Adaptive control of a wake flow using proper orthogonal decomposition

KONSTANTIN AFANASIEV[1] & MICHAEL HINZE[2]

PREPRINT NO. 648/1999

[1] Technische Universität Berlin, Fachbereich Mathematik, Straße des 17. Juni 136, D-10623 Berlin, email: afanasiev@pi.tu-berlin.de

[2] Technische Universität Berlin, Fachbereich Mathematik, Straße des 17. Juni 136, D-10623 Berlin, email: hinze@math.tu-berlin.de

Contents

Abstract

We present an effective quasi-optimal control method for the instationary Navier-Stokes equations. It in an adaptive manner utilizes the snapshot form of the proper orthogonal decomposition to construct a low dimensional approximation of the Navier-Stokes equations which then is used as subsidiary condition in the underlying optimization problem. The numerical solution of the reduced control problem is used as control for the full equations and to compute new snapshots. The set of the new snapshots then is added to the set of the snapshots already available, the low dimensional approximation of the Navier-Stokes equations is re-computed and the reduced optimization is restarted. This process is iterated until convergence is achieved.

As numerical example we present tracking-type control of the periodic flow around a cylinder. The numerical results of the quasi-optimal approach are compared to that of the optimal approach.

1 Introduction

In many applications it can be useful to control fluid flow and to optimize its characteristics. Typical applications include the reduction of drag and the lag or elimination of the transition from laminar to turbulent flow regime. The solutions of the underlying optimal control problems satisfy a system of coupled nonlinear partial differential equations involving the time-dependent Navier-Stokes equations. Solving these systems for realistic flows numerically is unworkable with most currently available computing facilities. Hence there is a need for suboptimal control strategies which incorporate the nonlinear character of the flow and which are amenable to controlling the flow using presently available computing environments.

In this work we present an iterative quasi-optimal approach to the control of the instationary Navier-Stokes equations. It is based on approximating the nonlinear dynamics of the Navier-Stokes equations by a reduced order model obtained by the snapshot form of the proper orthogonal decomposition [20] and carrying out an exact optimization for the reduced system. The numerical solution of the reduced control problem is used as control for the full equations and to compute new snapshots. The set of the new snapshots then is added to the set of the snapshots already available, the low dimensional approximation of the Navier-Stokes equations is re-computed and the reduced optimization is restarted. This process is iterated until convergence is achieved.

Related approaches to the control of the instationary Navier-Stokes equations can be found in [11, 15, 17, 21]. An excellent overview on diverse aspects of control for fluids can be obtained from [9]. For further references we also refer to [1, 4, 5, 6, 7, 8, 13, 14, 18].

The paper is organized as follows. In Section 2 we formulate the distributed control problem for the instationary Navier-Stokes equations and provide an appropriate analytical frame for its solution and for the proper formulation of Newton's method. Section 3 is devoted to the description of the reduced order model approach and its application to the instationary Navier-Stokes equations. Furthermore, in the same section the adaptive quasi-optimal control approach is introduced. Finally, in Section 4 we give some numerical examples including a comparison of optimal and quasi-optimal control computations for the two dimensional periodic flow around a cylinder.

2 The optimal control problem

We consider the optimal control problem

$$\left.\begin{array}{ll}
\min_{(y,u)\in W\times U} \; J(y,u) := \frac{1}{2}\int\limits_{Q_o} |y-z|^2\,dxdt + \frac{\gamma}{2}\int\limits_{Q_c} |u|^2\,dxdt & \\
\text{subject to} & \\
\quad \frac{\partial y}{\partial t} + (y\cdot\nabla)y - \nu\Delta y + \nabla p = Bu & \text{in } Q=(0,T)\times\Omega, \\
\quad \text{div } y = 0 & \text{in } Q, \\
\quad y(t,\cdot) = 0 & \text{on } \Sigma=(0,T)\times\partial\Omega, \\
\quad y(0,\cdot) = y_0 & \text{in } \Omega.
\end{array}\right\} \qquad (1)$$

where $Q_c := \Omega_c \times (0,T)$ and $Q_o := \Omega_o \times (0,T)$, with Ω_c and Ω_o subsets of the bounded spatial domain $\Omega \subset \mathbb{R}^2$ denoting control and observation volumes, respectively.

To define the spaces and operators required for the investigation of (1) we introduce the solenoidal spaces

$$H = \{v \in C_0^\infty(\Omega)^2 : \text{ div } v = 0\}^{-|\cdot|_{L^2}}, V = \{v \in C_0^\infty(\Omega)^2 : \text{ div } v = 0\}^{-|\cdot|_{H^1}},$$

with the superscripts denoting closures in the respective norms. Further we define

$$W = \{v \in L^2(V) : v_t \in L^2(V^*)\} \quad \text{and} \quad Z := L^2(V)\times H,$$

where W is endowed with the norm

$$|v|_W = (|v|^2_{L^2(V)} + |v_t|^2_{L^2(V^*)})^{1/2},$$

and set $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(V^*),L^2(V)}$, with V^* denoting the dual space of V. Here $L^2(V)$ is an abbreviation for $L^2(0,T;V)$ and similarly $L^2(V^*) = L^2(0,T;V^*)$. Recall that up to a set of measure zero in $(0,T)$ elements $v \in W$ can be identified with elements in $C([0,T];H)$. In (1) further $U = L^2(Q_c)$ denotes the Hilbert space of controls which is identified with its dual U^*. The the cost functional $J\colon L^2(H) \times U \to \mathbb{R}$ is assumed to be bounded from below, weakly lower semi-continuous, twice Fréchet differentiable with locally Lipschitzean second derivative, and radially unbounded in u, i.e. $J(y,u) \to \infty$ as $|u|_U \to \infty$, for every $y \in W$. We define the nonlinear mapping

$$e\colon W \times U \to Z^*$$

by

$$e(y,u) = \left(\tfrac{\partial y}{\partial t} + (y\cdot\nabla)y - \nu\Delta y - Bu, y(0) - y_0\right)$$

where $Bu(t,\cdot)$ denotes the Leray-projection of the extension by zero of $u(t,\cdot)$ to the whole of Ω and $y_0 \in H$. In variational form the constraints in (1) can be equivalently expressed as: given $u \in U$ find $y \in W$ such that $y(0) = y_0$ and

$$\langle y_t, v\rangle + \langle (y\cdot\nabla)y, v\rangle + \nu(\nabla y, \nabla v)_{L^2(L^2)} = \langle Bu, v\rangle \quad \forall v \in L^2(V). \tag{2}$$

It is well known, see [22] that for every $u \in U$ (2) admits a unique solution $y(u) \in W$. Therefore, with respect to existence (1) can equivalently be rewritten as

$$\min \hat{J}(u) = J(y(u),u) \text{ subject to } u \in U, \tag{3}$$

where $y(u) \in W$ satisfies $e(y(u),u) = 0$. It is proved in [1] that (3) admits a solution $(y^*,u^*) \in W \times U =: X$. We shall frequently refer to the linearized Navier-Stokes system and the adjoint equations given next:

$$\left\{\begin{array}{l} v_t + (v\cdot\nabla)y + (y\cdot\nabla)v - \nu\Delta v = f \quad \text{in } \Omega \text{ a.e. on } (0,T], \\ v(0) = v_0. \end{array}\right. \tag{4}$$

and

$$\left\{\begin{array}{l} -w_t + (\nabla y)^t w - (y\cdot\nabla)w - \nu\Delta v = g \quad \text{in } \Omega \text{ a.e. on } [0,T). \\ w(T) = 0. \end{array}\right. \tag{5}$$

The following proposition is proved in [13].

Proposition 2.1. Let $x = (y,u) \in W \times U$. Then $e_y(x)\colon W \to Z^*$ is a homeomorphism. Moreover, if the inverse of its adjoint $e_y^{-*}(x)\colon W^* \to Z$ is applied to an element $g \in W^* \cap L^\alpha(V^*)$, where $\alpha \in [1, 4/3]$ then setting $(w,w_0) := e_y^{-*}(x)g$ $\in L^2(V) \times H$ we have $w_t \in L^\alpha(V^*)$, $w(0) = w_0$ and w is the variational solution to (5).

The derivatives of the operator e were characterized in [13] and are cited for the convenience of the reader.

Proposition 2.2. The operator $e = (e^1, e^2) \colon X \to Z^*$ is twice continuously differentiable with Lipschitz continuous second derivative. The action of the first two derivatives of e^1 are given by

$$\langle e^1_x(x)(w,s), \phi\rangle = \langle w_t, \phi\rangle + \langle (w\cdot\nabla)y, \phi\rangle + \langle (y\cdot\nabla)w, \phi\rangle \\ + \nu(\nabla w, \nabla\phi)_{L^2(L^2)} - \langle Bs, \phi\rangle_{L^2(L^2)},$$

where $x = (y,u) \in X$, $(w,s) \in X$ and $\phi \in L^2(V)$, and

$$\langle e^1_{xx}(x)(w,s)(v,r), \phi\rangle = \langle e^1_{yy}(x)(w,v), \phi\rangle = \\ \langle (w\cdot\nabla)v, \phi\rangle + \langle (v\cdot\nabla)w\phi, v\rangle =: \langle v, H(\phi)w\rangle_{W,W^*}, \quad (6)$$

where $(v,r) \in X$.

As a consequence of variations of standard estimates for the Navier-Stokes and the linearized Navier-Stokes equations and the implicit function theorem the first derivative of the mapping $u \to y(u)$ at u in direction δu is given by

$$y'(u)\delta u = -e_y^{-1}(x)e_u(x)\delta u, \quad (7)$$

where $x = (y(u), u)$. By the chain rule we thus obtain

$$\langle \hat{J}'(u), \delta u\rangle_U = \langle J_u(x) - e_u^*(x)e_y^{-*}(x)J_y(x), \delta u\rangle_U.$$

Introducing the variable

$$\lambda = -e_y^{-*}(x)J_y(x) \in Z \quad (8)$$

we obtain the Riesz representation for the first derivative of $u \to \hat{J}(u)$:

$$\hat{J}'(u) = J_u(x) + e_u^*\lambda. \quad (9)$$

Here $\lambda = (\lambda^1, \lambda^0) \in Z$, $\lambda^1_t \in L^{4/3}(V^*)$, $\lambda^1 \in C(H)$ and λ^1 is the variational solution of

$$\left\{ \begin{array}{l} -\lambda^1_t + (\nabla y)^t\lambda^1 - (y\cdot\nabla)\lambda^1 - \nu\Delta\lambda^1 = -J_y(x) \\ \lambda^1(T) = 0, \end{array} \right. \quad (10)$$

where the first equation holds in $L^{4/3}(V^*) \cap W^*$.
The computation of the representation of the second derivative of $\hat{J}''$ of $\hat{J}$ is more involved. Introducing the Lagrangian $L \colon X \times Z \to \mathbb{R}$

$$L(x,\lambda) = J(x) + \langle e(x), \lambda\rangle_{Z^*,Z} \quad (11)$$

and the matrix operators

$$T(x) = \begin{pmatrix} -e_y^{-1}(x)e_u(x) \\ Id_U \end{pmatrix} \in \mathcal{L}(U, X) \quad (12)$$

we observe that the second derivative of $\hat{J}$ can be expressed as

$$\hat{J}''(u) = T^*(x) L_{xx}(x,\lambda) T(x), \tag{13}$$

where $x = (y(u), u)$ and the second derivative of L with respect to x is given by

$$L_{xx}(x,\lambda) = \begin{pmatrix} J_{yy}(x) + \langle e^1_{yy}(x)(\cdot,\cdot), \lambda^1 \rangle & 0 \\ 0 & J_{uu}(x) \end{pmatrix} \in \mathcal{L}(X, X^*).$$

As numerical solution method for the optimal control problem (3) we suggest Newton's algorithm which for the sake of reference is specified next.

Algorithm 2.1. (Newton Algorithm).

1. Choose $u^0 \in N(u^*)$, set $k = 0$.
2. Do until convergence
   i) solve $\hat{J}''(u^k)\delta u^k = -\hat{J}'(u^k)$,
   ii) update $u^{k+1} = u^k + \delta u^k$,
   iii) set $k = k + 1$.

Here u^* denotes a local solution of problem (3). The numerical complexity of Algorithm 2.1 applied to problem (3) is investigated in [12]. In the same paper also a local convergence prove is given. Numerical experiments show that despite the implementational complexity the performance of the algorithm compared to that of gradient-type methods with respect to computation time is superior [13]. However, to anticipate parts of the discussion of the numerical results in Section 4 the running time of Newton's method for optimal control computations for the cylinder flow on a realistic grid utilizing a ORIGIN 200 takes days. Moreover, several gradient iterations have to be applied to provide a suitable initial guess to ensure local convergence of Newton's algorithm.

3 Reduced order modeling (ROM) using proper orthogonal decomposition (POD)

The ROM approach to optimal control problems such as (1) is based on approximating the nonlinear dynamics by a Galerkin technique utilizing basis functions that contain characteristics of the expected controlled flow. This is in contrast to finite element based Galerkin schemes where the basis elements are not related to the physical properties of the system that they approximate. Consequently one expects that only a few basis elements will provide a reasonably good description of the controlled flow. Various ROM techniques differ in the choice of the basis functions [11, 15]. Here we use the snapshot variant of POD introduced by Sirovich in [20] to obtain a low-dimensional approximation of the Navier-Stokes equations and to define the control space.

First we describe the model reduction. To begin with let $y^1, \dots, y^n$ denote an ensemble of snapshots of the flow corresponding to different time instances. For the flow we make the Ansatz

$$y = \bar{y} + \sum_{i=1}^{n} \alpha_i \Phi_i \tag{14}$$

with modes Φ_i and mean $\bar{y}$ that are obtained as follows.

1. Compute mean $\bar{y} = \frac{1}{n} \sum\limits_{i=1}^{n} y^i$
2. Build correlation matrix $K = k_{ij}$, $k_{ij} = \int\limits_{\Omega} (y^i - \bar{y})(y^j - \bar{y})\, dx$
3. Compute eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $v^1, \dots, v^n$ of K
4. Set $\Phi_i := \sum\limits_{j=1}^{n} v_j^i (y^j - \bar{y})$
5. Normalize $\Phi_i = \frac{\Phi_i}{||\Phi_i||}$

Now choose the eigenvectors of the matrix K pairwise orthogonal (this is possible since K is symmetric). Then it follows that the modes Φ_i are pairwise orthonormal. Moreover, they are optimal with respect the L^2 scalar product in the sense that no other basis of $D := \text{span}\{y_1 - \bar{y}, \dots, y_n - \bar{y}\}$ can contain more energy in fewer elements, compare [2, 16]

In order to obtain a low-dimensional basis for the Galerkin Ansatz modes corresponding to small eigenvalues are neglected. To make this idea more precise let $D^M := \text{span}\{\Phi_1, \dots, \Phi_M\}$ $(1 \le M \le N := \text{dim} D)$ and define the relative information content of this basis by

$$\text{I}(M) \;:=\; \sum_{k=1}^{M} \lambda_k \,/\, \sum_{k=1}^{N} \lambda_k \,.$$

If the basis is required that contains $\delta\%$ of the total information contained in the space D, say the dimension M of the subspace D^M is determined by

$$M = \text{argmin}\,\{\text{I}(M);\, \text{I}(M) \ge \delta\}\,. \tag{15}$$

The reduced dynamical system is obtained by plugging in (14) into the Navier-Stokes System and using a subspace D^M containing sufficient information as test space. This results in

$$(y_t, \Phi_j) + \nu(\nabla y, \nabla \Phi_j) + ((y \cdot \nabla) y, \Phi_j) = (Bu, \Phi_j) \quad (j = 1, \dots, M).$$

which may be rewritten as

$$\dot{\alpha} + A\alpha = n(\alpha) + \mathcal{M}_1 \beta + r, \quad \alpha(0) = a_0.$$

Here the initial values y_0 are approximated by $\bar{y} + \sum_{k=1}^{M} a_0^k \Phi_k$ and the i-th component a_0^i of the M-vector a_0 is given by $(y_0 - \bar{y}, \Phi_i)$. The matrix A is the POD stiffness matrix and the inhomogeneity r results from the contribution of the mean $\bar{y}$ to the Ansatz in (14). The matrix $\mathcal{M}_1$ has the entries $m_{ij}^1 = \int_{\Omega_c} \Phi_i \Phi_j \, dx$, and the M-vector β results from the contribution of the controls, i.e. for them we make the Ansatz

$$u = \sum_{i=1}^{M} \beta_i \Phi_i \chi_{\Omega_c}. \tag{16}$$

Here, χ_{Ω_c} denotes the characteristic function of the control domain Ω_c. We note that $\mathcal{M}_1$ is the identity matrix in the casr $\Omega_c = \Omega$. The reduced optimization problem corresponding to (1) is obtained by plugging in (14) and (16) into the cost functional and utilizing the reduced dynamical system as subsidiary condition in the optimization process. Finally, writing $\int_{\Omega_o} |y - z|^2 \, dx = (\alpha - \alpha^z)^t \mathcal{M}_2 (\alpha - \alpha^z)$, where the entries of $\mathcal{M}_2$ are given by $m_{ij}^2 = \int_{\Omega_o} \Phi_i \Phi_j \, dx$ and the desired state z is written as $z = \bar{y} + \sum_{k=1}^{M} \alpha_k^z \Phi_k$ we obtain

$$\text{(ROM)} \begin{cases} \min J(\alpha, \beta) = \frac{1}{2} \int_0^T (\alpha - \alpha^z)^t \mathcal{M}_2 (\alpha - \alpha^z) \, dt + \frac{\gamma}{2} \int_0^T \beta^t \mathcal{M}_1 \beta \, dt \\ \text{s.t.} \\ \dot{\alpha} + A\alpha = n(\alpha) + \mathcal{M}_1 \beta + r, \quad \alpha(0) = a_0. \end{cases} \tag{17}$$

With these preparations we are in the position to formulate the quasi-optimal control strategy. We choose the value δ for (15) and a sequence of increasing numbers N_j.

Algorithm 3.1. (POD-based adaptive control)

1. Let a set of snapshots $y_i^0, i = 1, \ldots, N_0$ be given and set j=0.
2. Compute M according to (15) with N replaced by N_j.
3. Compute POD modes and solve the reduced optimization problem (17) for $u^j \in D_j^M$.
4. Compute the state y^j corresponding to the current control u^j and add the snapshots $y_i^{j+1}, i = N_j + 1, \ldots, N_{j+1}$ to the snapshot set $y_i^j, i = 1, \ldots, N_j$.
5. If $|u^{j+1} - u^j|$ is not sufficiently small, set j = j+1 and goto 2.

It cannot be claimed that the controls obtained with this algorithm in some sense converge to an optimal control. However, the numerical results to be presented justify the approach, especially with regard to the computing times.

4 Numerical results

Here we present numerical computations related to the approaches presented in the previous paragraphs. The flow configuration is given by the two-dimensional channel with a circular cylinder and coincides with that used in [19] as a benchmark for Navier-Stokes solver, see Fig. 1. The control gain is to track the Stokes

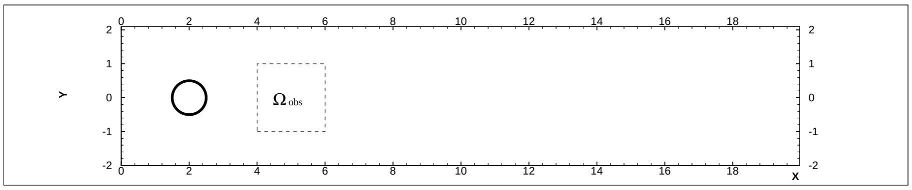

Figure 1: Flow domain and observation volume

flow in a given observation volume Ω_o by applying control forces in a control volume Ω_c. The Reynolds number Re$= 1/\nu$ for this configuration is determined by

$$\mathrm{Re} = \frac{\bar{U}d}{\mu}$$

with $\bar{U}$ denoting the bulk velocity at the inlet, d the diameter of the cylinder and μ denoting the molecular viscosity of the fluid. In all numerical experiments presented we choose a parabolic inflow profile at the inlet, $d = 1$, Re=100 and as channel length $l = 19.5d$. At the outflow boundary so called do-nothing boundary conditions are used, see [10]. This means that the boundary conditions used for the numerical computations are different from them given in (1). For the spatial discretization the Taylor-Hood finite element on a grid with 7808 triangles, 16000 velocity and 4096 pressure nodes is used. As time interval we use $[0, T]$ with $T = 3.4$ which coincides with the length of one period of the wake flow. For the time discretization we use a fractional step Θ-scheme [3] or a semi-implicit Euler-scheme on a grid containing 500 points, which corresponds to a time step size of $\delta t = 0.0068$. The total number of variables in the optimization problem (1) therefore is of order 5.4×10^7 (primal, adjoint and control variables). In all numerical examples $\gamma = 2.10^{-2}$.

For the POD we add 100 snapshots to the snapshot set in every iteration of Algorithm 3.1. The relative information content of the basis formed by the modes is required to be larger than 99.99%. In Fig. 2 the vorticity and the streamlines of the uncontrolled flow (top) and the Stokes flow (bottom) are presented. For the numerical solution of the reduced optimization problems the Schur-complement SQP-algorithm is used. All computations are performed on a ORIGIN 200.

Example 1 We first present the results for $\Omega_c = \Omega_o = \Omega$. Fig. 3 compares the evolution of the cost in $[0, T]$ for the optimal control and the control obtained by Algorithm 3.1. The adaptive algorithm terminates after 5 iterations and

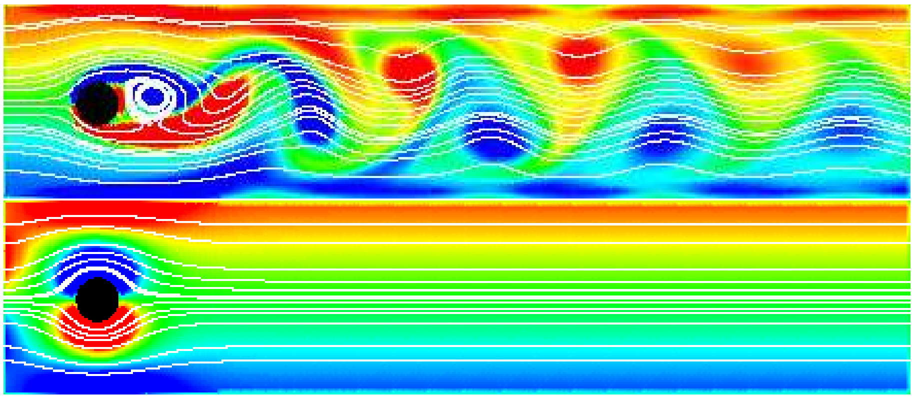

Figure 2: Uncontrolled flow (top) and Stokes flow (bottom)

achieves a remarkable cost reduction. To recover 99.99% of the energy stored in the snapshots in the first iteration 10 modes have to be taken, 20 in the second iteration, 26 in the third, 30 in the fourth, and 36 in the final iteration. The computation of the optimal control takes approximately 140 CPU hours with the Newton method including the initialization process with a step-size controlled gradient algorithm. To obtain a relative error $|\nabla J(u^n)|/|\nabla J(u^0)|$ lower than 10^{-2} 32 gradient iterations are needed. The value of the cost functional for the uncontrolled flow is $J(u^0) = 22.658437$, after 32 iterations this value is reduced to $J(u^{32}) = 1.138325$. Here, as initial control $u^0 = 0$ is taken. Note that every gradient step amounts to solving the non-linear Navier-Stokes equations in (1), the adjoint equations (10) and a further Navier-Stokes system for the computation of the step-size in the gradient algorithm, compare [13]. Newton's algorithm then is initialized with u^{32} and 3 Newton steps further reduce the value of the cost functional to $J(u^*) = 1.090321$. The numerical amount of work for Newton's method in the outer loop coincides with that of one gradient step. The Newton system 2.i) in Algorithm 3.1 has to be solved iteratively, where every iteration amounts to solving the systems (4) and (5), respectively with apropriate right-hand-sides f and g, respectively. Algorithm 3.1 takes 5 iterations and 8 CPU hours to obtain the quasi-optimal control $\tilde{u}^*$. The final value of the cost is $J(\tilde{u}^*) = 6.440180$. Fig. 4 shows a zoom on the evolutions of the control cost for both approaches. The optimal cost at $t = 0$ has the value 122.71 and is approximately 10 times larger than the quasi-optimal cost at the same time. Fig. 5 shows the streamlines and the vorticity of the flow controlled by the adaptive approach at $t = 3.4$ (top) and the mean flow $\bar{y}$ (bottom), the latter formed with the snapshots of all 5 iterations. The controlled flow no longer contains vortex sheddings and is approximately stationary. Recall that the controls are sought in the space of deviations from the mean flow. This explains the remaining recirculations behind the cylinder. In the previous example the observation volume Ω_o and the control volume Ω_c each cover the

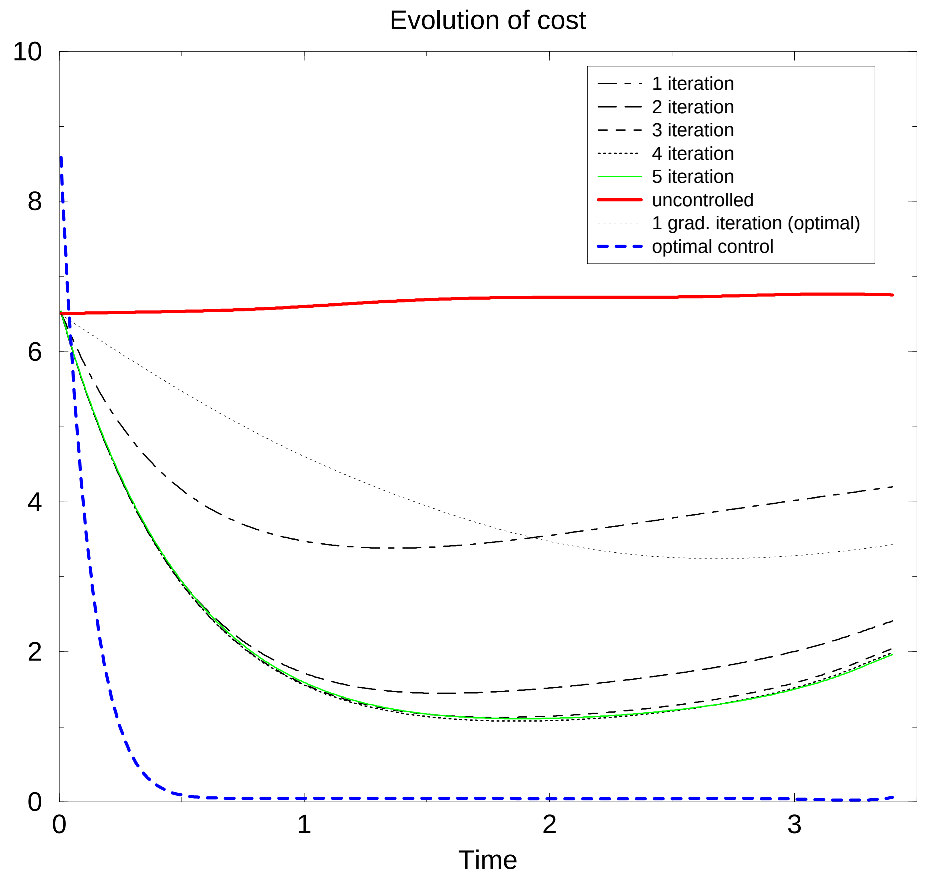

Figure 3: Example 1, Evolution of the cost $\frac{1}{2}(\alpha-\alpha^z)^t\mathcal{M}_2(\alpha-\alpha^z)+\frac{\gamma}{2}\beta^t\mathcal{M}_1\beta 1$

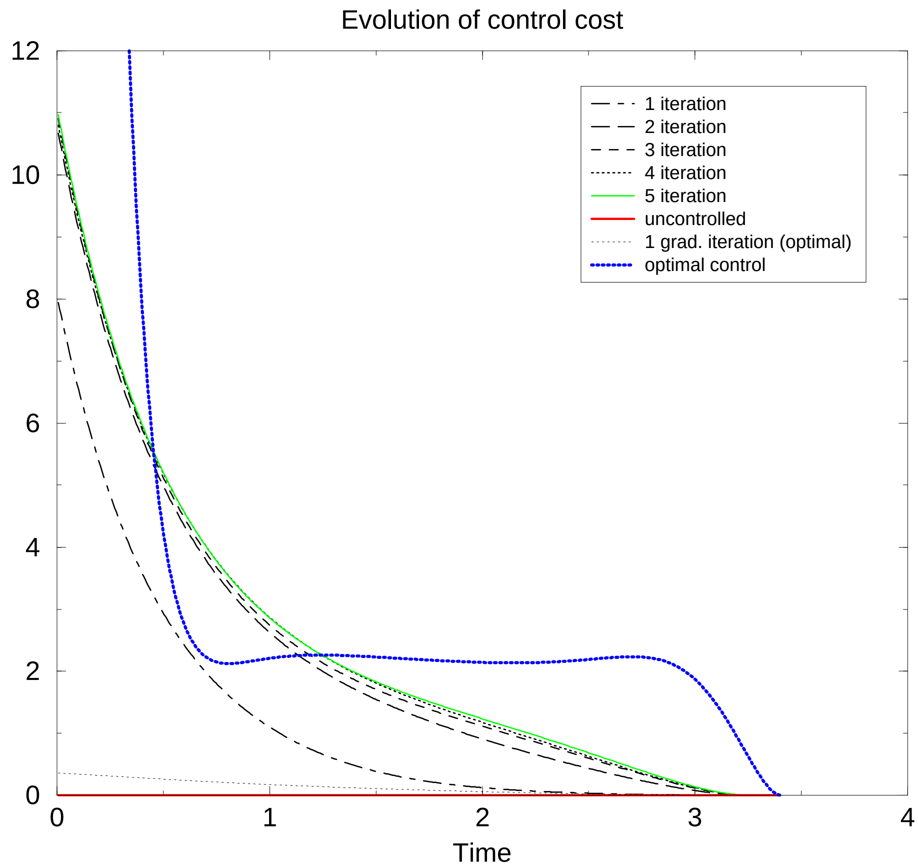

Figure 4: Example 1, Evolution of control control cost $|\beta|^2$

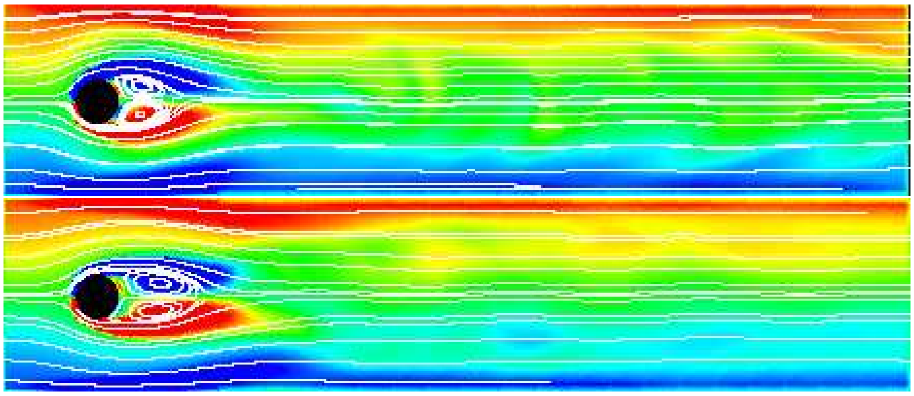

Figure 5: Example 1: Controlled flow (top) and mean flow $\bar{y}$ (bottom)

whole spatial domain. From the practical point of view this is not feasible. However, from the numerical standpoint this is a complicated situation, since the inhomogeneities in the primal and adjoint equations are large. We next present a numerical example with different observation and control volumes. This results in smaller control and observation volumes than in Example 1, and thus the primal and adjoint equations in the optimal control algorithm are numerically simpler to solve.

Example 2 Here $\Omega_c = \Omega_o = [4, 6] \times [-1, 1]$. Fig. 6 compares the evolution of the cost in $[0, T]$ for the optimal control and the adaptively improved control. The adaptive algorithm in this example terminates after 6 iterations and again achieves a remarkable cost reduction. The best value of the cost in the quasi-optimal case is nearly obtained already after the second iteration of Algorithm 3.1. The computation of the optimal control takes approximately 100 CPU hours utilizing a gradient algorithm with step-size control and initial control $u^0 = 0$. Using the same stopping criterion as in Example 1 the algorithm takes 28 iterations to reduce the cost from $J(u^0) = 4.087893$ to $J(u^{28}) = 0.637754$. The quasi-optimal control $\tilde{u}^*$ is computed by 6 iterations of Algorithm 3.1 within 7 CPU hours. The optimal function value in this case is $J(\tilde{u}^*) = 0.942525$. In Fig. 7 a zoom is shown that compares the evolutions of the control cost for both approaches. The value of the optimal cost at the beginning of the control horizon has the value 36.11 and is again approximately a magnitude larger than that of the quasi-optimal cost. An interesting observation is made when comparing the evolution of $|y-z|^2_{L^2(\Omega)}$ in the control horizon. As is indicated in Fig. 8 the quasi-optimal approach obtains a reduction of this difference similar to that of the optimal control approach, but at significantly lower control costs. This is due to the global nature of the Ansatz and the test functions in the POD method. As a consequence also local deviations have an immediate global impact. Of course, this is different for the optimal approach since the the finite element method is used to approximate the optimal controls.

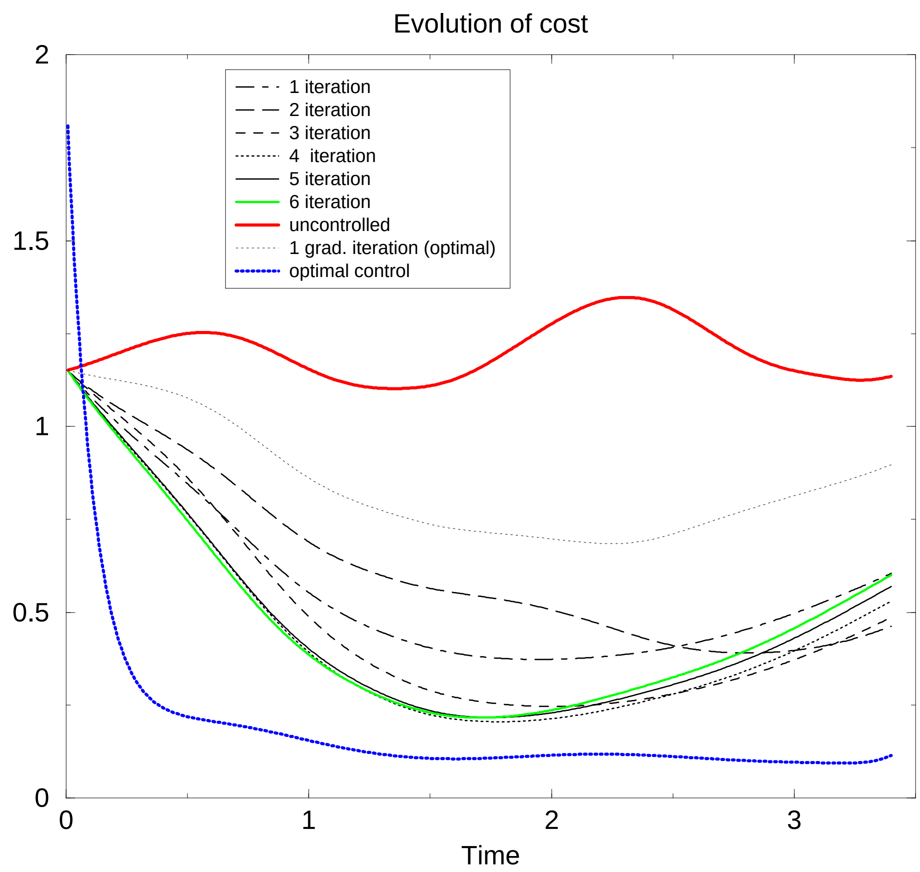

Figure 6: Example 2, Evolution of the cost $\frac{1}{2}(\alpha - \alpha^z)^t \mathcal{M}_2 (\alpha - \alpha^z) + \frac{\gamma}{2} \beta^t \mathcal{M}_1 \beta$

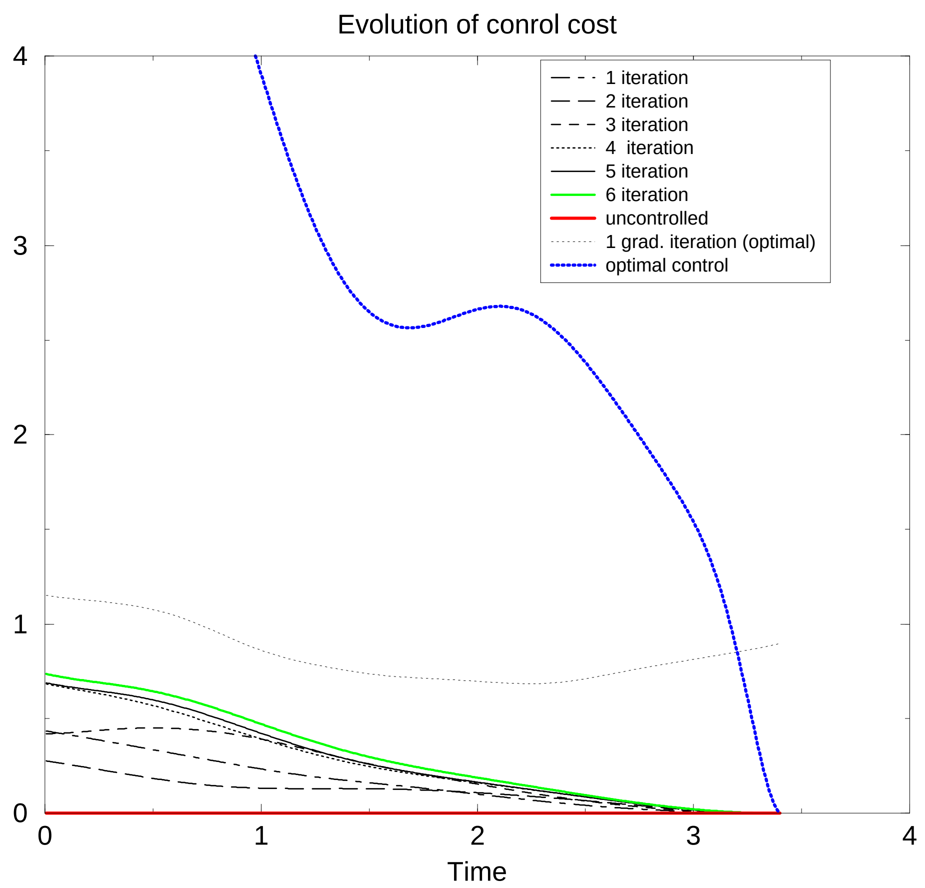

Figure 7: Example 2, Evolution of the control cost $\beta^t \mathcal{M}_1 \beta$

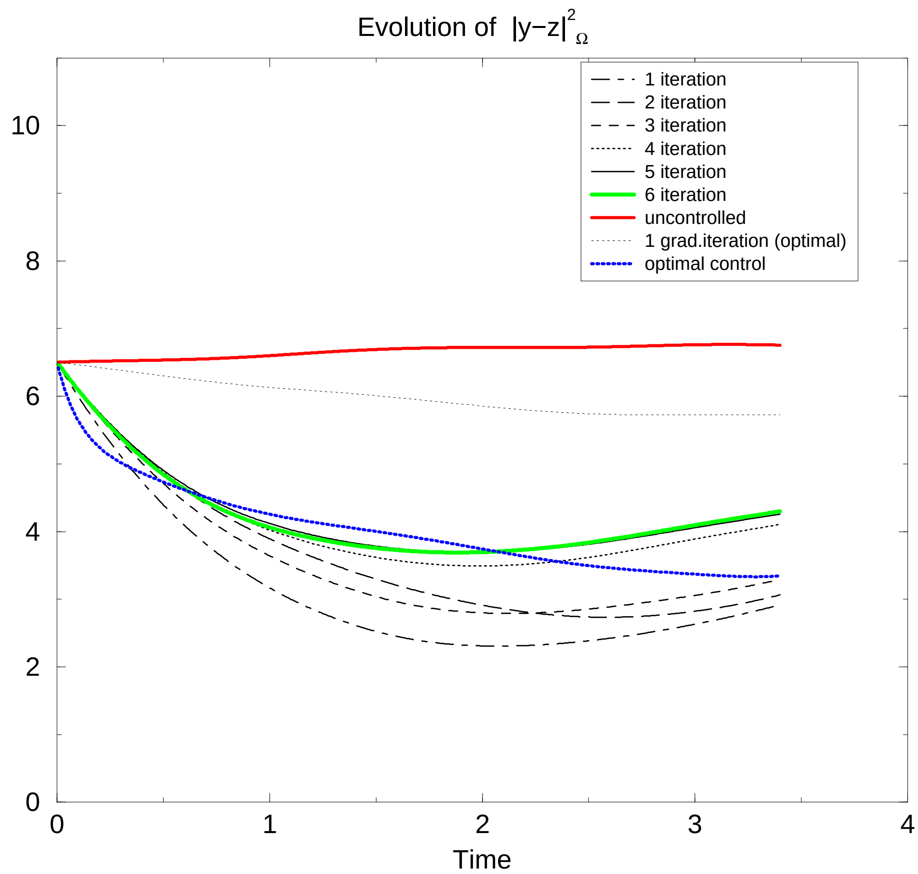

Figure 8: Example 2, Evolution of $|y - z|^2_{L^2(\Omega)}$

5 Conclusions

An adaptive quasi-optimal control approach for the wake flow around a cylinder is presented. The method is based on the snapshot form of the POD for the Navier-Stokes equations and iteratively computes controls by solving low-dimensional optimization problems. Compared to the optimal control approach the quasi-optimal method is much less time consuming and provides competitive controls. Moreover, due to the global nature of the POD basis functions already local measurements can be sufficient for achieving more global control goals.

Acknowledgements

The authors acknowledge the support of the Sonderforschungsbereich 557 **Beeinflussung komplexer turbulenter Scherströmungen**, sponsored by the Deutsche Forschungsgemeinschaft.

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