilde{y}_h\big) d\mu_h &= J_h(u). \end{aligned}$$

Since $u = u_0 - \frac{1}{\alpha} p \in W^{1,s}(\Omega)$ for all $\frac{2d}{d+2} \le s < \frac{d}{d-1}$ we obtain with the help of Lemma 3.3

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\tilde{y}^h - y|^2 + \frac{1}{2}\int_\Omega |\tilde{y}_h - y_h|^2 + \alpha \int_\Omega |u_h - u|^2 \\ (3.20) \quad &= J(u_h) - J(u) + J_h(u) - J_h(u_h) - \int_{\bar\Omega} (b - \tilde{y}^h) d\mu - \int_{\bar\Omega} \big(I_h b - \tilde{y}_h\big) d\mu_h \\ &\le Ch^{2+\frac{d}{2}-\frac{d}{s}} \Big(\|u_0\|_{H^1} \big(\|u\|_{W^{1,s}} + \|u_h\|_{W^{1,s}}\big) + \|u\|^2 + \|u_h\|^2 + \|y_0\|^2_{H^1} + \|u_0\|^2_{H^1} \Big) \\ &\quad + \int_{\bar\Omega} (\tilde{y}^h - b) d\mu + \int_{\bar\Omega} (\tilde{y}_h - I_h b) d\mu_h. \end{aligned}$$

Let us first consider the last two integrals. We have for $x \in \bar\Omega$

$$\begin{aligned} \tilde{y}^h(x) - b(x) &= (\tilde{y}^h(x) - y_h(x)) + (y_h(x) - (I_h b)(x)) + ((I_h b)(x) - b(x)) \\ &\le \|\mathcal{G}(u_h) - \mathcal{G}_h(u_h)\|_{L^\infty} + \|I_h b - b\|_{L^\infty}, \end{aligned}$$

since $y_h(x_j) \le b(x_j), j = 1, \ldots, m$ implies that $y_h \le I_h b$ in $\bar\Omega$. If we combine Lemma 3.4 with Lemma 3.5 we infer

$$\int_{\bar\Omega} (\tilde{y}^h - b) d\mu \le ch^{3-\frac{d}{s}} |\log h| \, \|u_h\|_{W^{1,s}} + Ch^2 |b|_{W^{2,\infty}} \le ch^{3-\frac{d}{s}} |\log h|.$$

Similarly we have from (1.5)

$$\begin{aligned} \tilde{y}_h(x) - (I_h b)(x) &= (\tilde{y}_h(x) - y(x)) + (y(x) - b(x)) + (b(x) - (I_h b)(x)) \\ &\le \|\mathcal{G}_h(u) - \mathcal{G}(u)\|_{L^\infty} + \|b - I_h b\|_{L^\infty}, \end{aligned}$$

so that (2.10) and Lemma 3.4 give

$$\int_{\bar\Omega} \big(y_h - I_h b\big) d\mu_h \le ch^{3-\frac{d}{s}} |\log h| \, \|u\|_{W^{1,s}} + Ch^2 |b|_{W^{2,\infty}} \le ch^{3-\frac{d}{s}} |\log h|.$$

Inserting these estimates into (3.20) and applying again Lemma 3.5 we derive

$$\|u - u_h\|^2 + \|y - y_h\|^2 \le ch^{3-\frac{d}{s}} |\log h|.$$

If we now choose s sufficiently close to $\frac{d}{d-1}$ we obtain

$$\|u - u_h\|^2 + \|y - y_h\|^2 \le C_\epsilon h^{4-d-2\epsilon}.$$

Finally, in order to obtain the error bound for y in H^1 we note that

$$\int_\Omega \big(\nabla(y-y_h)\cdot\nabla v_h + (y-y_h)v_h\big) = \int_\Omega (u-u_h)v_h$$

for all $v_h \in X_h$, from which one derives the desired estimate using standard finite element techniques and the bound on $\|u-u_h\|$. ∎

In general we only expect weak convergence of μ_h to μ. Nevertheless we have the following partial result.

Corollary 3.7. *Let $K \subset \bar\Omega$ be compact with $K \cap \operatorname{supp}\mu = \emptyset$. For every $\epsilon > 0$ there exists a constant C_ϵ such that*

$$\mu_h(K) \le C_\epsilon h^{2-\frac{d}{2}-\epsilon}.$$

Proof. By Lemma 5.2 in the Appendix there exists a nonnegative function $\phi \in C^2(\bar\Omega)$ which satisfies

$$\phi \ge 1 \text{ on } K,\ \phi = 0 \text{ on } \operatorname{supp}\mu,\ \partial_\nu\phi = 0 \text{ on } \partial\Omega.$$

Since $\mu_h \ge 0$ we obtain from (2.6)

$$\begin{aligned}
\mu_h(K) &\le \int_{\bar\Omega} \phi\, d\mu_h = \int_{\bar\Omega}(\phi - R_h\phi)d\mu_h + \int_{\bar\Omega} R_h\phi\, d\mu_h \\
&= \int_{\bar\Omega}(\phi - R_h\phi)d\mu_h + \int_\Omega \big(\nabla p_h\cdot\nabla R_h\phi + p_h R_h\phi\big) - \int_\Omega (y_h - P_h y_0)R_h\phi \\
&= \int_{\bar\Omega}(\phi - R_h\phi)d\mu_h + \int_\Omega \big(\nabla p_h\cdot\nabla\phi + p_h\phi\big) - \int_\Omega (y_h - P_h y_0)R_h\phi \\
&= \int_{\bar\Omega}(\phi - R_h\phi)d\mu_h + \int_\Omega p_h(-\Delta\phi + \phi) - \int_\Omega (y_h - P_h y_0)R_h\phi,
\end{aligned}$$

where R_h is again the Ritz projection. On the other hand, (1.3) and the fact that $\phi = 0$ on $\operatorname{supp}\mu$ imply

$$\int_\Omega (y-y_0)\phi - \int_\Omega p(-\Delta\phi+\phi) = 0.$$

Combining this relation with the first estimate we derive

$$\begin{aligned}
\mu_h(K) &\le \int_{\bar\Omega}(\phi - R_h\phi)d\mu_h + \int_\Omega (p_h - p)(-\Delta\phi+\phi) + \int_\Omega (y_h - P_h y_0)(\phi - R_h\phi) \\
&\quad + \int_\Omega (y - y_h - y_0 + P_h y_0)\phi \\
&\le \|\phi - R_h\phi\|_{L^\infty}\|\mu_h\|_{\mathcal{M}(\bar\Omega)} + \|p-p_h\|\|\phi\|_{H^2} + \big(\|y_h\| + \|P_h y_0\|\big)\|\phi - R_h\phi\| \\
&\quad + \big(\|y-y_h\| + \|y_0 - P_h y_0\|\big)\|\phi\| \\
&\le C\|\phi - R_h\phi\|_{L^\infty} + C_\epsilon h^{2-\frac{d}{2}-\epsilon} \le C_\epsilon h^{2-\frac{d}{2}-\epsilon}
\end{aligned}$$

in view of (1.4), (2.7) and Theorem 3.6. ∎

Remark 3.8. We mention here a second approach that differs from the one discussed above in the way in which the inequality constraints are realized. Denote by $D_1, \ldots, D_m$ the cells of the dual mesh. Each cell D_i is associated with a vertex x_i of $\mathcal{T}_h$ and we have

$$\bar\Omega = \cup_{i=1}^m D_i, \quad \operatorname{int}(D_i)\cap\operatorname{int}(D_j) = \emptyset, i \ne j.$$

In (2.3), we now impose the constraints

$$\fint_{D_j} (y_h - I_h b) \leq 0 \text{ for } j = 1, \ldots, m \tag{3.21}$$

on the discrete solution $y_h = \mathcal{G}_h(u)$. Here, we have abbreviated $\fint_{D_j} f = \frac{1}{|D_j|} \int_{D_j} f$. The measure μ_h that appears in Lemma 2.1 now has the form $\mu_h = \sum_{j=1}^m \mu_j \fint_{D_j} \cdot \, dx$, and the pointwise constraints in (2.8) are replaced by those of (3.21). The error analysis for the resulting numerical method can be carried out in the same way as shown above with the exception of Theorem 3.6, where the bounds on $\tilde{y}-b$ and $\tilde{y}_h - I_h b$ require a different argument. In this case, additional terms of the form

$$\|f - \fint_{D_j} f\|_{L^\infty(D_j)}$$

have to be estimated. Since these will in general only be of order $O(h)$, this analysis would only give $\|u - u_h\|, \|y - y_h\|_{H^1} = O(\sqrt{h})$. The numerical test example in §4 suggests that at least $\|u - u_h\| = O(h)$, but we are presently unable to prove such an estimate.

4 Numerical examples

Example 4.1. The following test problem is taken - in a slightly modified form - from [10], Example 6.2. Let $\Omega := B_1(0)$, $\alpha > 0$,

$$y_0(x) := 4 + \frac{1}{\pi} - \frac{1}{4\pi}|x|^2 + \frac{1}{2\pi} \log |x|, \quad u_0(x) := 4 + \frac{1}{4\alpha\pi}|x|^2 - \frac{1}{2\alpha\pi} \log |x|$$

and $b(x) := |x|^2 + 4$. We consider the cost functional

$$J(u) := \frac{1}{2} \int_\Omega |y - y_0|^2 + \frac{\alpha}{2} \int_\Omega |u - u_0|^2,$$

where $y = \mathcal{G}(u)$. By checking the optimality conditions of first order one verifies that $u \equiv 4$ is the unique solution of (1.2) with corresponding state $y \equiv 4$ and adjoint states

$$p(x) = \frac{1}{4\pi}|x|^2 - \frac{1}{2\pi} \log |x| \text{ and } \mu = \delta_0.$$

The finite element counterparts of y, u, p and μ are denoted by y_h, u_h, p_h and μ_h.
For an error functional $E(h)$ we define the experimental order of convergence as

$$\text{EOC} = \frac{\ln E(h_1) - \ln E(h_2)}{\ln h_1 - \ln h_2}.$$

To investigate EOCs for our model problem we choose a sequence of uniform partitions of Ω containing five refinement levels, starting with eight triangles forming a uniform octagon as initial triangulation of the unit disc. The corresponding grid sizes are $h_i = 2^{-i}$ for $i = 1, \ldots, 5$. As error functionals we take $E(h) = \|(u, y) - (u_h, y_h)\|$ and $E(h) = \|(u, y) - (u_h, y_h)\|_{H^1}$ and note, that the error $p - p_h$ is related to $u - u_h$ via (2.7). We solve problems (2.3) using the QUADPROG routine of the MATLAB OPTIMIZATION TOOLBOX. The required finite element matrices for the discrete state and adjoint systems are generated with the help of the MATLAB PDE TOOLBOX. Furthermore, for discontinuous functions f we use the quadrature rule

$$\int_\Omega f(x)dx \approx \sum_{T\in\mathcal{T}_h} f\left(x_{s(T)}\right)|T|,$$

where $x_{s(T)}$ denotes the barycenter of T. In all computations we set $\alpha = 1$.

In Table 1, we present EOCs for problem (2.3) (case $S = D$) and the approach sketched in Remark 3.8 (case $S = M$). As one can see, the error $\|u - u_h\|$ behaves in the case $S = D$ as predicted by Theorem 3.6, whereas the errors $\|y - y_h\|$ and $\|y - y_h\|_{H^1}$ show a better convergence behaviour. On the finest level we have $\|u - u_h\| = 0.003117033$, $\|y - y_h\| = 0.000123186$ and $|y - y_h|_{H^1} = 0.000083757$. Furthermore, all coefficients of μ_h are equal to zero, except the one in front of δ_0 whose value is 0.62820305383493. The errors $\|u - u_h\|$, $\|y - y_h\|$ and $\|y - y_h\|_{H^1}$ in the case $S = M$ show a better EOC than in the case $S = D$. This can be explained by the fact that the exact solutions y and u are very smooth, and that the relaxed form of the state constraints introduce a smearing effect on the numerical solutions at the origin. On the finest level we have $\|u - u_h\| = 0.001020918$, $\|y - y_h\| = 0.000652006$ and $|y - y_h|_{H^1} = 0.000037656$. Furthermore, the coefficient of μ_h corresponding to the patch containing the origin has the value 0.66505911271141.

Figures 1 and 2 present the numerical solutions y_h and u_h for $h = 2^{-5}$ in the case $S = D$ and $S = M$, respectively. We note that using equal scales on all axes would give completely flat graphs in all four figures.

	(S=D)	(S=M)	(S=D)	(S=M)	(S=D)	(S=M)
Level	$\|u - u_h\|$	$\|u - u_h\|$	$\|y - y_h\|$	$\|y - y_h\|$	$\|y - y_h\|_{H^1}$	$\|y - y_h\|_{H^1}$
1	0.788985	0.654037	0.536461	0.690302	0.860516	0.688531
2	0.759556	1.972784	1.147861	2.017836	1.272400	2.015602
3	0.919917	1.962191	1.389378	2.004383	1.457095	2.004286
4	0.966078	1.856687	1.518381	1.989727	1.564204	1.990566
5	0.986686	1.588722	1.598421	1.979082	1.632772	1.979945

Table 1: Experimental order of convergence

Example 4.2. The second test problem is taken from [11], Example 2. It reads

$$\min_{u\in L^2(\Omega)} J(u) = \frac{1}{2}\int_\Omega |y - y_0|^2 + \frac{1}{2}\int_\Omega |u - u_0|^2$$

subject to $y = \mathcal{G}(u)$ and $y(x) \geq b(x)$ in Ω.

Here, Ω denotes the unit square,

$$b(x) = \begin{cases} 2x_1 + 1, & x_1 < \frac{1}{2}, \\ 2, & x_1 \geq \frac{1}{2}, \end{cases} \quad y_0(x) = \begin{cases} x_1^2 - \frac{1}{2}, & x_1 < \frac{1}{2}, \\ \frac{1}{4}, & x_1 = \frac{1}{2}, \\ \frac{3}{4}, & x_1 > \frac{1}{2}, \end{cases}$$

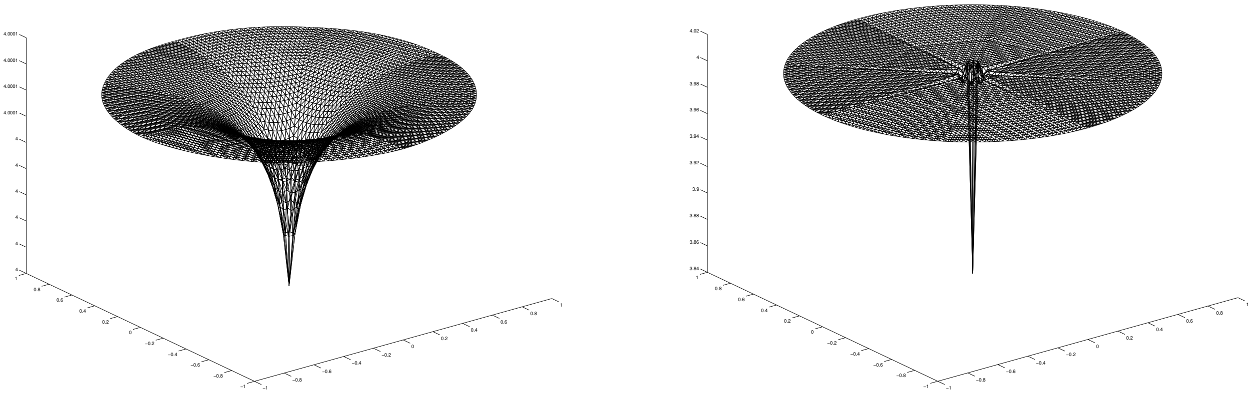

Figure 1: Numerically computed state y_h (left) and control u_h (right) for $h = 2^{-5}$ in the case $S = D$.

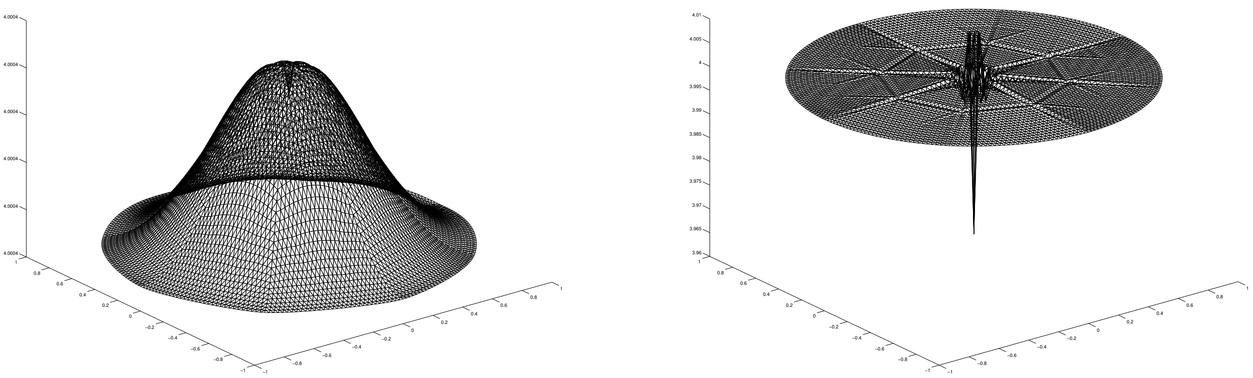

Figure 2: Numerically computed state y_h (left) and control u_h (right) for $h = 2^{-5}$ in the case $S = M$.

and

$$u_0(x) = \begin{cases} \frac{5}{2} - x_1^2, & x_1 < \frac{1}{2}, \\ \frac{9}{4}, & x_1 \geq \frac{1}{2}. \end{cases}$$

The exact solution is given by $y \equiv 2$ and $u \equiv 2$ in Ω. The corresponding Lagrange multiplier $p \in H^1(\Omega)$ is given by

$$p(x) = \begin{cases} \frac{1}{2} - x_1^2, & x_1 < \frac{1}{2}, \\ \frac{1}{4}, & x_1 \geq \frac{1}{2}. \end{cases}$$

The multiplier μ has the form

$$(4.22) \qquad \int_{\bar{\Omega}} f d\mu = \int_{\{x_1 = \frac{1}{2}\}} f ds + \int_{\{x_1 > \frac{1}{2}\}} f dx, \qquad f \in C^0(\bar{\Omega}).$$

In our numerical computations we use uniform grids generated with the POIMESH function of the MATLAB PDE TOOLBOX. Integrals containing y_0, u_0 are numerically evalu-

ated by substituting y_0, u_0 by their piecewise linear, continuous finite element interpolations $I_h y_0, I_h u_0$. The grid size of a grid containing l horizontal and l vertical lines is given by $h_l = \frac{\sqrt{2}}{l+1}$. Fig. 3 presents the numerical results for a grid with $h = \frac{\sqrt{2}}{36}$ in the case (S=D). The corresponding values of μ_h on the same grid are presented in Fig. 4. They reflect the fact that the measure consists of a lower dimensional part which is concentrated on the line $\{x \in \Omega \,|\, x_1 = \frac{1}{2}\}$ and a regular part with a density $\chi_{|\{x_1 > \frac{1}{2}\}}$. We again note that using equal scales on all axes would give completely flat graphs for y_h as well as for u_h.
We compute EOCs for the two different sequences of grid-sizes $s_o = \{h_1, h_3, \ldots, h_{19}\}$ and $s_e = \{h_0, h_2, \ldots, h_{18}\}$. We note that the grids corresponding to s_o contain the line $x_1 = \frac{1}{2}$. Table 2 presents EOCs for s_o, and Table 3 presents EOCs for s_e. For the sequence s_o we observe super-convergence in the case (S=D), although the discontinuous function y_0 for the quadrature is replaced by its piecewise linear, continuous finite element interpolant $I_h y_0$. Let us note that further numerical experiments show that the use of the quadrature rule (4.1) for integrals containing the function y_0 decreases the EOC for $\|u - u_h\|$ to $\frac{3}{2}$, whereas EOCs remain close to 2 for the other two errors $\|y - y_h\|$ and $\|y - y_h\|_{H^1}$. For this sequence also the case (S=M) behaves twice as good as expected by our arguments in Remark 3.8. For the sequence s_e the error $\|u - u_h\|$ in the case (S=D) approximately behaves as predicted by our theory, in the case (S=M) it behaves as for the sequence s_o. The errors $\|y - y_h\|$ and $\|y - y_h\|_{H^1}$ behave that well, since the the exact solutions y and u are very smooth. For h_{19} we have in the case (S=D) $\|u - u_h\| = 0.000103428$, $\|y - y_h\| = 0.000003233$ and $|y - y_h|_{H^1} = 0.000015155$, and in the case (S=M) $\|u - u_h\| = 0.011177577$, $\|y - y_h\| = 0.000504815$ and $|y - y_h|_{H^1} = 0.001547907$. We observe that the errors in the case $S = M$ are two magnitudes larger than in the case (S=D). This can be explained by the fact that an Ansatz for the multiplier μ with a linear combination of Dirac measures is better suited to approximate measures concentrated on singular sets than a piecewise constant Ansatz as in the case (S=M). Finally, Table 4 presents $\sum_{x_i \in \{x_1 = 1/2\}} \mu_i$ and $\sum_{x_i \in \{x_1 > 1/2\}} \mu_i$ for s_o in the case (S=D). As one can see $\sum_{x_i \in \{x_1 = 1/2\}} \mu_i$ tends to 1, the length of $\{x_1 = 1/2\}$, and $\sum_{x_i \in \{x_1 > 1/2\}} \mu_i$ tends to 1/2, the area of $\{x_1 > 1/2\}$. These numerical findings indicate that $\mu_h = \sum_{i=1}^{m} \mu_i \delta_{x_i}$ well approximates μ, since $\int_{\bar{\Omega}} d\mu_h = \sum_{i=1}^{m} \mu_i$, and that μ_h also well resolves the structure of μ, see (4.22). For all numerical computations of this example we have $\mu_i = 0$ for $x_i \in \{x_1 < 1/2\}$.

5 Appendix

Lemma 5.1. *Let $\frac{2d}{d+2} \le s \le 2$ and $v \in W^{1,s}(\Omega)$. Then*

$$\|v - P_h v\| \le C h^{1 + \frac{d}{2} - \frac{d}{s}} \|v\|_{W^{1,s}}.$$

Proof. The assertion is clear if $s = \frac{2d}{d+2}$ or if $s = 2$ so that we may assume $\frac{2d}{d+2} < s < 2$. Let us write

$$\int_\Omega |v - P_h v|^2 = \int_\Omega |v - P_h v|^{\frac{sd - 2d + 2s}{s}} |v - P_h v|^{\frac{d(2-s)}{s}}$$

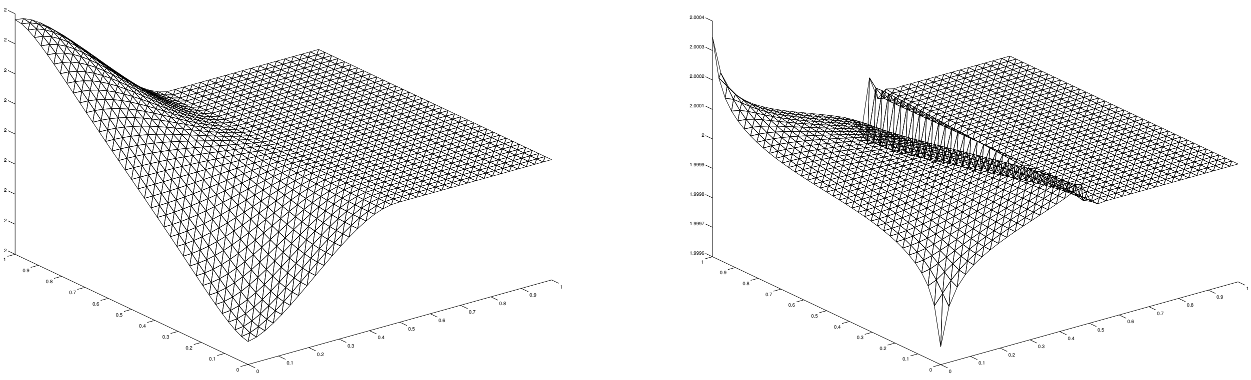

Figure 3: Numerically computed state y_h (left) and control u_h (right) for $h = \frac{\sqrt{2}}{36}$ in the case $S = D$.

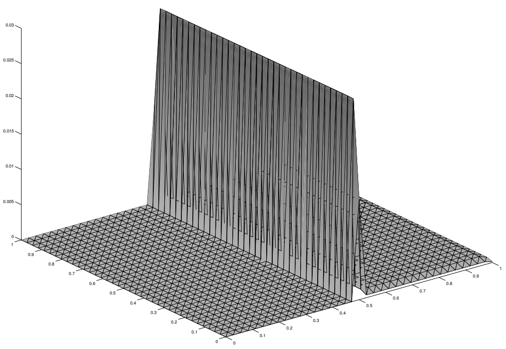

Figure 4: Numerically computed multiplier μ_h for $h = \frac{\sqrt{2}}{36}$ in the case $S = D$.

and apply Hölder's inequality with $p = \frac{s^2}{sd-2d+2s}, q = \frac{s^2}{(d-s)(2-s)}$ which implies

$$
\begin{aligned}
\|v - P_h v\|^2 &\leq \|v - P_h v\|_{L^s}^{\frac{sd-2d+2s}{s}} \|v - P_h v\|_{L^{\frac{ds}{d-s}}}^{\frac{d(2-s)}{s}} \\
&\leq \|v - P_h v\|_{L^s}^{\frac{sd-2d+2s}{s}} \left(\|v\|_{L^{\frac{ds}{d-s}}} + \|P_h v\|_{L^{\frac{ds}{d-s}}}\right)^{\frac{d(2-s)}{s}}.
\end{aligned}
$$

We infer from [6] that

$$
\|v - P_h v\|_{L^s} \leq Ch\|v\|_{W^{1,s}}, \quad \|P_h v\|_{L^{\frac{ds}{d-s}}} \leq C\|v\|_{L^{\frac{ds}{d-s}}}
$$

which, together with the continuous embedding $W^{1,s}(\Omega) \hookrightarrow L^{\frac{ds}{d-s}}(\Omega)$, gives

$$
\|v - P_h v\|^2 \leq ch^{\frac{sd-2d+2s}{s}} \|v\|_{W^{1,s}}^2
$$

so that the assertion follows. ∎

Level	(S=D) $\|u-u_h\|$	(S=M) $\|u-u_h\|$	(S=D) $\|y-y_h\|$	(S=M) $\|y-y_h\|$	(S=D) $\|y-y_h\|_{H^1}$	(S=M) $\|y-y_h\|_{H^1}$
1	1.669586	0.448124	1.417368	0.544284	1.594104	0.384950
2	1.922925	1.184104	1.990906	1.473143	1.992097	1.239771
3	2.000250	1.456908	2.101633	1.871948	2.080739	1.745422
4	2.029556	1.530303	2.125168	2.427634	2.108241	2.348036
5	2.041913	1.260744	2.124773	2.743918	2.116684	2.563363
6	2.047106	1.142668	2.117184	1.430239	2.117739	1.318617
7	2.048926	1.177724	2.107828	1.503463	2.115633	1.409563
8	2.049055	1.194893	2.098597	1.578342	2.112152	1.497715
9	2.048312	1.194802	2.090123	1.622459	2.108124	1.549495

Table 2: Experimental order of convergence, $x_1 = \frac{1}{2}$ grid line

Level	(S=D) $\|u-u_h\|$	(S=M) $\|u-u_h\|$	(S=D) $\|y-y_h\|$	(S=M) $\|y-y_h\|$	(S=D) $\|y-y_h\|_{H^1}$	(S=M) $\|y-y_h\|_{H^1}$
1	0.812598	0.460528	1.160789	2.154570	0.885731	1.473561
2	1.361946	0.406917	2.042731	0.597846	1.918942	0.405390
3	1.228268	1.031763	1.832573	1.392796	1.700124	1.088595
4	1.245030	1.262257	1.678233	1.621110	1.570580	1.392408
5	1.252221	1.416990	1.646124	1.844165	1.554434	1.686808
6	1.256861	1.505759	1.696309	2.128776	1.620231	2.021210
7	1.264456	1.489061	1.627539	2.507863	1.559065	2.415552
8	1.260157	1.316627	1.640964	2.989867	1.580113	2.818148
9	1.265599	1.169109	1.686579	1.601263	1.635084	1.460153

Table 3: Experimental order of convergence, $x_1 = \frac{1}{2}$ not a grid line

Lemma 5.2. *Suppose that K and $\tilde{K}$ are two disjoint compact subsets of $\bar{\Omega}$. Then there exists a nonnegative function $\phi \in C^2(\bar{\Omega})$ which satisfies*

$$\partial_\nu \phi = 0 \text{ on } \partial\Omega, \quad \phi \geq 1 \text{ on } K, \quad \phi = 0 \text{ on } \tilde{K}.$$

Proof. For $r > 0$ let us define $\Omega_r := \{x \in \bar{\Omega} \,|\, \mathrm{dist}(x, \partial\Omega) < r\}$. In view of the smoothness of $\partial\Omega$ there exists $\delta > 0$ such that for each $x \in \Omega_\delta$ there exists a unique point $y = y(x) \in \partial\Omega$ with

$$x = y - \mathrm{dist}(x, \partial\Omega)\nu(y)$$

Level	$\sum_{x_i \in \{x_1 = 1/2\}} \mu_i$	$\sum_{x_i \in \{x_1 > 1/2\}} \mu_i$
1	1.13331662624081	0.36552954225441
2	1.06315278164899	0.43644163287114
3	1.03989323182608	0.45990635060758
4	1.02893022155910	0.47095098878247
5	1.02265064139378	0.47727091447291
6	1.01855129775903	0.48139306499280
7	1.01569011772403	0.48426838085822
8	1.01359012331610	0.48637773715316
9	1.01198410389649	0.48799027450619

Table 4: Approximation of the multiplier in the case (S=D), $x_1 = \frac{1}{2}$ grid line

(see [7], 14.6). Since $K \cap \tilde{K} = \emptyset$ we may assume that $\mathrm{dist}(K, \tilde{K}) > \delta$. Let us define

$$\Gamma_K := \{y(x) \,|\, x \in K \cap \overline{\Omega_{\frac{\delta}{2}}}\}, \quad \Gamma_{\tilde{K}} := \{y(x) \,|\, x \in \tilde{K} \cap \overline{\Omega_{\frac{\delta}{2}}}\}.$$

Γ_K and $\Gamma_{\tilde{K}}$ are disjoint, compact subsets of $\partial\Omega$, since $\mathrm{dist}(K, \tilde{K}) > \delta$ and $x \mapsto y(x)$ is continuous. Let $\phi_1 \in C^2(\partial\Omega)$ be a nonnegative function satisfying $\phi_1 \geq 1$ on Γ_K, $\phi_1 = 0$ on $\Gamma_{\tilde{K}}$. By setting $\phi_1(x) = \phi_1(y(x))$ we extend ϕ_1 as a C^2 function to Ω_δ. Clearly, $\partial_\nu \phi_1 = 0$ on $\partial\Omega$. Let $\psi \in C^2(\bar{\Omega})$ be a nonnegative cut–off function with $\psi = 1$ in $\Omega_{\frac{\delta}{4}}$ and $\psi = 0$ in $\bar{\Omega} \setminus \Omega_{\frac{\delta}{2}}$. Then $\phi_2 := \psi\phi_1$ satisfies

$$\partial_\nu \phi_2 = 0 \text{ on } \partial\Omega, \quad \phi_2 \geq 1 \text{ on } K \cap \Omega_{\frac{\delta}{4}}, \quad \phi_2 = 0 \text{ on } \tilde{K}.$$

Finally, choose a nonnegative function $\phi_3 \in C^2(\bar{\Omega})$ with

$$\phi_3 \geq 1 \text{ on } K \cap (\bar{\Omega} \setminus \Omega_{\frac{\delta}{4}}), \quad \phi_3(x) = 0 \text{ if } \mathrm{dist}(x, K \cap (\bar{\Omega} \setminus \Omega_{\frac{\delta}{4}})) \geq \frac{\delta}{8}.$$

Then, $\partial_\nu \phi_3 = 0$ on $\partial\Omega$, $\phi_3 = 0$ on $\tilde{K}$ and $\phi := \phi_2 + \phi_3$ has the required properties. ∎

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