Three control methods for time – dependent Fluid Flow

MICHAEL HINZE[1] & KARL KUNISCH[2]

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[1]Technische Universität Berlin, Fachbereich Mathematik, Straße des 17. Juni 136, D-10623 Berlin, email: hinze@math.tu-berlin.de, supported by the Deutsche Forschungsgemeinschaft under Sfb 557 „Beeinflussung komplexer turbulenter Scherströmungen"

[2]Institut für Mathematik, Karl-Franzens-Universität Graz, A-8010 Graz, Austria, email: kunisch@kfunigraz.ac.at, research supported in part by the Fonds zur Förderung der wissenschaftlichen Forschung under SFB 03 „Optimierung und Kontrolle".

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Abstract

We summarize some of the recent developments in the field of optimal and suboptimal control strategies for time dependent incompressible Navier-Stokes equations. Besides the Newton method, reduced order modeling (ROM) and receding horizon control strategies are described. All approaches are applied to compute controls for the unsteady flow around a cylinder in two spatial dimensions.

Contents

1 Introduction

This work presents three active control methods for fluid flow governed by the instationary Navier-Stokes equations, namely optimal open loop control based on Newton's method, receding horizon control and reduced order modeling. We start by giving a brief review in active flow control emphasizing computational techniques.

1.1 Optimal control

In [1] Abergel and Temam apply optimal control theory to some problems in fluid mechanics and derive optimality conditions for various cost functionals. As solution procedure they propose gradient-type methods. A similar approach is taken by Glowinski in [23]. Optimal control strategies with focus on turbulent flows are discussed by Bewley, Choi, Temam and Moin in [8], compare also [9, 73]. In [12, 11] Chang and Collis combine large eddy simulation and flow control and obtain numerical results similar to those presented in [9], but at significantly lower numerical cost. Together with Ziane, Bewley and Temam extended the optimal control concept for the Navier-Stokes equations to robust control in [10]. Gunzburger and Manservisi investigate optimization problems for the instationary Navier-Stokes equations with distributed controls in [27, 29], and for boundary controls in [28], see also [42] and [63]. The long-time behavior of the controlled trajectories of the Navier-Stokes equations for a restricted class of cost functionals is investigated by Hou and Yan in [45]. They compare the optimal trajectory to that produced by a simple distributed closed-loop control law, for which decay can be estimated. In [7] Berggren presents a fully discrete boundary control problem for the instationary Navier-Stokes equations and utilizes a limited memory BFGS approach for its numerical solution. Justen investigates control problems for the thermally coupled Navier-Stokes equations in [52], and Clerc, Le Tallec, Mallet, Ravachol and Stoufflet consider optimal control of the parabolized Navier-Stokes system in [16]. Joslin, Gunzburger, Nicolaides, Erlebacher and Hussaini [51] present an optimal boundary control strategy for boundary layers which is based on tracking a prescribed normal stress distribution along an observation boundary. Protas and Styczek derive a boundary control problem with state observation on the boundary for the wake flow around a cylinder in [65]. The authors applied a gradient algorithm to compute boundary controls for a backward facing step flow in [36] and compared the result to that obtained by the instantaneous control strategy. In [40] we compare Newton- and SQP-methods for the numerical solution of optimal control problems governed by the instationary Navier Stokes equations and give convergence and rate of convergence proofs. The essential difference between these two methods consists in treating the Navier Stokes equations as implicit or alternatively as explicit constraints.

Concerning controllability for the instationary Navier-Stokes equations and related questions we refer for example to the work by Fursikov, Gunzburger & Hou, [21] and that by Fursikov [20], Coron [17], and Farbre [19], as well as the references cited therein.

Investigations concerning optimal control of the stationary Navier-Stokes equations are presented by Ghattas and Bark in [22], where they apply Newton and quasi-Newton SQP techniques to the solution of optimal control problems for the Navier-Stokes equations in two and three dimensions and compare their results to those obtained by the gradient method. Further investigations on control problems for the stationary Navier-Stokes equations are provided by Desai and Ito in [18], by Gunzburger, Hou and Svobodny in [25, 43], by Málek and Roubiček in [62], by Loncarcic for the Stokes equations in [61] and by Heinkenschloss in [32]. In [56] the second author and Marduel numerically investigate the optimal control problem with thermal controls for non-isothermal viscoelastic fluids in a suddenly expanding channel.

An excellent overview on diverse aspects of optimal control for fluids is given by Gunzburger in [24], and also by Sritharan in [70].

1.2 Suboptimal control

Solving open loop optimal control problems for realistic flows requires a significant and sometimes inhibiting amount of storage and computing time. Numerical solutions for closed loop control strategies appear to be altogether un tractable unless some well-considered reductions of the full problem statement are made.

For these reasons several concepts of suboptimal solutions have been proposed. We shall address two of them, namely receding horizon control and control by system reduction. Either of them can be utilized or adapted in an open as well as a closed loop context. The idea of receding horizon control techniques is to replace the open-loop optimal control problem on the full time horizon by a succession of optimal control problems on short control horizons. Control by system reduction keeps the control horizon fixed and replaces the full Navier-Stokes system by a suitable low order model, for which the optimal control problem then is solved exactly. The resulting reduced optimal control is used as control for the full system.

1.2.1 Receding horizon control, instantaneous control and other feedback control concepts

The (1,1)-receding horizon control strategy has been introduced as a suboptimal control procedure for the instationary Burgers equation by Choi, Temam, Moin and Kim in [15]. Here, (1,1) means that the length of the control horizon coincides with the step size of time integration. From then on this control concept has been utilized in a great number of numerical approaches to the control of fluid flow. In [13] Choi successfully applies it to unsteady flow around a cylinder. Together with Min he presents boundary control concepts which only utilize state information on the cylinder surface for the same flow configuration in [64]. In [60] Choi, Kim and Lee, utilize the control method for drag reduction of channel flows. Related work can be found in [38] and [35]. In [9] Bewley, Moin and Temam use receding horizon control to re-laminarize a turbulent channel flow, and Chang and Collis in [11] use large eddy simulation combined with receding horizon control and obtain numerical results similar to those obtained by Bewley et al in [9], but at significantly lower numerical cost.

Instead of utilizing the exact solution of (1,1)-receding horizon control as control input one may use the control obtained by applying exactly one gradient step to the solution of the (1,1)-receding horizon control problem. This approach is called instantaneous control. It has been utilized by Choi and the authors in [14] to reduce of the re-attachment length of laminar flow over a backward facing step. In the same work the interpretation of the method as closed-loop control approach utilizing boundary measurements only is given, compare also [34]. Kunisch and Marduel apply instantaneous control with thermal controls to transient non-isothermal viscoelastic fluids in [57].

There are only few analytical results available for receding horizon type control techniques applied to control of fluid flow. In [44] Hou and Yan prove stability of the (1,1)-receding horizon control approach with distributed controls for a tracking-type problem with adapted form of the cost functional. Stability of the instantaneous control technique is addressed by Volkwein and the first author for the instationary Burgers equation in [41]. The results presented there are extended to the instationary Navier-Stokes equations by the first author in [34]. A general framework for the stabilizing properties of receding horizon control for systems arising from discretizing dissipative partial differential equations is given in [47]. An interesting observation is made by Heinkenschloss in [30] where he shows that the first step of the Gauß-Seidel method applied to the numerical solution of quadratic optimization problems for discretized parabolic equations is equivalent to the application of instantaneous control to the parabolic equation, compare also [31]. Let us mention that receding horizon control is also an active area of research in the community of control of ordinary differential equations.

A different feedback approach is presented by Ito and Kang in [46], where a suboptimal solution to the optimal feedback control problem for systems governed by the Burgers and the Navier-Stokes equations is obtained by means of an appropriate Ansatz for the solution to the Hamilton Jacobi Bellman equation. Heuristic feedback control approaches to the instationary Navier-Stokes equations with distributed controls are investigated by Gunzburger and Manservisi in [26], and by Hou and Yan in [45]. Joshi, Speyer and Kim present a feedback control approach to stabilize disturbances in a plane Poiseuille flow in [50]. In a series of papers [53, 54, 55] Koumoutsakos develops a closed-loop boundary control mechanism for fluid flows which utilizes pressure information on the boundary only and applies it to control the vortex dipole. In [59] Lee, Kim, Babcock and Goodmann apply neural networks to turbulence control for drag reduction.

1.2.2 System reduction and control

In the system reduction approach to control of the instationary Navier-Stokes equations a Galerkin approximation for the spatial approximation of the full Navier-Stokes equations is made that utilizes global basis functions containing characteristics of the expected controlled flow and thus represents some of its dynamics. This is in contrast to finite element based Galerkin schemes where the basis elements are not related to the physical properties of the system which they approximate. Consequently one expects that only relatively few basis elements will provide a reasonably good description of the controlled flow. This results in a low order model for the Navier-Stokes equations which is used in the optimization process. The control computed for the reduced system is used as suboptimal control for the full equations.

In [71] Tang, Graham and Peraire apply the snapshot variant of principal orthogonal decomposition (POD) to the control of the unsteady wake flow around a cylinder. In their approach control is introduced into the system via cylinder rotation. The basis functions correspond to the solution of the time-dependent flow at predetermined times and are weighted by means of a singular value decomposition. Arian, Fahl and Sachs [3] present a sequential POD boundary control approach for fluid flows which successively improves the low order model and apply it to the control of driven cavity flows. Their approach takes up an idea presented by Afanasiev and the first author in [2] for adaptive distributed control of a wake flow using proper orthogonal decomposition. In [67] Ravindran uses the snapshot POD approach for boundary control of the backward-facing step flow.

System reduction and control utilizing the reduced basis method is introduced by Ito and Ravindran in [48, 49] where it is applied to the control of the flow over the backward facing step governed by stationary Navier-Stokes equations. In the reduced basis method the basis functions are solutions of the nonlinear system corresponding to different control regimes. In [67] Ravindran extends the method to boundary control of the instationary Navier-Stokes equations and compares his numerical results for the backward-facing step flow to those obtained by the snapshot variant of the POD method. In [37] the authors also present numerical experiments for boundary control of the backward facing step flow utilizing the reduced basis approach.

2 The optimal control problem

In this and the following two sections we consider some aspects of Newton's method applied to the optimal control problem

$$(1) \qquad \begin{cases} \min_{(y,u)\in W\times U} \;\; J(y,u) & \\ \text{subject to} & \\ \;\; \frac{\partial y}{\partial t} + (y\cdot\nabla)y - \nu\Delta y + \nabla p = Bu & \text{in } \; Q = (0,T)\times\Omega, \\ \;\; \text{div } y = 0 & \text{in } \; Q, \\ \;\; y(t,\cdot) = 0 & \text{on } \; \Sigma = (0,T)\times\partial\Omega, \\ \;\; y(0,\cdot) = y_0 & \text{in } \; \Omega, \end{cases}$$

where Ω is a bounded subset of $\mathbb{R}^2$ with sufficiently smooth boundary. At first solving (1) appears to be a standard task. However, the special structure of (1) as well as the goal of studying second order numerical methods suggest an independent analysis. Up to now only few contributions focused on second order methods for optimal control of fluids, see Ghattas *et al* [22] and Heinkenschloss [32]. These works are restricted to stationary problems, however.

To define the spaces and operators required for the investigation of (1) we introduce the solenoidal spaces

$$H = \{v \in C_0^\infty(\Omega)^2\colon \text{ div } v = 0\}^{-|\cdot|_{L^2}}, V = \{v \in C_0^\infty(\Omega)^2\colon \text{ div } v = 0\}^{-|\cdot|_{H^1}},$$

with superscripts denoting closures in the respective norms. Further we define

$$W = \{v \in L^2(V)\colon v_t \in L^2(V^*)\} \quad \text{and} \quad Z := L^2(V)\times H,$$

where W is endowed with the norm

$$|v|_W = (|v|^2_{L^2(V)} + |v_t|^2_{L^2(V^*)})^{1/2},$$

and set $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(V^*), L^2(V)}$, with V^* denoting the dual space of V. Here $L^2(V)$ is an abbreviation for $L^2(0,T;V)$ and similarly $L^2(V^*) = L^2(0,T;V^*)$. Recall that up to a set of measure zero in $(0,T)$ elements $v \in W$ can be identified with elements in $C([0,T];H)$. In (1) further U denotes the Hilbert space of controls which is identified with its dual U^*, B is a bounded linear operator from U to $L^2(V^*)$. The cost functional $J\colon W \times U \to \mathbb{R}$ is assumed to be bounded from below, weakly lower semi-continuous, twice Fréchet differentiable with locally Lipschitzean second derivative, and radially unbounded in u, i.e. $J(y,u) \to \infty$ as $|u|_U \to \infty$, for every $y \in W$. We define the nonlinear mapping

$$e\colon W \times U \to Z^*$$

by

$$e(y,u) = (\tfrac{\partial y}{\partial t} + (y \cdot \nabla)y - \nu \Delta y - Bu, y(0) - y_0).$$

In variational form the constraints in (1) can be equivalently expressed as: given $u \in U$ find $y \in W$ such that $y(0) = y_0$ and

$$(2) \quad \langle y_t, v \rangle + \langle (y \cdot \nabla)y, v \rangle + \nu (\nabla y, \nabla v)_{L^2(L^2)} = \langle Bu, v \rangle \quad \forall v \in L^2(V).$$

It is well known, see Temam [72] that for every $u \in U$ (2) admits a unique solution $y(u) \in W$. Therefore, with respect to existence (1) can equivalently be rewritten as

$$(3) \qquad \min \hat{J}(u) = J(y(u),u) \text{ subject to } u \in U,$$

where $y(u) \in W$ satisfies $e(y(u),u) = 0$. It is proved by Abergel *et al* [1] that (3) admits a solution $(y^*, u^*) \in W \times U =: X$.

We shall frequently refer to the linearized Navier-Stokes system and the adjoint equations given next:

$$(4) \qquad \begin{cases} v_t + (v \cdot \nabla)y + (y \cdot \nabla)v - \nu \Delta v = f & \text{in } \Omega \text{ a.e. on } (0,T], \\ v(0) = v_0, \end{cases}$$

and

$$(5) \qquad \begin{cases} -w_t + (\nabla y)^t w - (y \cdot \nabla)w - \nu \Delta v = g & \text{in } \Omega \text{ a.e. on } [0,T), \\ w(T) = 0. \end{cases}$$

The following proposition from [40] is essential for the analysis of Newton and quasi-Newton methods.

Proposition 2.1. Let $x = (y,u) \in W \times U$. Then $e_y(x)\colon W \to Z^*$ is a homeomorphism. Moreover, if the inverse of its adjoint $e_y^{-*}(x)\colon W^* \to Z$ is applied to an element $g \in W^* \cap L^\alpha(V^*)$, where $\alpha \in [1, 4/3]$ then setting $(w, w_0) := e_y^{-*}(x)g$ $\in L^2(V) \times H$ we have $w_t \in L^\alpha(V^*)$, $w(0) = w_0$ and w is the variational solution to (5).

3 Derivatives

To apply Newton's method to (3) we need second order information of the functional $\hat{J}$. This can be obtained by applying the implicit function theorem to the equation $e(y,u) = 0$. For the convenience of the reader we give the characterization of the derivatives of e, [40].

Proposition 3.1. The operator $e = (e^1, e^2)\colon X \to Z^*$ is twice continuously differentiable with Lipschitz continuous second derivative. The actions of the first two derivatives of e^1 are given by

$$\langle e^1_x(x)(w,s), \phi\rangle = \langle w_t, \phi\rangle + \langle (w\cdot\nabla)y, \phi\rangle + \langle (y\cdot\nabla)w, \phi\rangle \\ + \nu(\nabla w, \nabla\phi)_{L^2(L^2)} - \langle Bs, \phi\rangle_{L^2(L^2)},$$

where $x = (y,u) \in X, (w,s) \in X$ and $\phi \in L^2(V)$, and

$$\langle e^1_{xx}(x)(w,s)(v,r), \phi\rangle = \langle e^1_{yy}(x)(w,v), \phi\rangle = \\ \langle (w\cdot\nabla)v, \phi\rangle + \langle (v\cdot\nabla)w\phi, v\rangle =: \langle v, H(\phi)w\rangle_{W,W^*}, \tag{6}$$

where $(v,r) \in X$.

As a consequence of variations of standard estimates for the Navier-Stokes and the linearized Navier-Stokes equations and the implicit function theorem the first derivative of the mapping $u \to y(u)$ at u in direction δu is found to be

$$y'(u)\delta u = -e_y^{-1}(x)e_u(x)\delta u, \tag{7}$$

where $x = (y(u), u)$. By the chain rule we thus obtain

$$\langle \hat{J}'(u), \delta u\rangle_U = \langle J_u(x) - e_u^*(x)e_y^{-*}(x)J_y(x), \delta u\rangle_U.$$

Introducing the variable

$$\lambda = (\lambda^1, \lambda^0) = -e_y^{-*}(x)J_y(x) \in Z \tag{8}$$

we obtain the Riesz representation for the first derivative of $u \to \hat{J}(u)$:

$$\hat{J}'(u) = J_u(x) + e_u^*\lambda. \tag{9}$$

If in addition $J_y(x) \in L^{4/3}(V^*) \cap W^*$ we have $\lambda^1_t \in L^{4/3}(V^*)$, $\lambda^1 \in C([0,T]; H)$ and λ^1 is the variational solution of

$$\begin{cases} -\lambda^1_t + (\nabla y)^t\lambda^1 - (y\cdot\nabla)\lambda^1 - \nu\Delta\lambda^1 = -J_y(x) \\ \lambda^1(T) = 0, \end{cases} \tag{10}$$

where the first equation holds in $L^{4/3}(V^*) \cap W^*$.
The computation of the representation of the second derivative of $\hat{J}''$ of $\hat{J}$ is more involved. Introducing the Lagrangian $L\colon X \times Z \to \mathbb{R}$

$$L(x,\lambda) = J(x) + \langle e(x), \lambda\rangle_{Z^*,Z} \tag{11}$$

and the matrix of operators

$$T(x) = \begin{pmatrix} -e_y^{-1}(x)e_u(x) \\ Id_U \end{pmatrix} \in \mathcal{L}(U, X) \tag{12}$$

we observe that the second derivative of $\hat{J}$ can be expressed as

$$\hat{J}''(u) = T^*(x)L_{xx}(x,\lambda)T(x), \tag{13}$$

where $x = (y(u), u)$. The second derivative of L with respect to x is given by

$$L_{xx}(x,\lambda) = \begin{pmatrix} J_{yy}(x) + & \langle e^1_{yy}(x)(\cdot,\cdot), \lambda^1 \rangle & 0 \\ & 0 & J_{uu}(x) \end{pmatrix} \in \mathcal{L}(X, X^*).$$

4 Newton's algorithm

This section contains a description of Newton's method to solve (3). Throughout u^* denotes a (local) solution to (3). For the sake of reference we specify the Newton algorithm without globalization strategy.

Algorithm 4.1. (Newton Algorithm).

1. Choose $u^0 \in N(u^*)$, set $k = 0$.
2. Do until convergence
   i) solve $\hat{J}''(u^k)\delta u^k = -\hat{J}'(u^k)$,
   ii) update $u^{k+1} = u^k + \delta u^k$,
   iii) set $k = k + 1$.

Let us consider the linear system in 2.i). If U is infinite dimensional an approximation must be carried out before it can be realized numerically. The dimension is then that of the discretized control space. From the characterization of the Hessian $\hat{J}''(u^k)$ we conclude that its evaluation requires as many solutions to the linearized Navier–Stokes equation (7) with appropriate right hand sides as is the dimension of the discretized control space. If this dimension is large then direct evaluation of $\hat{J}''(u^k)$ is not feasible and as a consequence the linear system in 2.i) is solved iteratively, e.g. by a conjugate gradient technique. In this case we refer to 2.i) as the "inner " loop as opposed to the do–loop in 2. which is the "outer" loop of the Newton algorithm. The inner loop at iteration level k of the outer loop requires to

(i) evaluate $\hat{J}'(u^k)$, i.e. given u^k compute $y(u^k)$ from (2) and λ^1 from (10) with $x = (y(u^k), u^k)$,

(ii) iteratively evaluate the action of $\hat{J}''(u^k)$ on δ^k_j, the j–th iterate of the inner loop on the k–th level of the outer loop.

The iterate $q = \hat{J}''(u^k)\delta^k_j$ can be evaluated by successively applying the steps

a) solve (4) with $v_0 = 0$ and $f = B\delta_j^k$ in $L^2(V^*)$ for $v \in W$ where $y = y(u^k)$,

b) evaluate $J_{yy}(x)v + \langle e^1_{yy}(x)(v,\cdot), \lambda^1\rangle$,

c) solve (5) with $g = J_{yy}(x)v + \langle e^1_{yy}(x)(v,\cdot), \lambda^1\rangle$ in W^* for $w \in L^2(V)$,

d) and finally set $q := J_{uu}\delta u + B^* w$.

It can be shown that, provided $J_{yy}(x) \in \mathcal{L}(W \cup L^4(V), W^* \cap L^{4/3}(V^*))$, the functional appearing in b) is an element of $W^* \cap L^{4/3}(V^*)$, and hence (5) is meaningful in the sense of admitting a solution w in $L^2(V)$ with $w_t \in L^{4/3}(V^*)$. Summarizing, for the outer iteration of the Newton method one Navier–Stokes solve for $y(u^k)$ and one linearized Navier–Stokes solve for $\lambda(u^k)$ are required. For the inner loop one forward (– in time) as well as one backwards linearized Navier–Stokes solve per iteration are necessary.

Concerning initialization we observe that if an initial guess $(y_0, u_0) \in W \times U$ are available (with y_0 not necessarily $y(u_0)$) then alternatively to the initialization in Algorithm 4.1 this information can be used advantageously to compute the adjoint variable λ^1 required for the initial guess for the right hand side of the linear system as well as to carry out steps a) - c) for the evaluation of the Hessian. There is no necessity to compute $y(u_0)$ from u_0.

Local quadratic convergence of the Newton method can be guaranteed by a second order sufficient optimality at the local minimizer u^*. This condition in turn is satisfied, for example, if J is convex with respect to y and u separately, and $|J_y(x^*)|_{W^*}$, where $x^* = (y(u^*), u^*)$, is sufficiently small. The proof of the following theorem can then be given by standard arguments.

Theorem 4.1. If $J_{yy}(x^*)$ is positive semi–definite, $J_{uu}(x^*)$ is positive definite, and $|J_y(x^*)|_{W^*}$ is sufficiently small then there exists a neighborhood $\mathcal{U}(u^*)$ such that for every $u^0 \in \mathcal{U}(u^*)$ the iterates $\{u^n\}_{n\in\mathbb{N}}$ of Newton's Algorithm 4.1 converge quadratically to u^*.

It is worth noting that by choosing the initial approximation to the Hessian appropriately local super-linear convergence for quasi-Newton variants of Algorithm 4.1. can be proved. This is due to the fact that the Hessian $\hat{J}''(u)$ in (13) can be decomposed into the sum of a compact operator and $J_{uu}(y(u), u)$ [40].

5 Receding horizon control and instantaneous control

The essential feature of receding horizon control techniques is to replace the optimal control problem on the full time horizon by a sequence of optimal control problems on short control horizons and to move this horizon forward with respect to time. Instantaneous control is based on a time discretization of (1). At each discrete time level t_i a stationary control problem is solved approximately by applying exactly one step of the gradient algorithm and the resulting control g_i^* is used to steer the system from t_i to t_{i+1}, where a new control is determined. Therefore instantaneous control can be regarded as an inexact variant of (1,1)-receding horizon control. Of course, it cannot be claimed that the controls obtained by utilizing receding horizon control techniques approximate the optimal control for (1). However, these procedures are justified by the effectiveness that they exhibit for numerical examples [9, 11], the interpretation that they allow for suboptimal feedback control [13, 14] and their stabilizing effect [41], [47]. - In the remainder of this section we discuss the instantaneous control strategy.

To commence, let $m > 1$ be fixed and set $\delta t = \frac{T}{m}$, $t_i = i\delta t, i = 0, \dots, m$. As an intermediate step in the derivation consider the case where the Navier-Stokes equations in (1) are approximated by a Crank-Nicolson scheme. At the i-th level of the Crank-Nicolson based suboptimal strategy one approximately solves the following stationary optimal control problem, where the variables (y, p, g) correspond to $(y(t_i), p(t_i), g(t_i))$.

$$\left\{ \begin{array}{l} \min \hat{J}(u) = J(y,u) \\ \text{s.t.} \\ e_1((y,p),u) \;=\; 0, \end{array} \right. \tag{14}$$

where

(15)

$$e_1((y,p),u) = 0 \Longleftrightarrow \left\{ \begin{array}{rll} \frac{1}{\delta t} y + \frac{1}{2}(y \cdot \nabla) y + \nabla p = & \frac{\nu}{2}\Delta y + \tilde{R}(y(t_{i-1})) & \\ & +Bu & \text{in } \Omega \\ -\operatorname{div} y = & 0 & \text{in } \Omega \\ y = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

Here $\tilde{R}(y(t_{i-1})) = \frac{1}{\delta t} y(t_{i-1}) + \frac{\nu}{2}\Delta y(t_{i-1}) - \frac{1}{2}(y(t_{i-1}) \cdot \nabla) y(t_{i-1})$ is a known inhomogeneity.

For a given control u the gradient of the cost functional $\hat{J}$ in (14) at point u is given by

$$\hat{J}'(u) = J_u(y,u) - B^*\xi \tag{16}$$

where for a given control $u \in U$ the tuple (ξ, π) is a solution to $e^*_{1(y,p)}((y,p),u)(\xi,\pi) = -J_y(y,u)$, i.e.

$$\left\{ \begin{array}{rcll} \frac{1}{\delta t}\xi - \frac{1}{2}(y \cdot \nabla)\xi + \frac{1}{2}(\nabla y)^t \xi - \frac{\nu}{2}\Delta\xi + \nabla\pi & = & -J_y(y,u) & \text{in } \Omega \\ -\operatorname{div} \xi & = & 0 & \text{in } \Omega \\ \xi & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{17}$$

with (y, p) a solution of the primal equations (15) with volume force Bu. The instantaneous control strategy for the control problem (1) based on Crank-Nicolson time integration and one step of the gradient algorithm can be formulated as

Algorithm 5.1. (Instantaneous control with Crank-Nicolson)

1. Given an initial state y_0, set $j = 0$, $t_0 = 0$.

2. Given u_j^0
   i. Solve for (y,p) in (15) with $u = u_j^0$.
   ii. Solve for (ξ, π) in (17).
3. Set $\hat{J}'(u_j^0) = (J_u(y, u_j^0) - B^* u_j^0$.
4. Given $\rho > 0$, set $u_{j+1} = u_j^0 - \rho\, \hat{J}'(u_j^0)$.
5. Solve for (y_{j+1}, p_{j+1}) in (15) with $u = u_{j+1}$ and $\tilde{R}(y_j)$.
6. Set $t_{j+1} = t_j + h$, $j = j+1$.
7. Goto 2.

Note that (17) requires information of the velocity inside the domain as well as on its boundary. For computations this requires re calculation of the system matrix for the adjoint equation whenever the primal velocity field y changes. Even more importantly from the feedback-control point of view, one aims for replacing y by observable quantities of the system and computing u from (17) and (16). This would require knowledge of the velocity field y inside Ω which is restrictive. To circumvent these drawbacks the discretization is further simplified by replacing the implicit Crank-Nicolson scheme for the nonlinear equation by a semi-implicit time discretization scheme (see also [15]). In what follows the Euler method implicit in the viscous term and explicit in the nonlinear convective term is chosen, but different discretization schemes treating the non-linear term explicitly could be considered as well. In this case only knowledge of observable quantities is necessary to compute the adjoint, which in turn determines the control.

Consider the minimization problem

$$
(P_i) \begin{cases} \min \hat{J}(u) = J(y,u) \\ \text{s.t.} \\ e_2((y,p),u) \;=\; 0, \end{cases} \tag{18}
$$

where

$$
e_2((y,p),u) \;=\; 0 \iff \left\{ \begin{array}{rcll} y - c\Delta y + \delta t \nabla p & = & R(y(t_{i-1})) + Bu & \text{in } \Omega \\ -\operatorname{div} y & = & 0 & \text{in } \Omega \\ y & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{19}
$$

Here, $R(y(t_{i-1})) = y(t_{i-1}) - \delta t (y(t_{i-1}) \cdot \nabla) y(t_{i-1})$ and $c := \frac{\delta t}{\mathrm{Re}} = \delta t \nu$. Note that the information from time level t_{i-1} to t_i is passed solely through the inhomogeneity R. Again for a given control u the gradient of the cost functional $\hat{J}$ in (14) at point u is given by (16), where now (ξ, π) is a solution to $e^*_{2(y,p)}((y,p),u)(\xi,\pi) = -J_y(y,u)$, i.e.

$$
\left\{ \begin{array}{rcll} \xi - c\Delta \xi + \nabla \pi & = & -J_y(y,u) & \text{in } \Omega \\ -\delta t \operatorname{div} \xi & = & 0 & \text{in } \Omega \\ \xi & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{20}
$$

with (y, p) the solution of the primal equations $e((y, p), u) = 0$ in (19). Hence only observable state information is required for the computation of gradient. As a consequence instantaneous control with the semi-implicit Euler method as time discretization scheme can be interpreted as a suboptimal feedback algorithm for (1). Its algorithmic form is given by Algorithm 5.1 with (15) and (17) replaced by (19) and (20), respectively.

6 Reduced order modeling (ROM) and optimal control

The ROM approach to optimal control problems such as (1) is based on approximating the nonlinear dynamics by a Galerkin technique utilizing basis functions that contain characteristics of the expected controlled flow. This is in contrast to finite element based Galerkin schemes where the basis elements are not related to the physical properties of the system that they approximate. Consequently one expects that only a few basis elements will provide a reasonably good description of the controlled flow. Various ROM techniques differ in the choice of the basis functions [37, 48]. Here we use the snapshot variant of Proper Orthogonal Decomposition (POD) introduced by Sirovich in [69] to obtain a low-dimensional approximation of the Navier-Stokes equations and to define the control space.

To describe the model reduction let $y^1, \dots, y^n$ denote an ensemble of snapshots of the flow corresponding to different time instances. For the approximated flow we make the Ansatz

$$y = \bar{y} + \sum_{i=1}^{n} \alpha_i \Phi_i \tag{21}$$

with modes Φ_i that are obtained as follows:

1. Compute the mean $\bar{y} = \frac{1}{n} \sum\limits_{i=1}^{n} y^i$.
2. Build the correlation matrix $K = k_{ij}$, $k_{ij} = \int\limits_{\Omega} (y^i - \bar{y})(y^j - \bar{y})\, dx$.
3. Compute the eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $v^1, \dots, v^n$ of K.
4. Set $\Phi_i := \sum\limits_{j=1}^{n} v_j^i (y^j - \bar{y})$.
5. Normalize $\Phi_i = \frac{\Phi_i}{||\Phi_i||}$.

The modes Φ_i are pairwise orthonormal and are optimal with respect the L^2 scalar product in the sense that no other basis of $D := \mathrm{span}\{y_1 - \bar{y}, \dots, y_n - \bar{y}\}$ can contain more energy in fewer elements, compare [4, 58].

In order to obtain a low-dimensional basis for the Galerkin Ansatz modes corresponding to small eigenvalues are neglected. To make this idea more precise let $D^M := \text{span}\{\Phi_1, \ldots, \Phi_M\}$ $(1 \leq M \leq N := \dim D)$ and define the relative information content of this basis by

$$\mathrm{I}(M) \;:=\; \sum_{k=1}^{M} \lambda_k \,/\, \sum_{k=1}^{N} \lambda_k .$$

If the basis is required to describe $\gamma\%$ of the total information contained in the space D, then the dimension M of the subspace D^M is determined by

$$M = \operatorname{argmin}\left\{\mathrm{I}(M) : \mathrm{I}(M) \geq \gamma\right\}. \tag{22}$$

The reduced dynamical system is obtained by inserting (21) into the Navier-Stokes system and using a subspace D^M containing sufficient information as test space. This results in

$$(y_t, \Phi_j) + \nu(\nabla y, \nabla\Phi_j) + ((y \cdot \nabla)y, \Phi_j) = (Bu, \Phi_j) \quad (j = 1, \ldots, M).$$

which may be rewritten as

$$\dot{\alpha} + A\alpha = n(\alpha) + \beta + r, \quad \alpha(0) = a_0.$$

The components of a_0 are computed from $\bar{y} + \sum_{k=1}^{M}(y_0 - \bar{y}, \Phi_k)\Phi_k$. The matrix A is the POD stiffness matrix and the inhomogeneity r results from the contribution of the mean $\bar{y}$ to the Ansatz in (21). The M-vector β results from the contribution of the controls, which are also sought in the space D^M, i.e. we make the Ansatz

$$u = \sum_{i=1}^{M} \beta_i \Phi_i . \tag{23}$$

The reduced optimization problem corresponding to (1) is obtained by plugging (21) and (23) into the cost functional and utilizing the reduced dynamical system as constraint in the optimization process. Altogether we obtain

$$(\mathrm{ROM}) \begin{cases} \min \tilde{J}(\alpha, \beta) = J(y, u) \\ \text{s.t.} \\ \dot{\alpha} + A\alpha = n(\alpha) + \beta + r, \quad \alpha(0) = a_0. \end{cases} \tag{24}$$

With these preparations we are in the position to formulate the quasi-optimal control strategy. We choose the value γ for (22) and a sequence of increasing numbers N_j.

Algorithm 6.1. (POD-based adaptive control)

1. Let a set of snapshots $y_i^0, i = 1, \ldots, N_0$ be given and set j=0.
2. Compute M according to (22) with N replaced by N_j.

3. Compute POD modes and solve the reduced optimization problem (24) for $u^j \in D_j^M$.

4. Compute the state y^j corresponding to the current control u^j and add the snapshots $y_i^{j+1}, i = N_j + 1, \ldots, N_{j+1}$ to the snapshot set $y_i^j, i = 1, \ldots, N_j$.

5. If $|u^{j+1} - u^j|$ is not sufficiently small, set j = j+1 and goto 2.

In numerical tests this algorithm proved to be effective in that the control objective was reached with computing times that are significantly smaller than those for optimization without system reduction.

7 Numerical experiments

Here we present numerical results related to the approaches presented in sections 4.-6. The flow configuration is given by the two-dimensional channel with a cylindrical obstacle and coincides with that used in [68] as a benchmark for Navier-Stokes solvers, see Fig. 1. At the inlet a parabolic flow profile is pre-

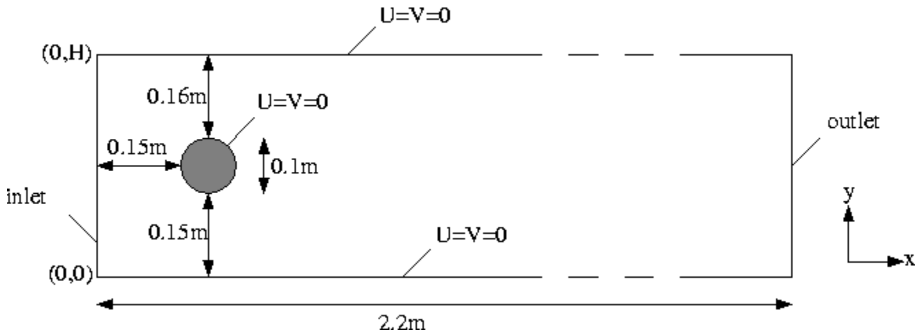

Figure 1: Flow configuration

scribed and at the outlet the so called 'do-nothing' boundary conditions are used [33]. As a consequence the boundary conditions for the Navier-Stokes equations in (1) have to be suitably modified. The control objective is to track the Navier Stokes flow to the Stokes flow by applying control forces in the spatial domain. Thus, the optimal control problem is given by

$$
(25) \qquad \begin{cases} \min_{(y,u)\in W\times U} \; J(y,u) := \frac{1}{2}\int_Q |y-z|^2\,dxdt + \frac{\alpha}{2}\int_Q |u|^2\,dxdt \\ \text{subject to} \\ \quad \frac{\partial y}{\partial t} + (y\cdot\nabla)y - \nu\Delta y + \nabla p = Bu & \text{in } Q = (0,T)\times\Omega, \\ \quad \text{div } y = 0 & \text{in } Q, \\ \quad y(t,\cdot) = y_d & \text{on } (0,T)\times\Gamma_d, \\ \quad \nu\partial_\eta y(t,\cdot) = p\eta & \text{on } (0,T)\times\Gamma_{out}, \\ \quad y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases}
$$

where $Q := \Omega \times (0,T)$ denotes the time-space cylinder, Γ_d the Dirichlet boundary at the inlet and Γ_{out} the outflow boundary. In this example the volume for the flow measurements and the control volume for the application of the volume forces each cover the whole spatial domain. From the practical point of view this is not feasible. However, from the numerical standpoint this case can present difficulties, since the inhomogeneities in the primal and adjoint equations are large.

We note that it is an open problem to prove existence of global smooth solutions in two space dimensions for the instationary Navier-Stokes equations with do-nothing boundary conditions. [6, 66].

The Reynolds number Re$= 1/\nu$ for the above configuration is determined by

$$\mathrm{Re} = \frac{\bar{U}d}{\mu},$$

with $\bar{U}$ denoting the bulk velocity at the inlet, d the diameter of the cylinder and μ the molecular viscosity of the fluid. In all numerical experiments to be presented we choose a parabolic inflow profile at the inlet, $d = 1$, Re=100 and the channel length is $l = 20d$. For the spatial discretization the Taylor-Hood finite elements on a grid with 7808 triangles, 16000 velocity and 4096 pressure nodes are used. As time interval in (25) we use $[0,T]$ with $T = 3.4$ which coincides with the length of one period of the wake flow. The time discretization is carried out by a fractional step Θ-scheme [5] or a semi-implicit Euler-scheme on a grid containing 500 points. This corresponds to a time step size of $\delta t = 0.0068$. The total number of variables in the optimization problem (1) therefore is of order 5.4×10^7 (primal, adjoint and control variables). The step size used for the instantaneous control strategy is the same as that used for the time integration. The value for α is chosen to be $\alpha = 2.10^{-2}$.

For the POD method we add 100 snapshots to the snapshot set in every iteration of Algorithm 6.1. The relative information content of the basis formed by the modes is required to be larger than 99.99%. The first ten modes generated from the uncontrolled snapshots are presented in Fig. 2. Fig. 3 shows the dynamics of the reduced Navier-Stokes system for varying number of modes. Graphically the dynamics are already recovered utilizing eight modes. Note, that the time horizon shown in this figure is $[34, 44]$ while the snapshots are taken only in the interval $[0, 3.4]$. The "amplitude" in Fig. 3 is defined as $\sum_{i=1}^{n} \alpha_i(t)|\Phi_i(x)|$, with x a point just behind the cylinder.

In Fig. 4 the vorticity and the streamlines of the uncontrolled flow (top) and the Stokes flow (bottom) are presented. For the numerical solution of the reduced optimization problems the Schur-complement SQP-algorithm is used. All computations are performed on an ORIGIN^{TM} 200.

We first present a comparison between the optimal control strategy based on Newton's method in Section 4 and Algorithm 6.1. In Fig. 5 selected iterates of the evolution of the cost in $[0,T]$ for both approaches are given. The adaptive algorithm terminates after 5 iterations which require about 8 CPU hours to obtain the quasi-optimal control $\tilde{u}^*$. It achieves a remarkable cost reduction

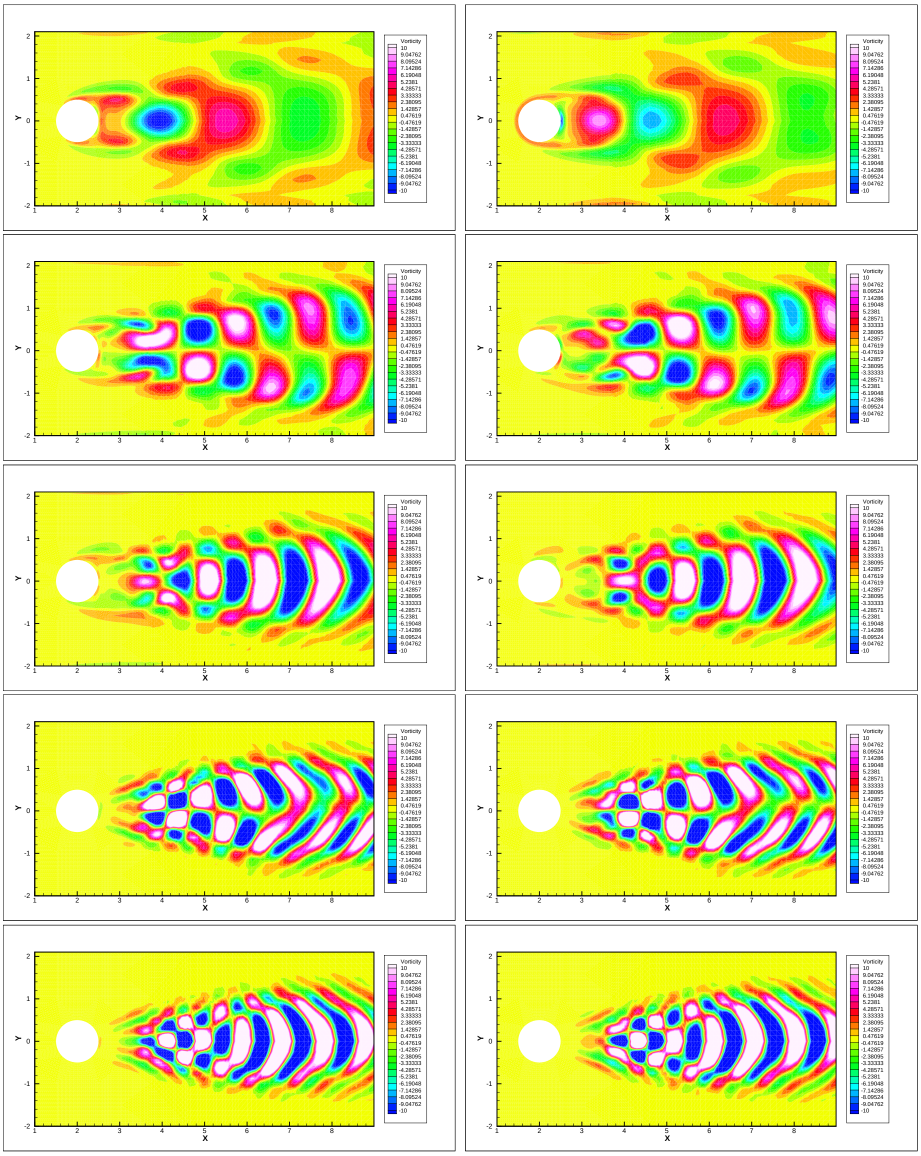

Figure 2: First 10 Modes generated form uncontrolled snapshots, vorticity

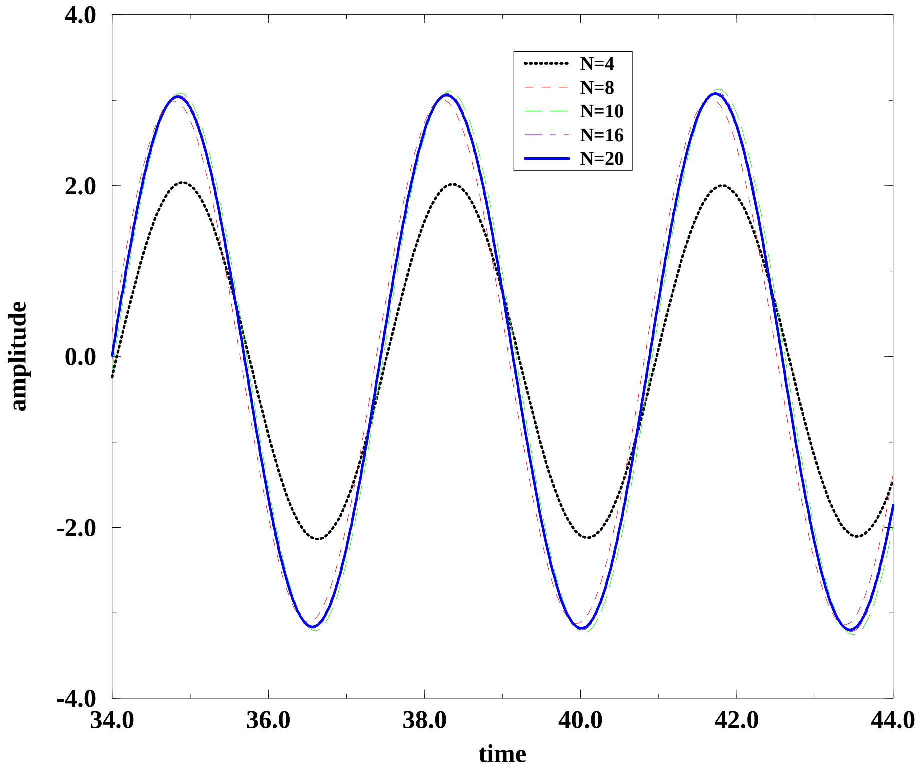

Figure 3: Development of amplitudes for varying number of snapshots

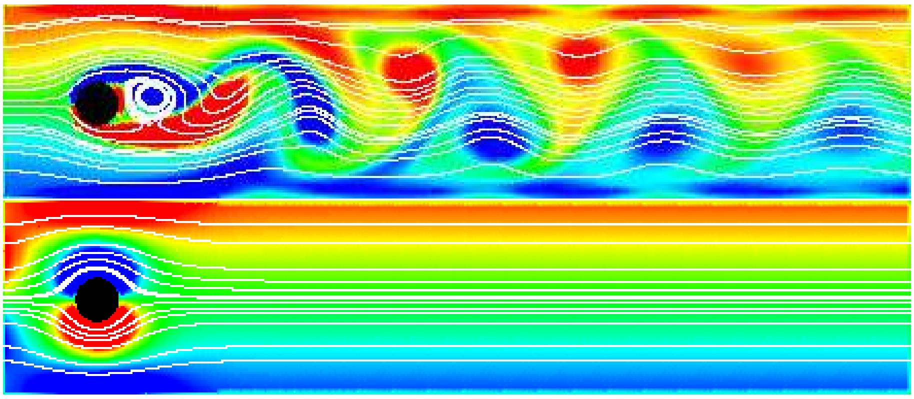

Figure 4: Uncontrolled flow (top) and Stokes flow (bottom)

decreasing the value of the cost functional for the uncontrolled flow $J(u^0) = 22.658437$ to $J(\tilde{u}^*) = 6.440180$. It is also worth recording that to recover 99.99% of the energy stored in the snapshots in the first iteration 10 modes have to be taken, 20 in the second iteration, 26 in the third, 30 in the fourth, and 36 in the final iteration. The computation of the optimal control with the Newton method takes approximately 140 CPU hours. This includes an initialization process with a step-size controlled gradient algorithm. To obtain a relative error $|\nabla J(u^n)|/|\nabla J(u^0)|$ lower than 10^{-2}, 32 gradient iterations are needed with $J(u^{32}) = 1.138325$. As initial control $u^0 = 0$ was taken. Note that every gradient step amounts to solving the non-linear Navier-Stokes equations in (1), the adjoint equations (10) and a further Navier-Stokes system for the computation of the step-size in the gradient algorithm, compare [40]. Newton's algorithm then is initialized with u^{32} and 3 Newton steps further reduce the value of the cost functional to $J(u^*) = 1.090321$. The controlled flow based on the Newton method is graphically almost indistinguishable from the Stokes flow in Fig.4.

The Newton method was also tested for control of driven cavity flow with Re $= 400$, [39]. In that case no initialization phase was necessary to guarantee convergence of the Newton method and in general superlinear convergence of the iterates could be observed. Fig. 6 shows a zoom on the evolutions of the control cost for both approaches. The optimal control cost at $t = 0$ has the value 122.71 and is approximately 10 times larger than the quasi-optimal control cost at the same time. Fig. 7 shows the streamlines and the vorticity of the flow controlled by the adaptive approach at $t = 3.4$ (top) and the mean flow $\bar{y}$ (bottom), the latter formed with the snapshots of all 5 iterations. The controlled flow no longer contains vortex sheddings and is approximately stationary. Recall that the controls are sought in the space of deviations from the mean flow. This explains the remaining recirculations behind the cylinder. We expect that they can be reduced if the Ansatz for the controls in (23) is based on a POD of the

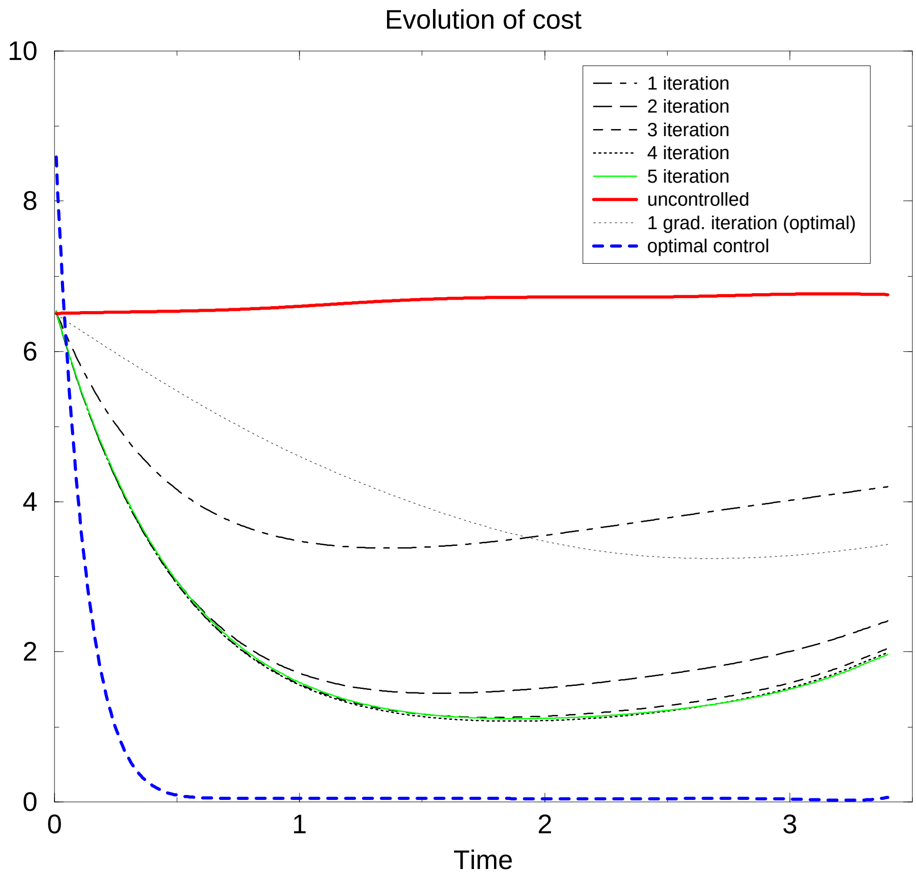

Figure 5: Evolution of cost

snapshots themselves rather than on a POD of the deviation from their mean.

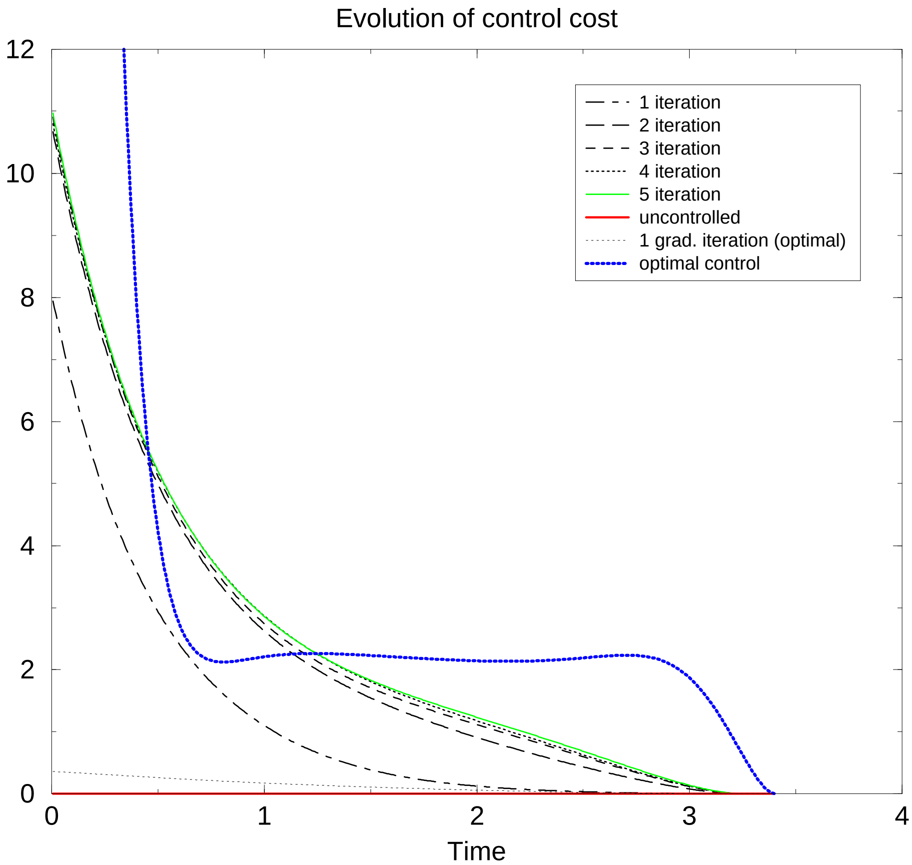

Figure 6: Evolution of control cost

We next present a numerical comparison between the instantaneous control strategy and the optimal control computed by Newton's method as already explained above. To compute the control u^I with the instantaneous control strategy we need about 6 cpu-hours on the same machine resulting in $J(u^I) = 18.6108$. The evolutions of the cost, of the control cost and of the control gain are shown in Fig. 8. As can be seen the instantaneous control strategy results in a strong control action at the beginning of the control horizon and obtains a remarkable reduction for the L^2-norm of the control gain. The optimal control strategy computes a control of moderate strength at the beginning of the control horizon and equi-distributes the control costs.

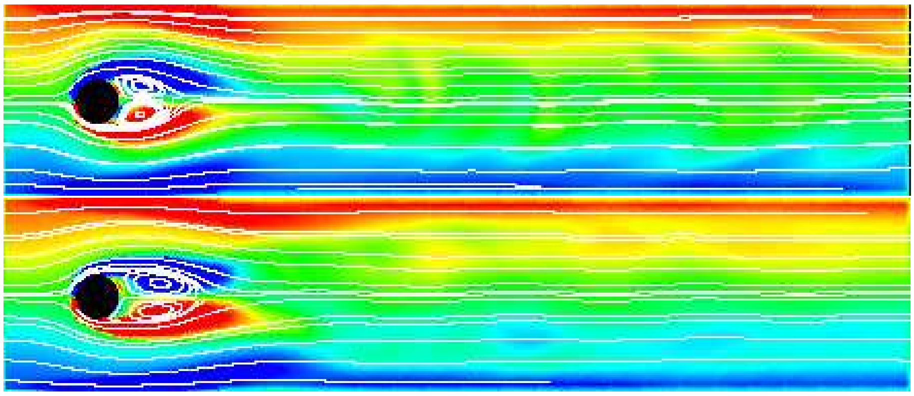

Figure 7: Example 1: POD controlled flow (top) and mean flow $\bar{y}$ (bottom)

8 Conclusions

Two suboptimal control concepts for the instationary Navier-Stokes equations are presented and compared to a Newton-based approach for control of the wake flow around a cylinder. The adaptive quasi-optimal control approach is based on the snapshot form of the POD for the Navier-Stokes equations and iteratively computes controls by solving low-dimensional optimization problems. Compared to the optimal control approach the quasi-optimal method is much less time consuming and provides competitive controls. The instantaneous control method provides controls that are expensive, in particular at the beginning of the control horizon, while yielding a remarkable reduction of the control gain.

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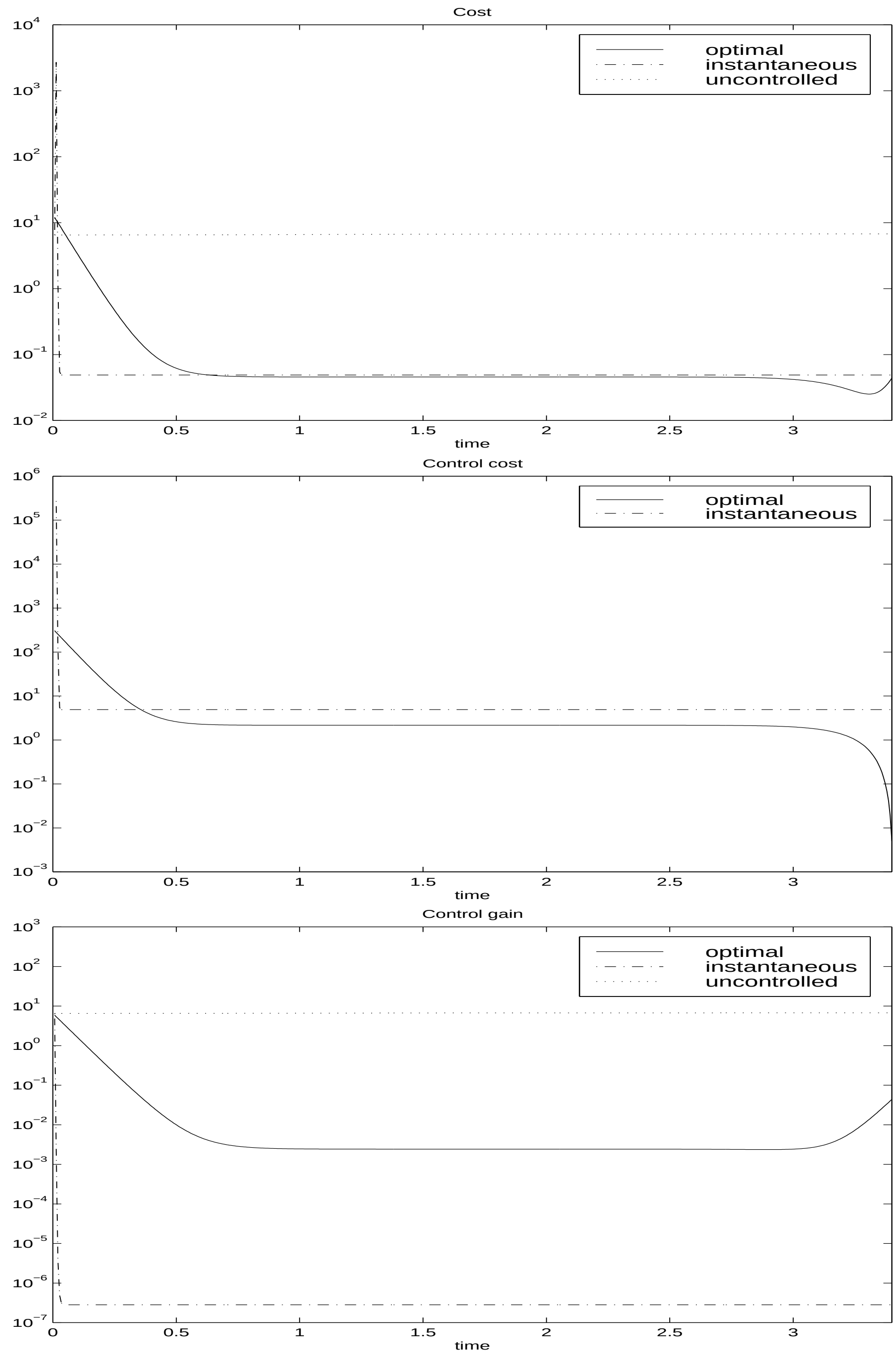

Figure 8: Comparison of Cost (top), control cost (middle) and control gain (bottom) between opitmal and instantaneous control

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